Localisation in Permutation Symmetric Fermionic Quantum Walks

A. P. Balachandran,[a,b] Anjali Kundalpady,[b] Pramod Padmanabhan,[c] Akash Sinha[c]

[a] *Physics Department, Syracuse University, Syracuse, NY, 13244-1130, USA*

[b] *Institute of Mathematical Sciences, CIT Campus, Taramani, Chennai 600113, India*

[c] *School of Basic Sciences, Indian Institute of Technology, Bhubaneswar, 752050, India*

apbal1938, anjali.kundalpady, pramod23phys, akash.sinha@gmail.com

Abstract

We investigate localisation in a quantum system with a global permutation symmetry and a superselected symmetry. We start with a systematic construction of many-fermion Hamiltonians with a global permutation symmetry using the conjugacy classes of the permutation group $\mathcal{S}_N$, with N being the total number of fermions. The resulting Hamiltonians are interpreted as generators of continuous-time quantum walk of indistinguishable fermions. In this setup we analytically solve the simplest example and show that all the states are localised without the introduction of any disorder coefficients. Furthermore, we show that the localisation is stable to interactions that preserve the global $\mathcal{S}_N$ symmetry making these systems candidates for a quantum memory. The models we propose can be realised on superconducting quantum circuits and trapped ion systems.

1 Introduction

A quantum theory with *superselection sectors* is tightly constrained, restricting the possible representations that the algebra of observables can take. In such theories, typically the operators are not charged under some symmetries, meaning they transform trivially under their action. The Hilbert space of such a system splits into different superselection sectors that are labelled by the irreducible representations of these symmetries. A physically important example of such a symmetry is the *permutation symmetry* generated by the statistics operator, s_{ij} that effects the exchange of particles located at i and j. This occurs while studying a system of identical and indistinguishable *bosons* (*fermions*) which correspond to the Abelian representations of the permutation group where the exchange operator takes the eigenvalues $+1$ (-1) respectively. In such theories the states are symmetrised (antisymmetrised) and the operators are invariant under this exchange symmetry.

In this work we explore localisation in such constrained systems. We find that with an additional global permutation symmetry $\mathcal{S}_N$, the fermionic system completely localises without any disorder. The requirement of the global $\mathcal{S}_N$ symmetry enforces the interaction of each fermion with every other fermion making the Hamiltonian similar in appearance to the SYK model [1–4]. Furthermore these systems are also stable under interaction terms that preserve the global $\mathcal{S}_N$ symmetry. This is contrasted with fermionic Hamiltonians without an explicit global $\mathcal{S}_N$ symmetry such as the *tight-binding* Hamiltonians, where we show that such localisation does not occur.

The models we write down can be interpreted as quantum walk Hamiltonians that have received a lot of attention in the past [5–10] in the context of search algorithms [11–13], as quantum simulators [14–17] and for universal quantum computation [18–21]. We are concerned with *continuous-time quantum walks* (CTQW) of identical particles [22–28] and especially those with an additional global symmetry. Symmetric quantum walks have been considered in the case of *discrete-time quantum walks* (DTQW) [29–31] and have been shown to feature topological phases [32–35] and localisation [36–40].

The contents are laid out as follows. We begin Sec. 2 with the space describing the fermions and the operators acting on them. The Hamiltonians with global $\mathcal{S}_N$ symmetry are constructed using simple ideas from the theory of permutation groups. Among the many possibilities, we consider the simplest such Hamiltonian in Sec. 2.1. The solution of the Hamiltonian is provided in Sec. 3 and the resulting features are compared with the models that lack a global $\mathcal{S}_N$ symmetry. We conclude with a few remarks and future directions in Sec. 4.

2 Construction

We begin with a brief description of the Fock space (spanned by the states diagonalising the number operator) describing N identical and indistinguishable fermions. The *vacuum*, $|\Omega\rangle$ denotes the state with no fermions. The state $|i\rangle$, for $i \in \{1, 2, \cdots, N\}$, describes the presence of a fermion on site i. These are the 1-fermion states and they span a N dimensional space henceforth denoted $\mathcal{H} \simeq \mathbb{C}^N$ with the canonical inner product. In this notation, multiparticle states such as, $|i\rangle \otimes |j\rangle$ live in $\mathcal{H} \otimes \mathcal{H}$. No relation between $|i\rangle \otimes |j\rangle$ and $|j\rangle \otimes |i\rangle$ is assumed a priori. However in the case of indistinguishable fermions, we work with normalised antisymmetrised states, $\frac{1}{\sqrt{2}} \left[|i\rangle \otimes |j\rangle - |j\rangle \otimes |i\rangle\right]$ that live in $\mathcal{H} \wedge \mathcal{H}$, with the $\wedge$ denoting antisymmetrisation. Thus the full Hilbert space becomes the antisymmetrised Fock space,

$$\bigoplus_{n=0}^{N} \mathcal{H}^{\wedge n}.$$

This space is finite and its dimension is seen from

$$\sum_{k=0}^{N} \begin{pmatrix} N \\ k \end{pmatrix} = 2^N,$$

with $\begin{pmatrix} N \\ k \end{pmatrix}$ being the dimension of $\underbrace{\mathcal{H} \wedge \mathcal{H} \wedge \cdots \wedge \mathcal{H}}_{k \text{ times}}$.

The creation $(a_j^\dagger)$ and annihilation (a_j) operators satisfying the fermionic (CAR) algebra,

$$\begin{aligned} \{a_j, a_k^\dagger\} &= \delta_{jk}, \\ \{a_j, a_k\} &= \{a_j^\dagger, a_k^\dagger\} = 0, \end{aligned} \tag{2.1}$$

are realized on this space. The index j take values in, $\{1, \cdots, N\}$. More generally we could add an extra index μ to each oscillator to denote an internal degree of freedom like a *colour* or *spin* index. For simplicity we will stick to just the indices $j \in \{1, 2, \cdots, N\}$ for the fermions, allowing their interpretation as lattice sites.

An arbitrary k-fermion state expressed as

$$a_{i_1}^\dagger a_{i_2}^\dagger \cdots a_{i_k}^\dagger |\Omega\rangle, \tag{2.2}$$

satisfies all the necessary antisymmetry properties as can be directly verified by using (2.1). With the help of these a and $a^\dagger$, we can mutate between different particle sectors. For example, the action of $a_i^\dagger$ on an arbitrary state from $\mathcal{H}^{\wedge k}$ yields

$$a_i^\dagger \left(a_{i_1}^\dagger a_{i_2}^\dagger \cdots a_{i_k}^\dagger |\Omega\rangle \right) = a_i^\dagger a_{i_1}^\dagger a_{i_2}^\dagger \cdots a_{i_k}^\dagger |\Omega\rangle \in \mathcal{H}^{\wedge k+1}. \tag{2.3}$$

The Fock space description ensures that the fermionic creation and annihilation operators commute with the superselected exchange symmetry of this system.

Next we move on to the action of the global permutation symmetry $\mathcal{S}_N$ on the site indices $\{1, 2, \cdots, N\}$. The action of these operators on the states and operators of the theory are obtained as follows. The vacuum is invariant under permutations, $s_{ij}|\Omega\rangle = |\Omega\rangle$ and the transformation rule of operators under conjugation by permutation generators is,

$$s_{jk} O_{\cdots j \cdots k \cdots} s_{jk}^{-1} = O_{\cdots k \cdots j \cdots},$$

where O is an operator with several indices including j and k (Note that $s_{jk}^{-1} = s_{jk}$). Using these properties we can deduce the action of the permutation group on arbitrary states of this system.

Permutation invariant operators acting on these states satisfy

$$s_i O_{i_1 \cdots i_p} s_i = O_{i_1 \cdots i_p}, \quad \forall\, i \in \{1, 2, \cdots, N-1\}, \tag{2.4}$$

where $s_i \equiv s_{i,i+1}$, are the transposition operators that generate the permutation group ($\mathcal{S}_N$) and satisfy

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \quad s_i^2 = 1, \quad s_i s_j = s_j s_i \text{ when } |i-j| \geq 2. \tag{2.5}$$

For the particular value of $N = 2$, let us consider the two operators

$$a_1^\dagger a_1 + a_2^\dagger a_2 \qquad \text{and} \qquad a_1^\dagger a_2 + a_2^\dagger a_1. \tag{2.6}$$

They clearly are invariant under the action of $\mathcal{S}_2$ and the objective is to construct such permutation invariant operators for arbitrary $\mathcal{S}_N$. A natural place to look for such objects are in the conjugacy classes of $\mathcal{S}_N$ which are left invariant as a set under the action of the group by definition. For the permutation group, the elements of a conjugacy class have the same *cycle* structure and their order is given by

$$\frac{N!}{\prod_{k=1}^{N} \left(k^{\nu_k}\right) \nu_k!}$$

where ν_k is the number of k-cycles. A clear invariant is the sum of the elements of a given conjugacy class with a particular cycle structure[1]. These statements are independent of the particular realisation of the transposition operators. For our current problem we will show a realisation using the fermionic creation and annihilation operators. The operators we use are such that the resulting Hamiltonians are hermitian and number-preserving as the permutation operators do not change the number of fermions.

We will obtain the fermionic realisation of $\mathcal{S}_N$ by showing the existence of the permutation operators for each particle sector. The fermionic realisation for the transposition permutation $\sigma \in \mathcal{S}_N$ in the k-fermion sector is given by,

$$\sigma = \sum_{i_1<i_2<\cdots i_k} a^\dagger_{\sigma(i_1)} a^\dagger_{\sigma(i_2)} \cdots a^\dagger_{\sigma(i_k)} a_{i_k} \cdots a_{i_2} a_{i_1}. \tag{2.7}$$

The permutation σ has a particular cycle structure. For example the fermionic realisations of the transpositions (ij) in the 1-fermion and 2-fermion sectors are given by

$$(ij)_1 = a^\dagger_j a_i + a^\dagger_i a_j + \sum_{\substack{k=1 \\ k\neq i,j}}^{N} a^\dagger_k a_k, \tag{2.8}$$

and

$$(ij)_2 = a^\dagger_j a^\dagger_i a_j a_i + \sum_{k\neq j} a^\dagger_j a^\dagger_k a_k a_i + \sum_{k\neq i} a^\dagger_k a^\dagger_i a_j a_k + \sum_{\substack{k,l=1 \\ k<l\neq\{i,j\}}}^{N} a^\dagger_k a^\dagger_l a_l a_k, \tag{2.9}$$

respectively. The suffix α, on $(ij)_\alpha$ denotes fermion number sector on which this transposition acts. Thus on the full Fock space the transposition is given by,

$$(ij) = \bigoplus_{\alpha=1}^{N} (ij)_\alpha. \tag{2.10}$$

A more non-trivial example is that of a 3-cycle permutation in the 1-fermion sector,

$$(ijk)_1 = a^\dagger_j a_i + a^\dagger_k a_j + a^\dagger_i a_k + \sum_{\substack{l=1 \\ l\neq\{i,j,k\}}}^{N} a^\dagger_l a_l. \tag{2.11}$$

Note that the hermitian conjugate of this term is $(ikj)_1$. Indeed the Hamiltonian identified as a sum of the elements of the conjugacy class will turn out hermitian. We can now write down an $\mathcal{S}_N$ invariant Hamiltonian for a given conjugacy class made of p-cycles as

$$H^{(p)} = \bigoplus_{\alpha=1}^{N} H^{(p)}_\alpha, \tag{2.12}$$

[1] These are precisely the generators of the center of the permutation group algebra $\mathbb{C}(\mathcal{S}_N)$.

where

$$H^{(p)}_\alpha = \sum_{i_1<i_2<\cdots<i_p} (i_1 i_2 \cdots i_p)_\alpha, \tag{2.13}$$

acts on the α-fermion sector. Such $\mathcal{S}_N$ invariant Hamiltonians (2.12) are true for any realisation of the permutation group. The fermionic realisation in (2.7) introduces simplifications to the Hamiltonian due to the CAR algebra (2.1).

Before going into these we first note that the operators corresponding to cycles of length larger than β annihilate the vectors in the β-fermionic sector. Thus the bilinear expression acting on the 1-fermion sector affects all possible fermion number sectors in a system of N fermions. It acts as an exchange operator on the 1-fermion states and has a non-trivial action on the remaining sectors.

In what follows we will restrict ourselves to the Hamiltonians constructed out of the 2-cycle conjugacy class. We will comment on models obtained from other conjugacy classes in Sec. 4 and carry out a more detailed investigation in a future work.

2.1 2-cycle Hamiltonian

Consider the conjugacy class made out of purely 2-cycles which are just the transpositions. They include the exchange of any two of the N indices and there are precisely $\frac{N(N-1)}{2}$ of them. The 2-cycle Hamiltonian in the 1-fermion sector is obtained using (2.8) and is bilinear in the fermion creation and annihilation operators,

$$H^{(2)}_1 = \sum_{i<j} \left[a^\dagger_i a_j + a^\dagger_j a_i\right] + \frac{(N-1)(N-2)}{2}\ \hat{N}. \tag{2.14}$$

The factor accompanying the number operator $\hat{N} = \sum_{k=1}^{N} a^\dagger_k a_k$ is a result of the substitution (2.8) for the 2-cycles. Clearly the term in the [ ] commutes with $\hat{N}$ and represents a fermion on a given site hopping to any other site. As noted earlier this Hamiltonian has a non-trivial action on every fermion number sector except on the 1-fermion sector where it acts as a permutation operator. It is clearly $\mathcal{S}_N$ invariant in its site indices[2]. The second term in (2.14) dominates for large N. Our goal is to study the localisation features of this system for large N and so the explicit presence of N in the Hamiltonian can lead to incorrect conclusions about

[2]A more rigorous proof is shown in App. A.

the origin of the localisation. To avoid this we will choose the term in [ ] as our Hamiltonian,

$$H = \sum_{i<j} \left[a_i^\dagger a_j + a_j^\dagger a_i\right], \tag{2.15}$$

where all the fermions interact with each other in a symmetric manner. This model can be solved exactly by a simple change of basis as we shall see in Sec. 3.

In addition to the above bilinear Hamiltonian, we consider the operators acting on the 2-fermion states which are quartic in the fermion creation and annihilation operators. This Hamiltonian can be thought of as an interaction term when added to the bilinear Hamiltonian in (2.15). However a crucial point is that this 2-cycle Hamiltonian can be simplified[3] using the CAR algebra in (2.1) resulting in,

$$H_2^{(2)} = \left(\frac{(N-2)(N-3)}{2} - 1\right)\left(\hat{N}^2 - \hat{N}\right) + 2\sum_{i<j} \left[a_i^\dagger a_j + a_j^\dagger a_i\right]\left(\hat{N} - 1\right). \tag{2.16}$$

This Hamiltonian continues to remain $\mathcal{S}_N$ invariant and acts on 2-fermion and higher states. These terms represent interactions but reduce to the product of bilinears due to the CAR algebra. As a consequence they commute with the Hamiltonian in (2.14) and thus merely shift their eigenvalues while sharing the eigenstates. This further implies that localised states of (2.15) are stable to such $\mathcal{S}_N$ preserving perturbations. This trend continues to hold for higher order perturbations obtained using the 2-cycle Hamiltonians acting on 3- and higher-fermion sectors (See App. B).

3 Solution

The bilinear Hamiltonian in (2.15) is solved with a simple change of variables in the space of creation and annihilation operators. Consider a new set of annihilation (creation) operators, A_α ($A_\alpha^\dagger$) defined as,

$$A_\alpha = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \omega^{j\alpha}\, a_j, \quad A_\alpha^\dagger = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \omega^{-j\alpha}\, a_j^\dagger, \tag{3.1}$$

with $\omega = e^{\frac{2\pi i}{N}}$ being a Nth-root of unity and $\alpha \in \{1, 2, \cdots, N\}$. These operators satisfy the CAR relations required of fermionic operators,

$$\begin{aligned} \{A_\alpha, A_\beta^\dagger\} &= \delta_{\alpha\beta}, \\ \{A_\alpha, A_\beta\} &= \{A_\alpha^\dagger, A_\beta^\dagger\} = 0. \end{aligned} \tag{3.2}$$

[3]The proof for this is shown in App. B.

In these variables, the 2-cycle Hamiltonian in (2.15) reduces to

$$H = N\ A_N^\dagger A_N - \hat{N}, \tag{3.3}$$

where $\hat{N} = \sum_{i=1}^{N} a_i^\dagger a_i = \sum_{\alpha=1}^{N} A_\alpha^\dagger A_\alpha$ commutes with the Hamiltonian. A number of fermionic symmetries for the Hamiltonian in (3.3) become apparent in this basis. We find that all bilinears $A_\alpha^\dagger A_\beta$[4] commute with the Hamiltonian when $\alpha, \beta \neq N$. In fact the permutation operators of (2.7) can be written as linear combinations of such bilinears and thus these are the operators that map the states of a given eigenspace into each other.

The spectrum can be found by labelling the eigenspaces with the set $\{1, 2, \cdots N\}$. The dimension of the k-fermion sector is $\frac{N\,!}{k\,!(N-k)\,!}$ and these are spanned by two kinds of eigenstates of the form

$$A_{\alpha_1}^\dagger A_{\alpha_2}^\dagger \cdots A_{\alpha_k}^\dagger |\Omega\rangle, \tag{3.4}$$

with no two α's equal to each other. The first set of eigenstates are those where at least one of the α's is N. There are a total of $\frac{(N-1)\,!}{(k-1)\,!(N-k)\,!}$ such states and they share the eigenvalue, $(N-k)$. The second set are those where none of the α's take the value N. These account for the remaining $\frac{(N-1)\,!}{k\,!(N-k-1)\,!}$ states and they come with the eigenvalue $-k$. In evaluating the spectrum we have used the identities,

$$\left[\hat{N}, A_\alpha^\dagger\right] = A_\alpha^\dagger, \quad \left[\hat{N}, A_\alpha\right] = -A_\alpha. \tag{3.5}$$

Having obtained the spectrum, we are in a position to compute the probability distributions. We will consider 1-fermion and 2-fermion walks which sufficiently illustrate the features of the permutation invariant systems considered here. Following this we will also discuss the general k-fermion sector. An important point to keep in mind is the role played by the global $\mathcal{S}_N$ symmetry in determining the structure of the spectrum. For instance, it is enough to find the time evolution of any single state in a particular number sector. The remaining states can be computed by the action of the appropriate $\mathcal{S}_N$ operators on this state. Furthermore, another crucial feature arises as a consequence of the global $\mathcal{S}_N$ symmetry, namely the restriction on the subspace that a given state is allowed to evolve into. For example the 2-fermion state $|1, 2(t)\rangle$ only evolves into the $|1, j\rangle$ and $|2, j\rangle$ states. There is no overlap with the states $|j, k\rangle$ when $j, k \notin \{1, 2\}$. In other words any state in this system does not explore the full Hilbert space under time evolution. This is not apparent from the A_α ($A_\alpha^\dagger$) basis but becomes more

[4]Removing the number operator from (3.3) will enhance the number of fermionic symmetries as now A_α and $A_\alpha^\dagger$ will also commute with the Hamiltonian when $\alpha \neq N$.

transparent in a new basis. We will demonstrate this for each of the fermion number sectors below.

1-fermion walks : The features we wish to illustrate are immediately seen in the following eigenbasis of the 1-fermion sector : there is one state of the form,

$$\sum_{j=1}^{N} a_j^\dagger |\Omega\rangle, \tag{3.6}$$

with eigenvalue $N-1$ and there are $N-1$ eigenstates of the form,

$$\left(a_1^\dagger - a_j^\dagger\right) |\Omega\rangle \ ; \ j \in \{2, 3, \cdots, N\}, \tag{3.7}$$

with eigenvalue -1. The non-degenerate state in (3.6) is symmetric under the action of $\mathcal{S}_N$, whereas the degenerate states in (3.7) are mapped into each other under the action of $\mathcal{S}_N$. More precisely the transposition operators in the 1-fermion sector (2.8) perform this mapping. These operators can be written as linear combinations of the bilinears $A_\alpha^\dagger A_\beta$ and as noted earlier these commute with the Hamiltonian (3.3).

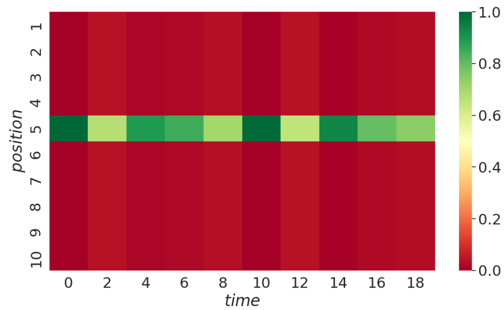
The non-zero probability distributions are found to be,

$$|\langle 1|1(t)\rangle|^2 = \frac{1}{N^2}\left[1 + (N-1)^2 + 2\,(N-1)\cos Nt\right], \tag{3.8}$$

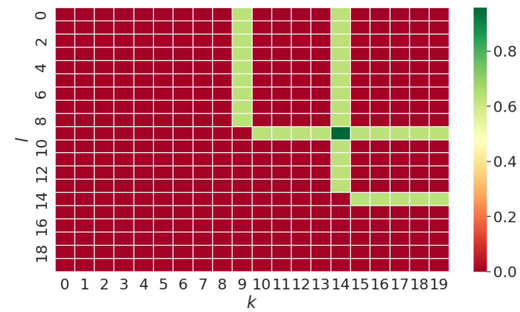
$$|\langle j|1(t)\rangle|^2 = \frac{2}{N^2}\left[1 - \cos Nt\right], \tag{3.9}$$

for $j \in \{2, 3, \cdots, N\}$ and $|j(t)\rangle = e^{-\mathrm{i}Ht}|j\rangle$ are the time evolved states. The non-oscillating terms of both these expressions highlight the localisation effect. For large N the term in (3.8) goes to 1 whereas the term in (3.9) goes to 0. These features are illustrated in Fig. 1a. The reason for the restricted evolution can also be seen from the explicit structure of the unitary evolution operator in the 1-fermion sector and this is shown in App. E.

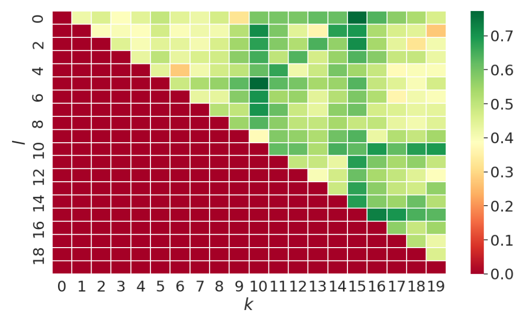
The phase of the oscillating term in (3.8) and (3.9) is the difference between the two energy levels in the 1-fermion sector. We have seen earlier that the addition of higher order interaction terms (2.10), will only shift these energy levels of the Hamiltonian (2.15) leaving the structure of the eigenstates intact. This implies that the localisation seen here is stable to the inclusion of such $\mathcal{S}_N$-symmetric interactions. This argument continues to hold even in the k-fermion sector as there are just two energy levels in each fermion number sector.



(a) 1-fermion walk in the symmetric case.
positions(1 to 10) v/s time(0 to 20)
initial position = 5



(b) 2-fermion walk in the symmetric case.
Probability on positions(0 to 19)
initial position (10,15)



(c) 2-fermion walk in the non-symmetric case.
Probability(to the power 0.1) on positions(0 to 19)
initial position (10,15)

Figure 1: (Color Online) The probability distributions for the 1-, and 2-fermion walks for the Hamiltonian (2.15) in (a) and (b) respectively. The 2-fermion walk for the tight-binding model on the circle $\sum_{j=1}^{N} a_j^\dagger a_{j+1} + h.c.$, is shown in (c). In all the plots the regions with shades of red correspond to probabilities close to zero and the shades of green correspond to probabilities close to one. In the non-symmetric case (c) the 2 fermions spread out to all other locations with probabilities of the order of 10^{-2}. To visualise this spread we have plotted the probabilities to the power of 0.1 in (c).

2-fermion walks : Using the eigenstates in (3.4) the amplitude for an initial 2-fermion state, $|i,j\rangle$ to end up in a state, $|k,l\rangle$ after a time t is found to be,

$$\langle k,l|i,j(t)\rangle = \frac{1}{N^2}\left[e^{-\mathrm{i}(N-2)t}\sum_{\alpha=1}^{N-1}\left(\omega^{i\alpha}-\omega^{j\alpha}\right)\left(\omega^{-k\alpha}-\omega^{-l\alpha}\right)\right.$$
$$\left. +\ e^{2\mathrm{i}t}\sum_{\substack{\alpha,\beta=1\\ \alpha<\beta}}^{N-1}\left(\omega^{i\alpha+j\beta}-\omega^{i\beta+j\alpha}\right)\left(\omega^{-k\alpha-l\beta}-\omega^{-l\alpha-k\beta}\right)\right], \tag{3.10}$$

where $|i,j(t)\rangle$ are the time evolved 2-fermion states. As mentioned earlier the restricted evolution is not apparent from (3.10) but it becomes more transparent in a changed basis for the 2-fermion states[5]. Consider the normalised eigenstates,

$$|[j]\rangle_2 = \frac{1}{\sqrt{N-1}}\,a_j^\dagger\sum_{\substack{i=1\\ i\neq j}}^{N} a_i^\dagger|\Omega\rangle\ ;\ j\in\{1,2,\cdots,N-1\}, \tag{3.11}$$

$$|[jkN]\rangle_2 = \frac{1}{\sqrt{3}}\,\left(a_j^\dagger a_k^\dagger + a_k^\dagger a_N^\dagger + a_N^\dagger a_j^\dagger\right)|\Omega\rangle\ ;\ j<k\in\{1,2,\cdots,N-1\}. \tag{3.12}$$

For these two sets of states, the notation $|[\ ]\rangle_2$ indicates that it is a linear combination of 2-fermion states. From these expressions we see that there are $N-1$ eigenstates of this form in (3.11) and they come with the eigenvalue $(N-2)$ and there are $\frac{(N-1)(N-2)}{2}$ states of the form (3.12) with eigenvalue -2. This is consistent with the previous solution. As in the 1-fermion sector, the $\mathcal{S}_N$ symmetries, generated using (2.9), map the degenerate eigenstates into each other. These operators can be written as products of the bilinears in $A_\alpha^\dagger A_\beta$ and hence commute with the Hamiltonian in (3.3).

These eigenstates are used to expand $|1,2\rangle = a_1^\dagger a_2^\dagger|\Omega\rangle$ as

$$|1,2\rangle = \frac{\sqrt{N-1}}{N}\Big(|[1]\rangle_2 - |[2]\rangle_2\Big) + \frac{\sqrt{3}(N-2)}{N}|[12N]\rangle_2 - \frac{\sqrt{3}}{N}\sum_{j=3}^{N-1}\Big(|[1jN]\rangle_2 - |[2jN]\rangle_2\Big). \tag{3.13}$$

The first three states in the above expression, $|[1]\rangle_2$, $|[2]\rangle_2$ and $|[12N]\rangle_2$, are linear combinations of $|1,j\rangle$, $|2,j\rangle$ with $j\in\{1,\cdots,N\}$. The states under the summation $|[1jN]\rangle_2$, $|[2jN]\rangle_2$, also contains the $|j,N\rangle$ states but these cancel while taking the difference of these two states. Thus from these arguments it is clear that the time evolved $|1,2\rangle$ state will not overlap with a state $|j,k\rangle$ where neither of j or k is $(1,2)$. This is verified in the plot for the probability distribution shown in Fig. 1b. This is to be contrasted with a similar plot for a Hamiltonian that is not

[5] The orthogonality and completeness of these states is discussed in App. C.

permutation invariant (see Fig. 1c), where the two fermions can now be found in states that do not follow the constraint for the symmetrised case [6]. Subsequently we can also compute the probability distributions

$$|\langle 1,2|1,2(t)\rangle|^2 = \frac{1}{N^2}\left[4+(N-2)^2+4\,(N-2)\cos Nt\right], \tag{3.14}$$

$$|\langle \psi|1,2(t)\rangle|^2 = \frac{2}{N^2}\left[1-\cos Nt\right]. \tag{3.15}$$

Here $|\psi\rangle$ denotes the allowed 2-fermion states that have an overlap with $|1,2\rangle$. These expressions present a clear indication of the localisation for large N as the term $\frac{1}{N^2}\left[4+(N-2)^2\right]$ in (3.14) tends to 1 and the term $\frac{2}{N^2}$ in (3.15) approaches 0.

k-fermion walks : This constraining feature continues to hold true for the amplitudes and the corresponding probability distributions in a general k-fermion sector. We will see below that a point in k-dimensional space occupied by k fermions moves to points where at least $k-1$ of the coordinates coincide with the initial state. As in the 2-fermion case this becomes apparent when we work with $\frac{(N-1)\,!}{(k-1)\,!(N-k)\,!}$ eigenstates of the form,

$$|[i_1 i_2\cdots i_{k-1}]\rangle_k = \frac{1}{\sqrt{N-k+1}}a^\dagger_{i_1}a^\dagger_{i_2}\cdots a^\dagger_{i_{k-1}}\sum_{\substack{j=1\\ j\neq\{i_1,i_2,\cdots i_{k-1}\}}}^{N} a^\dagger_j|\Omega\rangle, \tag{3.16}$$

with $i_1<i_2<\cdots<i_{k-1}\in\{1,\cdots N-1\}$. The second set of eigenstates[7] account for $\frac{(N-1)\,!}{k\,!(N-k-1)\,!}$ of them and take the form,

$$|[i_1 i_2\cdots i_k N]\rangle_k = \frac{1}{\sqrt{k+1}}\left[a^\dagger_{i_1}\cdots a^\dagger_{i_k}+(-1)^k\,a^\dagger_{i_2}\cdots a^\dagger_N+\cdots+(-1)^{k^2}\,a^\dagger_N\cdots a^\dagger_{i_{k-1}}\right]|\Omega\rangle. \tag{3.17}$$

An initial state of the form $|1,2,\cdots,k\rangle$ can be expanded using the above eigenstates such that each of them contain at least $k-1$ of $\{1,2,\cdots k\}$. Finally the probability distribution for the time evolved k-fermion state to overlap with its initial state is,

$$|\langle 1,2,3,\cdots,k|1,2,3,\cdots,k(t)\rangle|^2 = \frac{1}{N^2}\left[k^2+(N-k)^2+2k\,(N-k)\cos Nt\right], \tag{3.18}$$

generalising the result in (3.14). It is clear from this expression that the localisation feature continues to hold for the k-fermion states as well.

[6] A similar result for quantum walks on Cayley graphs of the symmetric group is in [41].

[7] The proof for this is in App. D.

4 Discussion

We have explored the question of localisation in a system with superselection sectors and a global discrete symmetry. It is important to understand the source of the localisation and the role played by these two symmetries in obtaining this feature. To this end we consider two possibilities. In the first situation we continue working with fermionic systems and reduce the explicit global $\mathcal{S}_N$ symmetry. This is done by 'marking' a single site, say 1, to modify the Hamiltonian in (2.15) to,

$$H = \beta \sum_{j=2}^{N} \left[a_1^\dagger a_j + a_j^\dagger a_1\right] + \sum_{\substack{j,k=2 \\ j<k}}^{N} \left[a_k^\dagger a_j + a_j^\dagger a_k\right]. \tag{4.1}$$

This model has a global $\mathcal{S}_{N-1}$ symmetry among the sites $\{2, 3, \cdots, N\}$. To analyse the consequences we rewrite the Hamiltonian in a different fashion. In the 1-fermion sector spanned by

$$a_i^\dagger|\Omega\rangle \to |i\rangle := (0, \cdots, \underbrace{1}_{i\text{-th position}}, \cdots, 0); \quad i = \{1, 2, \cdots, N\} \tag{4.2}$$

the Hamiltonian becomes an $N \times N$ matrix. However owing to the residual $\mathcal{S}_{N-1}$ symmetry, we further can reduce the dimension of the matrix. For example, we can work in the space spanned by

$$\left\{|1\rangle, |2\rangle, \frac{1}{\sqrt{N-2}}\left(|3\rangle + \cdots + |N\rangle\right)\right\} \tag{4.3}$$

The resulting Hamiltonian is

$$H = \begin{pmatrix} 0 & \beta & \beta\sqrt{N-2} \\ \beta & 0 & \sqrt{N-2} \\ \beta\sqrt{N-2} & \sqrt{N-2} & N-3 \end{pmatrix} \tag{4.4}$$

Using this the probability that after evolving $|1\rangle$ in time, we still find it at $|1\rangle$ comes to be

$$|\langle 1|1(t)\rangle|^2 = 1 - \frac{2(N-1)\beta^2\left(1 - \cos\left[t\sqrt{((N-2)^2 + 4(N-1)\beta^2)}\right]\right)}{(N-2)^2 + 4(N-1)\beta^2} \tag{4.5}$$

Similarly we can compute $|\langle 2|2(t)\rangle|^2$, and we show the plots for both of these in Fig.[2, 3]. The essential difference between these two is that, we can localize $|1\rangle$ even for small N by tuning

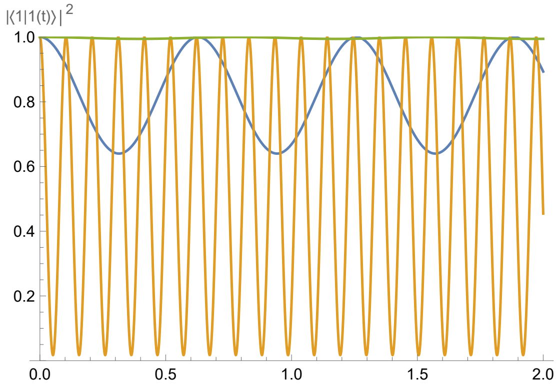


Figure 2: $|\langle 1|1(t)\rangle|^2$ vs t

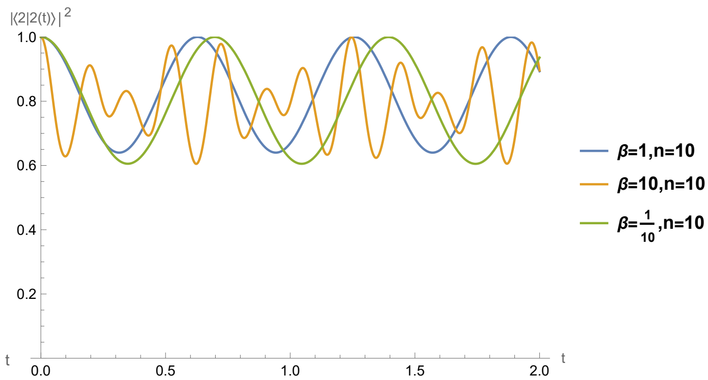


Figure 3: $|\langle 2|2(t)\rangle|^2$ vs t

β, which is not possible for the other states that continue to localise for large N values as in the case of the full global $\mathcal{S}_N$ symmetry.

To examine the role of the superselected symmetry, consider a spin chain system with no superselected symmetry and just a global $\mathcal{S}_N$ symmetry. We obtain such an example by using another realisation of the transposition operators, (ij),

$$(ij) = \frac{1 + X_iX_j + Y_iY_j + Z_iZ_j}{2}, \tag{4.6}$$

where X, Y and Z are the Pauli matrices. This system acts on N sites with each site occupied by a two-level system, $\{|\uparrow\rangle, |\downarrow\rangle\} \simeq \mathbb{C}^2$. In this case the 2-cycle Hamiltonian generalizes the XXX spin chain,

$$H = \frac{1}{2}\sum_{\substack{j,k=1 \\ j<k}}^{N} \left[X_jX_k + Y_jY_k + Z_jZ_k\right] + \frac{N(N-1)}{2}. \tag{4.7}$$

The 2-cycle Hamiltonian is just the sum of the $\frac{N(N-1)}{2}$ transpositions acting on $\otimes_{j=1}^{N}\mathbb{C}^2_j$. The Hilbert space of this system splits into sectors labelled by the number of $|\downarrow\rangle$'s as the Hamiltonian, being the sum of permutation operators, cannot mix states containing different number of $|\downarrow\rangle$'s. Thus the problem we have now is similar to the one encountered in the fermionic realisation. For example consider the sector where there is a single $|\downarrow\rangle$ with the remaining sites filled by a $|\uparrow\rangle$. There are N such states and the Hamiltonian in this sector reduces to the form,

$$(H_1)_{ij} = \left(\frac{(N-1)(N-2)}{2} - 1\right)\delta_{ij} + 1 \tag{4.8}$$

which is similar to the one obtained in the fermion case, (Appendix E.1) modulo the function of N appearing with the δ_{ij}. We expect to see localisation here as well suggesting that this property is true regardless of the realisation chosen for the global permutation symmetry. It is not hard to see that this pattern continues for the other sectors of this Hilbert space and so we conclude that the symmetric XXX chain will contain localised states for large N.

To summarise we see from these considerations that the localisation observed here is a property of the global permutation symmetry and does not depend on its realisation. This suggests that, in group theoretic Hamiltonians that preserve some symmetry group G, the states will localise without any disorder providing a stable quantum memory protected by the global symmetry. This would be an interesting and systematic way to obtain localised phases. We also note that these setups are not merely a theoretical exercise and can in fact be seen in experimental systems such as superconducting quantum circuits [42] and trapped ions [43].

We end with a few remarks and possible future works.

1. The *out-of-time-order correlators* (OTOC's) indicate chaotic behavior in thermal systems [44]. The Lyapunov exponent can be read off from such expressions [3]. The saturation of this quantity implies behavior similar to that predicted by the AdS/CFT correspondence and is like an SYK model [1–4]. The systems discussed in this work describe the opposite behavior and thus the results on the OTOC's do not hold for them.
2. An extension worthy of mention is adding an internal symmetry index $\alpha = 1, 2$ transforming by the spin $\frac{1}{2}$ representation of $SU(2)$ to the operators a_i. Then since this representation is pseudoreal, $a_i^\alpha \, (\sigma_2)_{\alpha,\alpha'} \, a_j^{\alpha'}$ is $SU(2)$ invariant and so is its adjoint (Here σ_2 is the second Pauli matrix.). So we can add such $SU(2)$ or colour singlet Majorana terms which are also permutation invariant and incorporate features of the SYK model.
3. Furthermore it would be interesting to consider the algebra of observables that are permutation invariant and study the corresponding Hilbert spaces built using the GNS construction [45–50]. These can then be used to explore the entanglement entropy and thermalisation properties of these systems and those derived from them.

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A $H_1^{(2)}$ in different particle sectors

Let us consider the operator

$$H_1^{(2)} = \sum_{\substack{\rho \in \text{2-cycle} \\ \text{conjugacy class}}} \sum_{i=1}^{N} a^\dagger_{\rho(i)} a_i \tag{A.1}$$

This is invariant in every particle sector under global permutations among the site indices. In one-particle sector this can be seen very easily :

$$\left(\sigma_1 H_1^{(2)} \sigma_1^{-1}\right) a^\dagger_m |\Omega\rangle = H_1^{(2)} a^\dagger_m |\Omega\rangle \tag{A.2}$$

Here σ_1 is the realization of $\sigma \in \mathcal{S}_N$ in the one-particle sector. However before proceeding further let us consider the action of $H_1^{(2)}$ on an arbitrary k-particle state, $a^\dagger_{i_1} \cdots a^\dagger_{i_k}|\Omega\rangle$. We can write

$$\begin{aligned} H_1^{(2)} a^\dagger_{i_1} \cdots a^\dagger_{i_k}|\Omega\rangle &= \sum_{\substack{\rho \in \text{2-cycle} \\ \text{conjugacy class}}} \left(a^\dagger_{\rho(i_1)} a_{i_1} + \cdots a^\dagger_{\rho(m)} a_m + \cdots + a^\dagger_{\rho(i_k)} a_{i_k}\right) a^\dagger_{i_1} \cdots a^\dagger_{i_k}|\Omega\rangle \qquad \text{(rests give 0)} \\ &= \sum_{\substack{\rho \in \text{2-cycle} \\ \text{conjugacy class}}} \left(a^\dagger_{\rho(i_1)} \cdots a^\dagger_{i_k} + a^\dagger_{i_1} \cdots a^\dagger_{\rho(m)} \cdots a^\dagger_{i_k} + a^\dagger_{i_1} \cdots a^\dagger_{\rho(i_k)}\right)|\Omega\rangle \end{aligned} \tag{A.3}$$

We will now demonstrate the invariance of $H_1^{(2)}$ in the k-particle sector.

$$\begin{aligned} \sigma_k H_1^{(2)} \sigma_k^{-1} |i_1, \cdots, i_k\rangle &= \sigma_k H_1^{(2)} |\sigma^{-1}(i_1), \cdots, \sigma^{-1}(i_k)\rangle \\ &= \sigma_k \sum_{\substack{\rho \in \text{2-cycle} \\ \text{conjugacy class}}} \left(|\rho\sigma^{-1}(i_1), \cdots, \sigma^{-1}(i_k)\rangle + \cdots + |\sigma^{-1}(i_1), \cdots, \rho\sigma^{-1}(i_k)\rangle\right) \\ &= \sum_{\substack{\rho \in \text{2-cycle} \\ \text{conjugacy class}}} \left(|\sigma\rho\sigma^{-1}(i_1), \cdots, i_k\rangle + \cdots + |i_1, \cdots, \sigma\rho\sigma^{-1}(i_k)\rangle\right) \\ &= \sum_{\substack{\rho' \in \text{2-cycle} \\ \text{conjugacy class}}} \left(|\rho'(i_1), \cdots, i_k\rangle + \cdots + |i_1, \cdots, \rho'(i_k)\rangle\right) \\ &= H_1^{(2)} |i_1, \cdots, i_k\rangle \end{aligned} \tag{A.4}$$

where $\sigma\rho\sigma^{-1} = \rho' \in$ 2-cycle conjugacy class also and σ_k is the realization of $\sigma \in \mathcal{S}_N$ in the k-particle sector.

B Proof of (2.16)

The quartic Hamiltonian originating from the two-cycle conjugacy class is

$$H_2^{(2)} = \sum_{\sigma}\sum_{i<j} a^{\dagger}_{\sigma(i)} a^{\dagger}_{\sigma(j)} a_j a_i = \frac{1}{2}\sum_{i,j}\sum_{\sigma} a^{\dagger}_{\sigma(i)} a^{\dagger}_{\sigma(j)} a_j a_i \tag{B.1}$$

We notice that all σ's belonging to the two-cycle conjugacy class can be grouped as

- $i \to i \quad j \to j$ there are ${}^{N-2}C_2$ such elements.
- $i \to j \quad j \to i$ there is one such element.
- $i \to i \quad j \to m(\neq i, j)$ there are $N-2$ such elements.
- $i \to m(\neq i, j) \quad j \to j$ there are $N-2$ such elements.

Considering this, we can simplify (B.1) as

$$H_2^{(2)} = \frac{1}{2}\sum_{i,j}\left[{}^{N-2}C_2 a_i^{\dagger} a_j^{\dagger} a_j a_i + a_j^{\dagger} a_i^{\dagger} a_j a_i + \sum_{m\neq i,j}\left(a_i^{\dagger} a_m^{\dagger} + a_m^{\dagger} a_j^{\dagger}\right) a_j a_i\right] \tag{B.2}$$

With $A = \sum_i a_i$, $\hat{N} = \sum_i a_i^{\dagger} a_i$ being the number operator and using the relations $[\hat{N}, A/a] = -A/a$, we obtain

$$\begin{aligned} H_2^{(2)} &= \frac{1}{2}\left({}^{N-2}C_2 - 3\right)\left(\hat{N}^2 - \hat{N}\right) + \left(A^{\dagger} A \hat{N} - A^{\dagger} A\right) \\ &= \frac{1}{2}\left(\frac{(N-2)(N-3)}{2} - 1\right)\left(\hat{N}^2 - \hat{N}\right) + 2\sum_{i<j}\left[a_i^{\dagger} a_j + a_j^{\dagger} a_i\right]\left(\hat{N} - 1\right) \end{aligned} \tag{B.3}$$

So, we can write $H_2^{(2)}$ in terms of $H_1^{(2)}$ and $\hat{N}$ and we expect this trend to continue for higher order Hamiltonians like hextic Hamiltonian and so on.

C Orthogonality and Completeness of (3.11), (3.12)

The inner product between 2-fermion eigenstates $\langle u|v\rangle$ is zero when $|u\rangle \in$ (3.11) and $|v\rangle \in$ (3.12). However when both $|u\rangle$ and $|v\rangle$ belong to any particular eigenvalue then they are not orthogonal to each other. Nevertheless we can show that these states are complete using the action of the permutation operators from $\mathcal{S}_N$. To do this we first note that the states in (3.11) are mapped into each other,

$$s_{jk}|[j]\rangle_2 = |[k]\rangle_2 \; ; \; j, k \in \{1, 2, \cdots, N-1\}, \tag{C.1}$$

using the transpositions s_{jk} and the states in (3.12) are mapped into each other,

$$s_{j_1 j_2} s_{k_1 k_2} |[j_1 k_1 N]\rangle_2 = |[j_2 k_2 N]\rangle_2 \tag{C.2}$$

using the permutations $s_{j_1 j_2} s_{k_1 k_2}$. Combining these identities with the expression of the 2-fermion state $|1,2\rangle$ in (3.13) we see that any other 2-fermion state $|j,k\rangle$, with $j,k \neq \{1,2\}$, can be obtained as

$$\begin{aligned} |j,k\rangle &= s_{1j} s_{2k} |1,2\rangle \\ &= \frac{\sqrt{N-1}}{N}\Big(|[j]\rangle_2 - |[k]\rangle_2\Big) + \frac{\sqrt{3}(N-2)}{N}|[jkN]\rangle_2 - \frac{\sqrt{3}}{N}\sum_{m=3}^{N-1}\Big(|[jmN]\rangle_2 - |[kmN]\rangle_2\Big). \end{aligned} \tag{C.3}$$

Notice that we have used $s_{1j}|[2]\rangle_2 = |[2]\rangle_2$ and $s_{2k}|[1]\rangle_2 = |[1]\rangle_2$. On the other hand the states of the form $|1,j\rangle$ ($|2,j\rangle$), for $j \in \{3, \cdots N\}$, are obtained by applying s_{2j} (s_{1j}) on $\pm|1,2\rangle$ respectively. Thus any 2-fermion state can be written as a linear combination of the 2-fermion eigenstates in (3.11) and (3.12) showing their completeness. These arguments can be extended to a general k-fermion sector as well.

D Proof of (3.17)

We have the action of quadratic all-to-all on a general k-particle state as

$$H_1^{(2)} a^\dagger_{i_1} \cdots a^\dagger_{i_k} |\Omega\rangle = \sum_\sigma \left(a^\dagger_{\sigma(i_1)} \cdots a^\dagger_{i_k} + a^\dagger_{i_1} \cdots a^\dagger_{\sigma(i_j)} \cdots a^\dagger_{i_k} + \cdots + a^\dagger_{i_1} \cdots a^\dagger_{\sigma(i_k)} \right) |\Omega\rangle \tag{D.1}$$

We concentrate on a single term

$$\sum_\sigma a^\dagger_{i_1} \cdots a^\dagger_{\sigma(i_j)} \cdots a^\dagger_{i_k} \tag{D.2}$$

As done previously, we can group the σ's as

- $i_j \to i_j$ there are ${}^{N-1}C_2$ such elements.
- $i_j \to m \neq i_j$ there are $N-1$ such elements.

Then we finally have

$$\sum_{\sigma} a^\dagger_{i_1}\cdots a^\dagger_{\sigma(i_j)}\cdots a^\dagger_{i_k} = {}^{N-1}C_2\, a^\dagger_{i_1}\cdots a^\dagger_{i_j}\cdots a^\dagger_{i_k} + a^\dagger_{i_1}\cdots\left(\sum_{m\neq i_j} a^\dagger_m\right)\cdots a^\dagger_{i_k}$$
$$= \left({}^{N-1}C_2 - 1\right)\, a^\dagger_{i_1}\cdots a^\dagger_{i_j}\cdots a^\dagger_{i_k} + a^\dagger_{i_1}\cdots\left(\sum_{m} a^\dagger_m\right)\cdots a^\dagger_{i_k}$$
$$= \left({}^{N}C_2 - N\right) a^\dagger_{i_1}\cdots a^\dagger_{i_k} + a^\dagger_{i_1}\cdots \underbrace{A^\dagger}_{j\text{-th position}} \cdots a^\dagger_{i_k} \tag{D.3}$$

Therefore we have

$$H_1^{(2)} a^\dagger_{i_1}\cdots a^\dagger_{i_k}|\Omega\rangle = k\left({}^{N}C_2 - N\right) a^\dagger_{i_1}\cdots a^\dagger_{i_k}|\Omega\rangle + A^\dagger a^\dagger_{i_2}\cdots a^\dagger_{i_k}|\Omega\rangle + \cdots + a^\dagger_{i_1}\cdots \underbrace{A^\dagger}_{j\text{-th position}} \cdots a^\dagger_{i_k}|\Omega\rangle$$
$$+ \cdots + a^\dagger_{i_1}\cdots a^\dagger_{i_{k-1}} A^\dagger|\Omega\rangle \tag{D.4}$$

Now let us consider the following situation :

$$H_1^{(2)} a^\dagger_{i_1}\cdots \sum_{i_k} a^\dagger_{i_k}|\Omega\rangle = k\left({}^{N}C_2 - N\right) a^\dagger_{i_1}\cdots \sum_{i_k} a^\dagger_{i_k}|\Omega\rangle + A^\dagger a^\dagger_{i_2}\cdots \sum_{i_k} a^\dagger_{i_k}|\Omega\rangle +$$
$$\cdots + a^\dagger_{i_1}\cdots \underbrace{A^\dagger}_{j\text{-th position}} \cdots \sum_{i_k} a^\dagger_{i_k}|\Omega\rangle + \cdots + a^\dagger_{i_1}\cdots a^\dagger_{i_{k-1}} \sum_{i_k} A^\dagger|\Omega\rangle$$
$$= k\left({}^{N}C_2 - N\right) a^\dagger_{i_1}\cdots A^\dagger|\Omega\rangle + N a^\dagger_{i_1}\cdots A^\dagger|\Omega\rangle \tag{D.5}$$

Thus we have the following eigenstates of $H_1^{(2)}$

$$H_1^{(2)} a^\dagger_{i_1}\cdots A^\dagger|\Omega\rangle = \left(k\,{}^{N}C_2 - N(k-1)\right) a^\dagger_{i_1}\cdots A^\dagger|\Omega\rangle \tag{D.6}$$

Now let us consider another kind of states

$$|i_1,\cdots,i_k,N;k\rangle = a^\dagger_{i_1}\cdots a^\dagger_{i_k}|\Omega\rangle + \cdots + (-1)^{jk} a^\dagger_{j+1}\cdots a^\dagger_{i_k} \underbrace{a^\dagger_N}_{k-j+1\text{-th position}} a^\dagger_{i_1}\cdots a^\dagger_{j-1}|\Omega\rangle +$$
$$\cdots + (-1)^{k^2} a^\dagger_N a^\dagger_{i_1}\cdots a^\dagger_{i_{k-1}}|\Omega\rangle \tag{D.7}$$

Let us focus on two particular expressions

$$H_1^{(2)} a^\dagger_{i_1}\cdots a^\dagger_{i_k}|\Omega\rangle \quad \text{and} \quad H_1^{(2)}(-1)^{jk} a^\dagger_{j+1}\cdots a^\dagger_{i_k} \underbrace{a^\dagger_N}_{k-j+1\text{-th position}} a^\dagger_{i_1}\cdots a^\dagger_{j-1}|\Omega\rangle \tag{D.8}$$

We should have two terms coming from the above

$$a^\dagger_{i_1}\cdots \underbrace{A^\dagger}_{j\text{-th position}} \cdots a^\dagger_{i_k}|\Omega\rangle \quad \text{and} \quad (-1)^{jk} a^\dagger_{j+1}\cdots a^\dagger_{i_k} \underbrace{A^\dagger}_{k-j+1\text{-th position}} a^\dagger_{i_1}\cdots a^\dagger_{j-1}|\Omega\rangle \tag{D.9}$$

They have same index content. Rearrangement of the indices in the second of these yields

$$
\begin{aligned}
&(-1)^{(k-j+1)(j-1)+(k-j)+jk} a_{i_1}^\dagger \cdots \underbrace{A^\dagger}_{j\text{-th position}} \cdots a_{i_k}^\dagger |\Omega\rangle \\
&= (-1)^{-j^2+2jk+j-1} a_{i_1}^\dagger \cdots \underbrace{A^\dagger}_{j\text{-th position}} \cdots a_{i_k}^\dagger |\Omega\rangle \\
&= (-1)^{-j^2+j}(-1)^{2jk-1} a_{i_1}^\dagger \cdots \underbrace{A^\dagger}_{j\text{-th position}} \cdots a_{i_k}^\dagger |\Omega\rangle \\
&= -a_{i_1}^\dagger \cdots \underbrace{A^\dagger}_{j\text{-th position}} \cdots a_{i_k}^\dagger |\Omega\rangle
\end{aligned}
\tag{D.10}
$$

Thus these terms always cancel and the states

$$|i_1, \cdots, i_k, N; k\rangle \tag{D.11}$$

are eigenstates of $H_1^{(2)}$

$$H_1^{(2)} |i_1, \cdots, i_k, N; k\rangle = k\left({}^N C_2 - N\right) |i_1, \cdots, i_k, N; k\rangle \tag{D.12}$$

E Time evolution operator in 1-particle sector

The one-particle Hilbert space spanned by $\{a_i^\dagger|\Omega\rangle; i = 1, 2, \cdots, N\}$ can equivalently be described by the space spanned by $\{|i\rangle = (0, \cdots, \underbrace{1}_{i\text{-th position}}, \cdots, 0)^T\}$. In this framework, the bilinear Hamiltonian in (2.15) is given by

$$H : (H)_{ij} = 1 - \delta_{ij} \implies H = \mathcal{I} - \mathbb{I} \tag{E.1}$$

where the matrix $\mathcal{I}$ has entries $\mathcal{I}_{ij} = 1$ and the matrix $\mathbb{I}$ is the $N \times N$ identity matrix with entries $\mathbb{I}_{ij} = \delta_{ij}$. They satisfy the following properties

$$\mathcal{I}^m = N^{m-1}\mathcal{I} \quad \text{and} \quad \mathbb{I}^m = \mathbb{I} \tag{E.2}$$

Then one can simplify :

$$e^{-iHt} = \left[\left(\mathbb{I} - \frac{\mathcal{I}}{N}\right) + \frac{\mathcal{I}}{N} e^{-iNt}\right] e^{it} \tag{E.3}$$

Further, the probability can be calculated as

$$|\langle j|j(t)\rangle|^2 = \left|\left[\left(1-\frac{1}{N}\right)+\frac{1}{N}e^{-iNt}\right]e^{it}\right|^2 = \frac{1}{N^2}\left[1+(N-1)^2+2(N-1)\cos(Nt)\right] \tag{E.4}$$

$$|\langle j'|j(t)\rangle|^2_{j'\neq j} = \left|\left[-\frac{1}{N}+\frac{1}{N}e^{-iNt}\right]e^{it}\right|^2 = \frac{2}{N^2}\left[1-\cos(Nt)\right] \tag{E.5}$$

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