

Perturbed Iterative Algorithms with Errors for Completely Generalized Strongly Nonlinear Implicit Quasivariational Inclusions

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In this paper, we introduce a new class of completely generalized strongly nonlinear implicit quasivariational inclusions and prove its equivalence with a class of fixed point problems by using some properties of maximal monotone mappings. We also prove the existence of solutions for the completely generalized strongly nonlinear implicit quasivariational inclusions and the convergence of iterative sequences generated by the perturbed algorithms with errors.

Keywords: Completely generalized strongly nonlinear implicit quasivariational inclusion; Mann type iterative sequence with errors; Ishikawa type iterative sequence with errors

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1 INTRODUCTION

It is well known that variational inequality theory and complementarity problem theory are very powerful tool of the current mathematical technology. In recent years, classical variational inequality and complementarity problem have been extended and generalized to study a

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wide class of problems arising in mechanics, physics, optimization and control, nonlinear programming, economics and transportation equilibrium and engineering sciences, etc. Various quasi-(implicit)variational inequalities, generalized quasi-(implicit)variational inequalities, quasi-(implicit)complementarity problem and generalized quasi-(implicit)-complementarity problem are very important generalizations of these classical problems. For details we refer to [1,3–18,20–29] and the references therein.

Recently, Huang [15,16] introduced and studied the Mann and Ishikawa type perturbed iterative sequences with errors for the generalized implicit quasivariational inequalities and inclusions. Inspired and motivated by recent research works in this field, in this paper, we introduce a new class of completely generalized strongly nonlinear implicit quasivariational inclusions and prove its equivalence with a class of fixed point problems by using some properties of maximal monotone mappings. We also show the existence of solutions for this completely generalized strongly nonlinear implicit quasivariational inclusions and the convergence of iterative sequences generated by the perturbed algorithms with errors.

2 PRELIMINARIES

Let H be a real Hilbert space endowed with a norm $\|\cdot\|$ and an inner product $\langle \cdot, \cdot \rangle$. Let $f, p, g, m: H \rightarrow H$ and $N: H \times H \rightarrow H$ be single-valued mappings. Suppose that $M: H \times H \rightarrow 2^H$ is a set-valued mapping such that, for each fixed $y \in H$, $M(\cdot, y): H \rightarrow 2^H$ is a maximal monotone mapping and $\text{Range}(g - m) \cap \text{dom}(M(\cdot, y)) \neq \emptyset$ for each $y \in H$. We consider the following problem:

Find $u \in H$ such that

$$\begin{aligned} (g - m)(u) \cap \text{dom}(M(\cdot, u)) &\neq \emptyset, \\ 0 \in N(f(u), p(u)) + M((g - m)(u), u), \end{aligned} \tag{2.1}$$

where $g - m$ is defined as

$$(g - m)(u) = g(u) - m(u)$$

for each $u \in H$. The problem (2.1) is called the *completely generalized strongly nonlinear implicit quasivariational inclusion*.

Now, we give some special cases of the problem (2.1) as follows:

(I) If $M(\cdot, y) = \partial\varphi(\cdot, y)$ for each $y \in H$, where $\varphi : H \times H \rightarrow R \cup \{+\infty\}$ such that for each fixed $y \in H$, $\varphi(\cdot, y) : H \rightarrow R \cup \{+\infty\}$ is a proper convex lower semicontinuous function on H and $\text{Range}(g - m) \cap \text{dom}(\partial\varphi(\cdot, y)) \neq \emptyset$ for each $y \in H$ and $\partial\varphi(\cdot, y)$ denotes the subdifferential of function $\varphi(\cdot, y)$, then the problem (2.1) is equivalent to finding $u \in H$ such that

$$\begin{aligned} &(g - m)(u) \cap \text{dom}(\partial\varphi(\cdot, u)) \neq \emptyset, \\ \langle N(f(u), p(u)), v - (g - m)(u) \rangle &\geq \varphi((g - m)(u), u) - \varphi(v, u) \end{aligned} \tag{2.2}$$

for all $v \in H$, which is called the *generalized strongly nonlinear implicit quasivariational inclusion*.

(II) If $N(u, v) = u - v$ for all $u, v \in H$, then the problem (2.1) is equivalent to finding $u \in H$ such that

$$\begin{aligned} &(g - m)(u) \cap \text{dom}(M(\cdot, u)) \neq \emptyset, \\ 0 \in f(u) - p(u) + M((g - m)(u), u), \end{aligned} \tag{2.3}$$

which is called the *generalized nonlinear implicit quasivariational inclusion*, which was considered by Huang [16].

(III) If $N(u, v) = u - v$ for all $u, v \in H$, $m = 0$ and $M(x, y) = M(x)$ for all $y \in H$, where $M : H \rightarrow 2^H$ is a maximal monotone mapping, then the problem (2.1) is equivalent to finding $u \in H$ such that

$$\begin{aligned} &g(u) \cap \text{dom}(M(u)) \neq \emptyset, \\ 0 \in f(u) - p(u) + M(g(u)), \end{aligned} \tag{2.4}$$

which was studied by Adly [1].

(IV) If $N(u, v) = u - v$ for all $u, v \in H$ and $M(\cdot, y) = \partial\varphi(\cdot, y)$ for each $y \in H$, where $\varphi : H \times H \rightarrow R \cup \{+\infty\}$ such that, for each fixed $y \in H$, $\varphi(\cdot, y) : H \rightarrow R \cup \{+\infty\}$ is a proper convex lower semicontinuous function on H and $\text{Range}(g - m) \cap \text{dom}(\partial\varphi(\cdot, y)) \neq \emptyset$ for each $y \in H$ and $\partial\varphi(\cdot, y)$ denotes the subdifferential of function $\varphi(\cdot, y)$, then the problem (2.1) is equivalent to finding $u \in H$ such that

$$\begin{aligned} &(g - m)(u) \cap \text{dom}(\partial\varphi(\cdot, u)) \neq \emptyset, \\ \langle f(u) - p(u), v - (g - m)(u) \rangle &\geq \varphi((g - m)(u), u) - \varphi(v, u) \end{aligned} \tag{2.5}$$

for all $v \in H$, which is called the *generalized quasivariational inclusion*, which was considered by Ding [10].

(V) If $N(u, v) = u - v$ for all $u, v \in H$ and $M(\cdot, y) = \partial\varphi$ for all $y \in H$, where $\partial\varphi$ denotes the subdifferential of a proper convex lower semi-continuous function $\varphi: H \rightarrow R \cup \{+\infty\}$, then the problem (2.1) is equivalent to finding $u \in H$ such that

$$\begin{aligned} (g - m)(u) \cap \text{dom}(\partial\varphi) &\neq \emptyset, \\ \langle f(u) - p(u), v - (g - m)(u) \rangle &\geq \varphi((g - m)(u)) - \varphi(v) \end{aligned} \quad (2.6)$$

for all $v \in H$, which was studied by Kazmi [18].

(VI) If $K: H \rightarrow 2^H$ is a set-valued mapping such that, for each $x \in H$, $K(x)$ is a closed convex subset of H and, for each fixed $y \in H$, $M(\cdot, y) = \partial\varphi(\cdot, y)$, $\varphi(\cdot, y) = I_{K(y)}(\cdot)$ is the indicator function of $K(y)$ defined by

$$I_{K(y)}(x) = \begin{cases} 0, & \text{if } x \in K(y), \\ +\infty, & \text{if } x \notin K(y), \end{cases}$$

then the problem (2.1) is equivalent to finding $u \in H$ such that

$$\begin{aligned} g(u) - m(u) &\in K(u), \\ \langle N(f(u), p(u)), v - (g - m)(u) \rangle &\geq 0, \end{aligned} \quad (2.7)$$

for all $v \in K(u)$.

It is clear that the completely generalized strongly nonlinear implicit quasivariational inclusion problem (2.1) includes many kinds of variational inequalities, quasivariational inequalities, complementarity problems and quasi-(implicit)complementarity problems of [1,6,13–16,25–27] as special cases.

3 PERTURBED ITERATIVE ALGORITHMS

LEMMA 3.1 $u \in H$ is a solution of the problem (2.1) if and only if there exists $u \in H$ such that

$$u = F(u) \stackrel{\text{def}}{=} u - (g - m)(u) + J_\alpha^{M(\cdot, u)}((g - m)(u) - \alpha N(f(u), p(u))), \quad (3.1)$$

where $\alpha > 0$ is a constant and $J_\alpha^{M(\cdot, u)}(I + \alpha M(\cdot, u))^{-1}$ is the so-called proximal mapping on H .

Proof Let $u \in H$ be a solution of the problem (3.1). From the definition of the proximal mapping $J_\alpha^{M(\cdot, u)}$, it follows that

$$(g - m)(u) - \alpha N(f(u), p(u)) \in (g - m)(u) + \alpha M((g - m)(u), u)$$

and so

$$N(p(u), f(u)) \in M((g - m)(u), u).$$

Thus $u \in H$ is a solution of the problem (2.1).

Conversely, if $u \in H$ is a solution of (2.1), we have $u \in H$ such that

$$(g - m)(u) \cap \text{dom}(M(\cdot, u)) \neq \emptyset$$

and

$$0 \in N(f(u), p(u)) + M((g - m)(u), u).$$

Hence we have

$$(g - m)(u) - \alpha N(f(u), p(u)) \in (g - m)(u) + \alpha M((g - m)(u), u).$$

From the definition of the proximal mapping $J_\alpha^{M(\cdot, u)}$, we know that $u \in H$ is a solution of the problem (3.1). This completes the proof.

Based on Lemma 3.1, we now suggest and analyze the following new general and unified algorithms for the problem (2.1).

(A) Mann Type Perturbed Iterative Algorithm with Errors (MTPIAE)

Let $f, p, g, m : H \rightarrow H$ and $N : H \times H \rightarrow H$ be mappings. Given $u_0 \in H$, the iterative sequence $\{u_n\}$ are defined by

$$u_{n+1} = (1 - \alpha_n - \gamma_n)u_n + \alpha_n[u_n - (g - m)(u_n) + J_\alpha^{M^n(\cdot, u_n)}((g - m)(u_n) - \alpha N(f(u_n), p(u_n)))] + \gamma_n e_n \tag{3.2}$$

for $n = 0, 1, 2, \dots$, where $M^n : H \times H \rightarrow 2^H$ is a set-valued mapping such that, for each $y \in H$, $M^n(\cdot, y) : H \rightarrow 2^H$ is a maximal monotone mapping for $n = 0, 1, 2, \dots$, $\alpha > 0$ is a constant, $\{e_n\}$ is a bounded

sequence of the element of H introduced to take into account possible inexact computation and the sequences $\{\alpha_n\}$, $\{\gamma_n\}$ in $[0, 1]$ satisfying the following conditions:

- (1) $\alpha_n + \gamma_n \leq 1$ for $n = 0, 1, 2, \dots$,
- (2) $\alpha_n \rightarrow 0$ as $n \rightarrow \infty$,
- (3) $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=0}^{\infty} \gamma_n < \infty$.

(B) Ishikawa Type Perturbed Iterative Algorithm with Errors (ITPIAE)

Let $f, p, g, m: H \rightarrow H$ and $N: H \times H \rightarrow H$ be mappings. Given $u_0 \in H$, the iterative sequence $\{u_n\}$ are defined by

$$\begin{aligned} u_{n+1} &= (1 - \alpha_n - \gamma_n)u_n + \alpha_n[v_n - (g - m)(v_n) \\ &\quad + J_{\alpha}^{M^n(\cdot, v_n)}((g - m)(v_n) - \alpha N(f(v_n), p(v_n)))] + \gamma_n e_n, \\ v_n &= (1 - \beta_n - \delta_n)u_n + \beta_n[u_n - (g - m)(u_n) \\ &\quad + J_{\alpha}^{M^n(\cdot, u_n)}((g - m)(u_n) - \alpha N(f(u_n), p(u_n)))] + \delta_n f_n \end{aligned} \quad (3.3)$$

for $n = 0, 1, 2, \dots$, where $M^n: H \times H \rightarrow 2^H$ is a set-valued mapping such that, for each $y \in H$, $M^n(\cdot, y): H \rightarrow 2^H$ is a maximal monotone mapping for $n = 0, 1, 2, \dots$, $\alpha > 0$ is a constant, $\{e_n\}$, $\{f_n\}$ are two bounded sequences in H introduced to take into account possible in exact computation and the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ in $[0, 1]$ satisfying the following conditions:

- (1) $\alpha_n + \gamma_n \leq 1$, $\beta_n + \delta_n \leq 1$ for all $n = 0, 1, 2, \dots$,
- (2) $\alpha_n \rightarrow 0$, $\delta_n \rightarrow 0$ as $n \rightarrow \infty$,
- (3) $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=0}^{\infty} \gamma_n < \infty$.

(C) Mann Type Perturbed Iterative Algorithm (MTPIA)

Let $f, p, g, m: H \rightarrow H$ and $N: H \times H \rightarrow H$ be mappings. Given $u_0 \in H$, the iterative sequence $\{u_n\}$ are defined by

$$\begin{aligned} u_{n+1} &= (1 - \alpha_n)u_n + \alpha_n[u_n - (g - m)(u_n) \\ &\quad + J_{\alpha}^{M^n(\cdot, u_n)}((g - m)(u_n) - \alpha N(f(u_n), p(u_n)))] \end{aligned} \quad (3.4)$$

for $n = 0, 1, 2, \dots$, where $M^n : H \times H \rightarrow 2^H$ is a set-valued mapping such that, for each $y \in H$, $M^n(\cdot, y) : H \rightarrow 2^H$ is a maximal monotone mapping for $n = 0, 1, 2, \dots$, $\alpha > 0$ is a constant, the sequence $\{\alpha_n\}$ in $[0, 1]$ satisfying the following conditions:

- (1) $\alpha_n \rightarrow 0$ as $n \rightarrow \infty$,
- (2) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

(D) Ishikawa Type Perturbed Iterative Algorithm (ITPIA)

Let $f, p, g, m : H \rightarrow H$ and $N : H \times H \rightarrow H$ be mappings. Given $u_0 \in H$, the iterative sequence $\{u_n\}$ are defined by

$$\begin{aligned}
 u_{n+1} &= (1 - \alpha_n)u_n + \alpha_n[v_n - (g - m)(v_n) \\
 &\quad + J_{\alpha}^{M^n(\cdot, v_n)}((g - m)(v_n) - \alpha N(f(v_n), p(v_n)))], \\
 v_n &= (1 - \beta_n)u_n + \beta_n[u_n - (g - m)(u_n) \\
 &\quad + J_{\alpha}^{M^n(\cdot, u_n)}((g - m)(u_n) - \alpha N(f(u_n), p(u_n)))]
 \end{aligned}
 \tag{3.5}$$

for $n = 0, 1, 2, \dots$, where $M^n : H \times H \rightarrow 2^H$ is a set-valued mapping such that, for each $y \in H$, $M^n(\cdot, y) : H \rightarrow 2^H$ is a maximal monotone mapping for $n = 0, 1, 2, \dots$, $\alpha > 0$ is a constant, the sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}$ in $[0, 1]$ satisfying the following conditions:

- (1) $\alpha_n \rightarrow 0$ as $n \rightarrow \infty$,
- (2) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Remark 3.1

- (1) If $\beta_n = \delta_n = 0$ for all $n = 0, 1, 2, \dots$, then ITPIAE reduces to MTPIAE.
- (2) If $\gamma_n = \delta_n = 0$ for all $n = 0, 1, 2, \dots$, then ITPIAE reduces to ITPIA.
- (3) If $\gamma_n = 0$ for all $n = 0, 1, 2, \dots$, then MTPIAE reduces to MTPIA.
- (4) Our ITPIAE includes several known algorithms of [1,6,10,15,16,18] as special cases.

4 EXISTENCE AND CONVERGENCE THEOREMS

In this section, we show the existence of a solution of the problem (2.1) and the convergence of iterative sequence generated by ITPIAE.

DEFINITION 4.1 Let $N : H \times H \rightarrow H$ and $g : H \rightarrow H$.

(1) g is said to be strongly monotone if there exists some $\delta > 0$ such that

$$\langle g(u_1) - g(u_2), u_1 - u_2 \rangle \geq \delta \|u_1 - u_2\|^2$$

for all $u_i \in H, i = 1, 2$;

(2) g is said to be Lipschitz continuous if there exists some $\sigma > 0$ such that

$$\|g(u_1) - g(u_2)\| \leq \sigma \|u_1 - u_2\|$$

for all $u_i \in H, i = 1, 2$;

(3) g is said to be strongly monotone with respect to the first argument of N if there exists some $\mu > 0$ such that

$$\langle N(g(u_1), \cdot) - N(g(u_2), \cdot), u_1 - u_2 \rangle \geq \mu \|u_1 - u_2\|^2$$

for all $u_i \in H, i = 1, 2$;

(4) g is said to be Lipschitz continuous with respect to the first argument of N if there exists some $\xi > 0$ such that

$$\|N(g(u_1), \cdot) - N(g(u_2), \cdot)\| \leq \xi \|u_1 - u_2\|$$

for all $u_i \in H, i = 1, 2$.

In a similar way, we can define strong monotonicity and Lipschitz continuity of g with respect to the second argument of N .

DEFINITION 4.2 Let $\{S^n\}$ and S be maximal monotone mappings for $n = 1, 2, \dots$. The sequence $\{S^n\}$ is said to be graph-convergence to S (write $S^n \xrightarrow{G} S$) if, for every $(x, y) \in \text{Graph}(S)$, there exists a sequence $(x_n, y_n) \in \text{Graph}(S^n)$ such that $x_n \rightarrow x$ and $y_n \rightarrow y$ in H .

LEMMA 4.1 [2] Let $\{S^n\}$ and S be maximal monotone mappings for $n = 0, 1, 2, \dots$. Then $S^n \xrightarrow{G} S$ if and only if $J_\lambda^{S^n}(x) \rightarrow J_\lambda^S(x)$ for every $x \in H$ and $\lambda > 0$.

LEMMA 4.2 [19] Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be three nonnegative real sequences such that, for all $n = 0, 1, 2, \dots$,

$$a_{n+1} \leq (1 - t_n)a_n + b_n + c_n$$

with $0 \leq t_n \leq 1$, $\sum_{n=0}^{\infty} t_n = \infty$, $b_n = o(t_n)$ and $\sum_{n=0}^{\infty} c_n < \infty$. Then $\lim_{n \rightarrow \infty} a_n = 0$.

THEOREM 4.1 *Let $N: H \times H \rightarrow H$ be a mapping, $g, m: H \rightarrow H$ be Lipschitz continuous with constants σ and μ , respectively, $f, p: H \rightarrow H$ be Lipschitz continuous with constants η and ϵ with respect to the first and second arguments of N , respectively, f be strongly monotone with constant β with respect to the first argument of N and $g - m$ be strongly monotone with constant δ . Assume that*

$$\langle m(v) - m(u), g(u) - g(v) \rangle \leq \lambda \|u - v\|^2 \tag{4.1}$$

for all $u, v \in H$ and for some constant λ such that $\lambda_0 \leq \lambda \leq \sigma\mu$, where

$$\lambda_0 = \inf\{s: \langle m(v) - m(u), g(u) - g(v) \rangle \leq s \|u - v\|^2 \text{ for all } u, v \in H\}.$$

Suppose that there exists a constant $\xi > 0$ such that, for each $x, y, z \in H$,

$$\|J_\alpha^{M(\cdot, x)}(z) - J_\alpha^{M(\cdot, y)}(z)\| \leq \xi \|x - y\|. \tag{4.2}$$

If the following conditions hold:

$$\left| \alpha - \frac{\beta + \epsilon(k - 1)}{\eta^2 - \epsilon^2} \right| < \frac{\sqrt{(\beta + (k - 1)\epsilon)^2 - (\eta^2 - \epsilon^2)k(2 - k)}}{\eta^2 - \epsilon^2}, \tag{4.3}$$

$$\beta > (1 - k)\epsilon + \sqrt{(\eta^2 - \epsilon^2)k(2 - k)}, \quad \eta > \epsilon, \tag{4.4}$$

$$\alpha\epsilon < 1 - k, \quad k = \xi + 2\sqrt{1 - 2\delta + \sigma^2 + \mu^2 + 2\lambda}, \quad k < 1, \tag{4.5}$$

then the problem (2.1) has a unique solution $u^* \in H$. Moreover, suppose that $M^n: H \times H \rightarrow 2^H$ is a set-valued mapping such that, for each $y \in H$, $M^n(\cdot, y): H \rightarrow 2^H$ is a maximal monotone mapping for $n = 0, 1, 2, \dots$, $M^n(\cdot, y) \xrightarrow{G} M(\cdot, y)$ and, for each $x, y, z \in H$,

$$\|J_\alpha^{M^n(\cdot, x)}(z) - J_\alpha^{M^n(\cdot, y)}(z)\| \leq \xi \|x - y\|. \tag{4.6}$$

Then $\{u_n\}$ strongly convergences to it u^* , where $\{u_n\}$ is defined by ITPIAE.

Proof First we prove that there exists $u^* \in H$ which is a unique solution of the problem (2.1). By Lemma 3.1, it is enough to show that the mapping $F: H \rightarrow H$ defined by (3.1) has a unique fixed point $u^* \in H$. In fact, for any $u, v \in H$, we have

$$F(u) = u - (g - m)(u) + J_\alpha^{M(\cdot, u)}((g - m)(u) - \alpha N(f(u), p(u)))$$

and

$$F(v) = v - (g - m)(v) + J_\alpha^{M(\cdot, v)}((g - m)(v) - \alpha N(f(v), p(v))).$$

From the definition of $J_\alpha^{M(\cdot, u)}$ and (4.2), we have

$$\begin{aligned} & \|F(u) - F(v)\| \\ &= \|u - (g - m)(u) + J_\alpha^{M(\cdot, u)}((g - m)(u) - \alpha N(f(u), p(u))) \\ &\quad - [v - (g - m)(v) + J_\alpha^{M(\cdot, v)}((g - m)(v) - \alpha N(f(v), p(v)))]\| \\ &\leq \|u - v - ((g - m)(u) - (g - m)(v))\| \\ &\quad + \|J_\alpha^{M(\cdot, u)}((g - m)(u) - \alpha N(f(u), p(u))) \\ &\quad - J_\alpha^{M(\cdot, u)}((g - m)(v) - \alpha N(f(v), p(v)))\| \\ &\quad + \|J_\alpha^{M(\cdot, u)}((g - m)(v) - \alpha N(f(v), p(v))) \\ &\quad - J_\alpha^{M(\cdot, v)}((g - m)(v) - \alpha N(f(v), p(v)))\| \\ &\leq 2\|u - v - ((g - m)(u) - (g - m)(v))\| \\ &\quad + \|u - v - \alpha(N(f(u), p(u)) - N(f(v), p(u)))\| \\ &\quad + \alpha\|N(f(v), p(u)) - N(f(v), p(v))\| + \xi\|u - v\|. \end{aligned} \tag{4.7}$$

By (4.1), the Lipschitz continuity of f, g, m and strong monotonicity of $g - m$ and f , we obtain

$$\begin{aligned} & \|u - v - ((g - m)(u) - (g - m)(v))\|^2 \\ &= \|u - v\|^2 - 2\langle u - v, (g - m)(u) - (g - m)(v) \rangle \\ &\quad + \|m(u) - m(v)\|^2 + \|g(u) - g(v)\|^2 \\ &\quad + 2\langle m(v) - m(u), g(u) - g(v) \rangle \\ &\leq (1 - 2\delta + \sigma^2 + \mu^2 + 2\lambda)\|u - v\|^2 \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} & \|u - v - \alpha(N(f(u), p(u)) - N(f(v), p(u)))\|^2 \\ & \leq (1 - 2\alpha\beta + \alpha^2\eta^2)\|u - v\|^2. \end{aligned} \tag{4.9}$$

Further, since p is Lipschitz continuous with respect to the second argument of N , we get

$$\|N(f(v), p(u)) - N(f(v), p(v))\| \leq \epsilon\|u - v\|. \tag{4.10}$$

From (4.7)–(4.10), it follows that

$$\|F(u) - F(v)\| \leq h\|u - v\|, \tag{4.11}$$

where

$$h = 2\sqrt{1 - 2\delta + \sigma^2 + \mu^2 + 2\lambda} + \xi + \sqrt{1 - 2\alpha\beta + \alpha^2\eta^2} + \alpha\epsilon.$$

From (4.3)–(4.5), we know that $0 < h < 1$ and so F has a unique fixed point $u^* \in H$. By Lemma 3.1 and (4.11), we know that u^* is a unique solution of the problem (2.1).

Next we prove that the iterative sequence $\{u_n\}$ defined by ITPIAE strongly converges to u^* . Since u^* is a solution of the problem (2.1), we have

$$F(u^*) = u^* - (g - m)(u^*) + J_\alpha^{M(\cdot, u^*)}((g - m)(u^*) - \alpha N(f(u^*), p(u^*))). \tag{4.12}$$

It follows from (3.3), (4.12) and Lemma 3.1 that

$$\begin{aligned} & \|u_{n+1} - u^*\| \\ & = \|(1 - \alpha_n - \gamma_n)u_n + \alpha_n[v_n - (g - m)(v_n) + J_\alpha^{M^n(\cdot, v_n)}((g - m)(v_n) \\ & \quad - \alpha N(f(v_n), p(v_n)))] + \gamma_n e_n - (1 - \alpha_n - \gamma_n)u^* \\ & \quad - \alpha_n[u^* - (g - m)(u^*) + J_\alpha^{M^n(\cdot, u^*)}((g - m)(u^*) \\ & \quad - \alpha N(f(u^*), p(u^*)))] + \gamma_n u^*\| \end{aligned}$$

$$\begin{aligned}
&\leq (1 - \alpha_n - \gamma_n)\|u_n - u^*\| + \alpha_n\|v_n - (g - m)(v_n) \\
&\quad - (u^* - (g - m)(u^*))\| + \gamma_n\|e_n - u^*\| + \alpha_n\|J_\alpha^{M^n(\cdot, v_n)}((g - m)(v_n) \\
&\quad - \alpha N(f(v_n), p(v_n))) - J_\alpha^{M(\cdot, u^*)}((g - m)(u^*) - \alpha N(f(u^*), p(u^*)))\| \\
&\leq (1 - \alpha_n - \gamma_n)\|u_n - u^*\| + \alpha_n\|v_n - u^* - ((g - m)(v_n) \\
&\quad - (g - m)(u^*))\| + \gamma_n\|e_n - u^*\| + \alpha_n\|J_\alpha^{M^n(\cdot, v_n)}((g - m)(v_n) \\
&\quad - \alpha N(f(v_n), p(v_n))) - J_\alpha^{M^n(\cdot, v_n)}((g - m)(u^*) - \alpha N(f(u^*), p(u^*)))\| \\
&\quad + \alpha_n\|J_\alpha^{M^n(\cdot, v_n)}((g - m)(u^*) - \alpha N(f(u^*), p(u^*))) \\
&\quad - J_\alpha^{M^n(\cdot, u^*)}((g - m)(u^*) - \alpha N(f(u^*), p(u^*)))\| \\
&\quad + \alpha_n\|J_\alpha^{M^n(\cdot, u^*)}((g - m)(u^*) - \alpha N(f(u^*), p(u^*))) \\
&\quad - J_\alpha^{M(\cdot, u^*)}((g - m)(u^*) - \alpha N(f(u^*), p(u^*)))\| \\
&\leq (1 - \alpha_n - \gamma_n)\|u_n - u^*\| + 2\alpha_n\|v_n - u^* - ((g - m)(v_n) \\
&\quad - (g - m)(u^*))\| + \xi\alpha_n\|v_n - u^*\| + \gamma_n\|e_n - u^*\| \\
&\quad + \alpha_n\|v_n - u^* - \alpha(N(f(v_n), p(v_n)) - N(f(u^*), p(v_n)))\| \\
&\quad + \alpha\alpha_n\|N(f(u^*), p(v_n)) - N(f(u^*), p(u^*))\| \\
&\quad + \alpha_n\|J_\alpha^{M^n(\cdot, u^*)}((g - m)(u^*) - \alpha N(f(u^*), p(u^*))) \\
&\quad - J_\alpha^{M(\cdot, u^*)}((g - m)(u^*) - \alpha N(f(u^*), p(u^*)))\|. \tag{4.13}
\end{aligned}$$

By (4.1), the Lipschitz continuity of f, g, m and strong monotonicity of $g - m$ and f , we obtain

$$\begin{aligned}
&\|v_n - u^* - ((g - m)(v_n) - (g - m)(u^*))\|^2 \\
&\leq (1 - 2\delta + \sigma^2 + \mu^2 + 2\lambda)\|v_n - u^*\|^2 \tag{4.14}
\end{aligned}$$

and

$$\begin{aligned}
&\|v_n - u^* - \alpha(N(f(v_n), p(v_n)) - N(f(u^*), p(v_n)))\|^2 \\
&\leq (1 - 2\alpha\beta + \alpha^2\eta^2)\|v_n - u^*\|^2. \tag{4.15}
\end{aligned}$$

Further since p is Lipschitz continuous with respect to the second argument of N , we get

$$\|N(f(u^*), p(v_n)) - N(f(u^*), p(u^*))\| \leq \epsilon \|v_n - u^*\|. \tag{4.16}$$

From (4.13)–(4.16), it follows that

$$\begin{aligned} \|u_{n+1} - u^*\| &\leq (1 - \alpha_n - \gamma_n)\|u_n - u^*\| + h\alpha_n\|v_n - u^*\| \\ &\quad + \alpha_nb_n + \gamma_n\|e_n - u^*\|, \end{aligned} \tag{4.17}$$

where

$$\begin{aligned} b_n &= \|J_\alpha^{M^n(\cdot, u^*)}((g - m)(u^*) - \alpha N(f(u^*), p(u^*))) \\ &\quad - J_\alpha^{M(\cdot, u^*)}((g - m)(u^*) - \alpha N(f(u^*), p(u^*)))\|. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \|v_n - u^*\| &\leq (1 - \beta_n - \delta_n)\|u_n - u^*\| + h\beta_n\|u_n - u^*\| \\ &\quad + \beta_nb_n + \delta_n\|f_n - u^*\|. \end{aligned} \tag{4.18}$$

It follows from (4.17) and (4.18) that

$$\begin{aligned} \|u_{n+1} - u^*\| &\leq (1 - \alpha_n - \gamma_n + h\alpha_n(1 - \beta_n - \delta_n + h\beta_n))\|u_n - u^*\| \\ &\quad + h\alpha_n(\beta_nb_n + \delta_n\|f_n - u^*\|) + \alpha_nb_n + \gamma_n\|e_n - u^*\|. \end{aligned}$$

Let $M_1 = \sup\{\|f_n - u^*\|: n \geq 0\}$ and $M_2 = \sup\{\|e_n - u^*\|: n \geq 0\}$. Since $0 < h < 1$, we have

$$\begin{aligned} \|u_{n+1} - u^*\| &\leq (1 - \alpha_n(1 - h))\|u_n - u^*\| \\ &\quad + \alpha_n(h\delta_nM_1 + (h\beta_n + 1)b_n) + \gamma_nM_2. \end{aligned} \tag{4.19}$$

Using Lemma 4.1, we know that $b_n \rightarrow 0$ as $n \rightarrow \infty$. It follows from (4.19) and Lemma 4.2 that $u_n \rightarrow u^*$ as $n \rightarrow \infty$. This completes the proof.

Remark 4.1

- (1) If $\beta_n = \delta_n = 0$ for all $n = 0, 1, 2, \dots$, then Theorem 4.1 gives the conditions under which the sequence $\{u_n\}$ defined by MTPIAE strongly converges to u^* .

- (2) If $\gamma_n = \delta_n = 0$ for all $n = 0, 1, 2, \dots$, then Theorem 4.1 gives the conditions under which the sequence $\{u_n\}$ defined by ITPIA strongly converges to u^* .
- (3) If $\gamma_n = \beta_n = \delta_n = 0$ for all $n = 0, 1, 2, \dots$, then Theorem 4.1 gives the conditions under which the sequence $\{u_n\}$ defined by MTPIA strongly converges to u^* .

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