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Some Subclasses of Spirallike Multivalent Functions Associated with a Differential Operator

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Abstract: In this paper we study convolution properties of spirallike multivalent functions defined by using a differential operator and higher order derivatives. Using convolution product relations we determine necessary and sufficient conditions for multivalent functions to belong to these classes, and our results generalized many previous results obtained by different authors. We obtain convolution and inclusion properties for new subclasses of multivalent functions defined by using the Dziok-Srivatava operator. Moreover, using a result connected with the Briot-Bouquet differential subordination, we obtain an inclusion relation between some of these classes of functions.

Keywords: analytic function; convolution product; generalized hypergeometric function; Dziok-Srivastava linear operator; spirallike function; differential subordination; Briot-Bouquet differential subordination

MSC: 30C45; 30C80

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1. Introduction

With the aid of the convolution (Hadamard) product Dziok and Srivastava [1] defined a well-known operator that will help us to define a few general subclasses of multivalent functions that generalize and unify many of some previous ones introduced and studied by different authors. We determine necessary and sufficient conditions in term of convolution relations such that a multivalent function belong to these classes, that generalize some of the previous results of Sarkar et al. [2] (Theorems 2.1 and 2.2), of Ahuja [3] (Theorems 2.1 and 2.2) and of Padmanabhan and Ganesan [4] (Theorems 1–4). Moreover, using the *Briot-Bouquet* differential subordination we found an inclusion property for these classes. The importance of the results besides in the wide generalities of the convolution and the inclusion theorems that could be useful for those that will follow a special study of some special cases of the subclasses we defined. We would like to emphasize that some other fundamental results regarding introductory and new results of the *Geometric Function Theory* could be found in the recent monographs [5,6].

Let $\mathcal{A}(p)$ denote the class of functions of the for

$$f(z) = z^p + \sum_{k=p+1}^{+\infty} a_k z^k,\ z \in \mathbb{D} \quad (p \in \mathbb{N} := \{1, 2, \dots\}) \tag{1}$$

which are analytic in the open unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$.

If f and g are analytic functions in $\mathbb{D}$ we say that f is *subordinate* to g, written $f(z) \prec g(z)$, if there exist a function w which is analytic in $\mathbb{D}$, with $w(0) = 0$, and $|w(z)| < 1$, $z \in \mathbb{D}$, such that $f(z) = g(w(z))$ for all $z \in \mathbb{D}$.

Furthermore, if the function g is univalent in $\mathbb{D}$, then we have the equivalence

$$f(z) \prec g(z) \text{ if and only if } f(0) = g(0) \text{ and } f(\mathbb{D}) \subset g(\mathbb{D}).$$

For the function $f \in \mathcal{A}(p)$ given by (1), and g given by

$$g(z) = z^p + \sum_{k=p+1}^{+\infty} b_k z^k,\ z \in \mathbb{D} \quad (p \in \mathbb{N}),$$

the *Hadamard (or convolution) product* of f and g is defined by

$$(f * g)(z) := z^p + \sum_{k=p+1}^{+\infty} a_k b_k z^k,\ z \in \mathbb{D}.$$

For complex parameters $a_1, \ldots, a_l$ and $b_1, \ldots, b_s$, with $b_j \notin \mathbb{Z}_0^- := \{0, -1, -2, \ldots\}$ for all $j \in \{1, 2, \ldots, s\}$, we now define *the generalized hypergeometric function*

$$ {}_lF_s(a_1, \ldots, a_l; b_1, \ldots, b_s; z) := \sum_{k=0}^{+\infty} \frac{(a_1)_k \ldots (a_l)_k}{(b_1)_k \ldots (b_s)_k} \cdot \frac{z^k}{k!},\ z \in \mathbb{D},$$
$$(l \leq s+1,\ l, s \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}),$$

where $(\theta)_\nu$ is *the Pochhammer symbol*, defined in terms of the *Gamma function* by

$$(\theta)_\nu := \frac{\Gamma(\theta + \nu)}{\Gamma(\theta)} = \begin{cases} 1, & \text{if} \quad \nu = 0,\ \theta \in \mathbb{C}^* := \mathbb{C} \setminus \{0\} \\ \theta(\theta+1)\ldots(\theta+\nu-1), & \text{if} \quad \nu \in \mathbb{N},\ \theta \in \mathbb{C}. \end{cases}$$

Corresponding to a function $h_p(a_1, \ldots, a_l; b_1, \ldots, b_s; z)$ given by

$$h_p(a_1, \ldots, a_l; b_1, \ldots, b_s; z) := z^p \cdot {}_lF_s(a_1, \ldots, a_l; b_1, \ldots, b_s; z),\ z \in \mathbb{D},$$

Dziok and Srivastava [1] considered the linear operator

$$H_p(a_1, \ldots, a_l; b_1, \ldots, b_s; z) : \mathcal{A}(p) \to \mathcal{A}(p)$$

defined by the following Hadamard product

$$H_p(a_1, \ldots, a_l; b_1, \ldots, b_s; z) f(z) := h_p(a_1, \ldots, a_l; b_1, \ldots, b_s; z) * f(z),\ z \in \mathbb{D}.$$

Thus, for the function f defined by (1) we have

$$H_p(a_1, \ldots, a_l; b_1, \ldots, b_s; z) f(z) = z^p + \sum_{k=p+1}^{+\infty} \Gamma_{k-p}[a_1; b_1] a_k z^k,\ z \in \mathbb{D}, \tag{2}$$

where

$$\Gamma_{k-p}[a_1; b_1] := \frac{(a_1)_{k-p} \ldots (a_l)_{k-p}}{(b_1)_{k-p} \ldots (b_s)_{k-p}} \cdot \frac{1}{(k-p)!},$$

and for convenience we write

$$H_{p,l,s}(a_1) := H_p(a_1, \ldots, a_l; b_1, \ldots, b_s; z).$$

It can be easily verified from the definition (2) that

$$z\Big(H_{p,l,s}(a_1) f(z)\Big)' = a_1 H_{p,l,s}(a_1+1) f(z) - (a_1 - p) H_{p,l,s}(a_1) f(z),\ z \in \mathbb{D}. \tag{3}$$

It should be remarked that the linear operator $H_{p,l,s}(a_1)$ is a generalization of other linear operators (see [7–13]).

In the following two definitions we will recall some previously defined subclasses of $\mathcal{A}(p)$ that we will use in our article.

Definition 1 (see [14])**.** *Let $p \in \mathbb{N}$, $q \in \mathbb{N}_0$, with $p > q$, and $0 \le \alpha < p - q$.*

(i) We say the function $f \in \mathcal{A}(p)$ is in the class $S_p(q,\alpha)$ if it satisfies the inequality

$$\operatorname{Re} \frac{zf^{(1+q)}(z)}{f^{(q)}(z)} > \alpha,\ z \in \mathbb{D}.$$

(ii) We say the function $f \in \mathcal{A}(p)$ is in the class $K_p(q,\alpha)$ if it satisfies the inequality

$$\operatorname{Re}\left(1 + \frac{zf^{(2+q)}(z)}{f^{(1+q)}(z)}\right) > \alpha,\ z \in \mathbb{D}.$$

From the above definition it follows that

$$f \in K_p(q,\alpha)\ F \in S_p(q,\alpha),$$

$$\text{where } F \in \mathcal{A}(p),\ \text{such that } F^{(q)}(z) = \frac{zf^{(1+q)}(z)}{p-q},\ z \in \mathbb{D},$$

and we note that $S_p(0,\alpha) =: S_p(\alpha)$ and $K_p(0,\alpha) =: K_p(\alpha)$ (see [15]).

Definition 2 (see [14])**.** *Let $\lambda \in \mathbb{R}$, with $|\lambda| < \frac{\pi}{2}$, $p \in \mathbb{N}$, $q \in \mathbb{N}_0$, with $p > q$, and $0 \le \alpha < p - q$.*

(i) We say that the function $f \in \mathcal{A}(p)$ is in the class $S_p^\lambda(q,\alpha)$ if it satisfies the inequality

$$\operatorname{Re}\left(e^{i\lambda} \frac{zf^{(1+q)}(z)}{f^{(q)}(z)}\right) > \alpha \cos\lambda,\ z \in \mathbb{D}.$$

(ii) We say the function $f \in \mathcal{A}(p)$ is in the class $C_p^\lambda(q,\alpha)$ if it satisfies the inequality

$$\operatorname{Re}\left[e^{i\lambda}\left(1 + \frac{zf^{(2+q)}(z)}{f^{(1+q)}(z)}\right)\right] > \alpha \cos\lambda,\ z \in \mathbb{D}.$$

It is easy to check that

$$f \in C_p^\lambda(q,\alpha) \text{ if and only if } F \in S_p^\lambda(q,\alpha),$$

$$\text{where } F \in \mathcal{A}(p),\ \text{such that } F^{(q)}(z) = \frac{zf^{(1+q)}(z)}{p-q},\ z \in \mathbb{D}.$$

and we remark that $S_p^0(q,\alpha) =: S_p(q,\alpha)$, $C_p^0(q,\alpha) =: K_p(q,\alpha)$, $S_p^\lambda(0,\alpha) =: S_p^\lambda(\alpha)$ and $C_p^\lambda(0,\alpha) =: C_p^\lambda(\alpha)$ (see [16]).

The next three definitions will introduce some classes connected to the studies of this article.

Definition 3. *Let $\lambda \in \mathbb{R}$, with $|\lambda| < \frac{\pi}{2}$, $p \in \mathbb{N}$, $q \in \mathbb{N}_0$, with $p > q$, and $0 \le \alpha < p - q$. Suppose that ϕ is a univalent function in the unit disk $\mathbb{D}$ with $\phi(0) = 1$, such that*

$$\operatorname{Re}\phi(z) > 1 - \frac{1}{p-q-\alpha},\ z \in \mathbb{D}. \tag{4}$$

We define the classes $S^{\lambda}_{p,q,\alpha}(\phi)$ and $C^{\lambda}_{p,q,\alpha}(\phi)$ by

$$S^{\lambda}_{p,q,\alpha}(\phi) := \left\{ f \in \mathcal{A}(p) : e^{i\lambda} \frac{zf^{(1+q)}(z)}{f^{(q)}(z)} \prec \phi(z)(p-q-\alpha)\cos\lambda \right.$$
$$\left. +\alpha\cos\lambda + i(p-q)\sin\lambda \right\},$$

and

$$C^{\lambda}_{p,q,\alpha}(\phi) := \left\{ f \in \mathcal{A}(p) : e^{i\lambda} \left(1 + \frac{zf^{(2+q)}(z)}{f^{(1+q)}(z)} \right) \prec \phi(z)(p-q-\alpha)\cos\lambda \right.$$
$$\left. +\alpha\cos\lambda + i(p-q)\sin\lambda \right\}.$$

From the above definition we have

$$f \in C^{\lambda}_{p,q,\alpha}(\phi) \text{ if and only if } F \in S^{\lambda}_{p,q,\alpha}(\phi), \tag{5}$$
$$\text{where } F \in \mathcal{A}(p), \text{ such that } F^{(q)}(z) = \frac{zf^{(1+q)}(z)}{p-q},\ z \in \mathbb{D}.$$

Remark 1. *The subclasses defined in the above definition generalize some previous ones like we could see for the next special cases of ϕ.*

(i) For $\phi(z) = \dfrac{1+z}{1-z}$, we obtain the subclasses

$$S^{\lambda}_{p,q,\alpha}\left(\frac{1+z}{1-z}\right) =: S^{\lambda}_{\alpha}(p,q) \quad \textit{and} \quad C^{\lambda}_{p,q,\alpha}\left(\frac{1+z}{1-z}\right) =: C^{\lambda}_{\alpha}(p,q) \quad \textit{(see Aouf [14])},$$

which, for $q = 0$ reduces to

$$S^{\lambda}_{p,0,\alpha}\left(\frac{1+z}{1-z}\right) =: S^{\lambda}_{p}(\alpha) \quad \textit{and} \quad C^{\lambda}_{p,0,\alpha}\left(\frac{1+z}{1-z}\right) =: C^{\lambda}_{p}(\alpha)$$

(see Libera [17], Srivastava et al. [16]).

(ii) For $\phi(z) = \dfrac{1+Az}{1+Bz}$, with $-1 \le B < A \le 1$, we get the next extensions of the above subclasses, i.e.,

$$S^{\lambda}_{p,0,\alpha}\left(\frac{1+Az}{1+Bz}\right) =: S^{\lambda}(A,B,p,\alpha) \quad \textit{(see Aouf [18])},$$

$$C^{\lambda}_{p,0,\alpha}\left(\frac{1+Az}{1+Bz}\right) =: C^{\lambda}(A,B,p,\alpha), \quad S^{0}_{p,0,\alpha}\left(\frac{1+Az}{1+Bz}\right) =: S^{*}_{p}(A,B,\alpha),$$

and

$$C^{0}_{p,0,\alpha}\left(\frac{1+Az}{1+Bz}\right) =: K_{p}(A,B,\alpha) \quad \textit{(see Aouf [15])}.$$

To define the next generalizations of the above subclasses, we shall prove the following lemma:

Lemma 1. *Suppose the function $f \in S^{\lambda}_{p,q,\alpha}(\phi)$.*

(i) Then, $F(z) := \dfrac{f^{(q)}(z)}{\delta(p,q)z^{p-q-1}}$ is a λ-spirallike (univalent) function in $\mathbb{D}$.

(ii) Consequently, for all $\gamma \in \mathbb{C}$ the multivalued functions $\left[\dfrac{f^{(q)}(z)}{\delta(p,q)z^{p-q}}\right]^{\gamma}$ has an analytic branch in $\mathbb{D}$ with $\left.\left[\dfrac{f^{(q)}(z)}{\delta(p,q)z^{p-q}}\right]^{\gamma}\right|_{z=0} = 1.$

Proof. If we let $f \in S^{\lambda}_{p,q,\alpha}(\phi) \subset \mathcal{A}(p)$, then $F \in \mathcal{A}(1)$ and differentiating the definition formula of F we get

$$\frac{zF'(z)}{F(z)} = \frac{zf^{(1+q)}(z)}{f^{(q)}(z)} - p + q + 1,\ z \in \mathbb{D}.$$

Since $f \in S^{\lambda}_{p,q,\alpha}(\phi)$ we have

$$e^{i\lambda}\frac{zF'(z)}{F(z)} \prec [(p-q-\alpha)\phi(z) + \alpha - p + q + 1]\cos\lambda + i\sin\lambda =: H(z), \tag{6}$$

and using the assumption (4) it follows that

$$\operatorname{Re} H(z) = \cos\lambda[(p-q-\alpha)\operatorname{Re}\phi(z) + \alpha - p + q + 1] > 0,\ z \in \mathbb{D}.$$

Therefore, the subordination (6) yields that

$$\operatorname{Re}\left(e^{i\lambda}\frac{zF'(z)}{F(z)}\right) > 0,\ z \in \mathbb{D},$$

and this last inequality shows that F is λ-spirallike and univalent function in $\mathbb{D}$ (for details, see [19]).

It follows $\dfrac{F(z)}{z} \neq 0$ for all $z \in \mathbb{D}$, i.e., $\dfrac{f^{(q)}(z)}{\delta(p,q)z^{p-q}} \neq 0$, $z \in \mathbb{D}$, and this implies the second part of our lemma. □

Definition 4. *Let $\lambda \in \mathbb{R}$, with $|\lambda| < \dfrac{\pi}{2}$, $p \in \mathbb{N}$, $q \in \mathbb{N}_0$, with $p > q$, and $0 \leq \alpha < p - q$. Suppose that ϕ is a univalent function in the unit disk $\mathbb{D}$ with $\phi(0) = 1$, and satisfies the condition (4), and let $\gamma \in \mathbb{C}$ be an arbitrary complex number.*

(i) The function $g \in \mathcal{A}(p)$ is said to be in the class $S^{\lambda,\gamma}_{p,q,\alpha}(\phi)$ if there exists a function $f \in S^{\lambda}_{p,q,\alpha}(\phi)$ such that

$$g^{(q)}(z) = \delta(p,q)z^{p-q}\left[\frac{f^{(q)}(z)}{\delta(p,q)z^{p-q}}\right]^{\gamma}, \quad \text{where}$$

$$\delta(p,q) := \frac{p!}{(p-q)!} \quad \text{and} \quad \left.\left[\frac{f^{(q)}(z)}{\delta(p,q)z^{p-q}}\right]^{\gamma}\right|_{z=0} = 1.$$

(ii) The function $g \in \mathcal{A}(p)$ is said to be in the class $C^{\lambda,\gamma}_{p,q,\alpha}(\phi)$, if $G \in S^{\lambda,\gamma}_{p,q,\alpha}(\phi)$ where $G \in \mathcal{A}(p)$, such that $G^{(q)}(z) = \dfrac{zg^{(1+q)}(z)}{p-q}$, $z \in \mathbb{D}$.

Remark that, according to the second part of Lemma 1, since $f \in S^{\lambda}_{p,q,\alpha}(\phi)$ then the multi-valued functions $\left[\dfrac{f^{(q)}(z)}{\delta(p,q)z^{p-q}}\right]^{\gamma}$ has an analytic branch in $\mathbb{D}$ with $\left.\left[\dfrac{f^{(q)}(z)}{\delta(p,q)z^{p-q}}\right]^{\gamma}\right|_{z=0} = 1$, hence the power function of the above definition is correctly defined.

Remark 2. *We note that for appropriate choices of the parameters in the above defined classes of functions we obtain a few earlier studied classes as follows:*

(i) For the special case $\gamma = 1$ the classes $S^{\lambda,\gamma}_{p,q,\alpha}(\phi)$ and $C^{\lambda,\gamma}_{p,q,\alpha}(\phi)$ coincide with the classes $S^{\lambda}_{p,q,\alpha}(\phi)$ and $C^{\lambda}_{p,q,\alpha}(\phi)$, respectively;

(ii) For $q = \alpha = 0$, $p = 1$ and $\phi(z) = \dfrac{1+Az}{1+Bz}$, with $-1 \le B < A \le 1$, we have $S^{\lambda,\gamma}_{1,0,\alpha}[A,B] =: S^{\lambda,\gamma}[A,B]$ which was studied by Ahuja [20];

(iii) For $q = \alpha = 0$ we have $S^{\lambda,\gamma}_{p,0,0}(\phi) = S^{\lambda,\gamma}_{p}(\phi)$ and $C^{\lambda,\gamma}_{p,0,0}(\phi) =: C^{\lambda,\gamma}_{p}(\phi)$ which was studied by Sarkar et al. [2].

Definition 5. *Let $\lambda \in \mathbb{R}$, with $|\lambda| < \dfrac{\pi}{2}$, $p \in \mathbb{N}$, $q \in \mathbb{N}_0$, with $p > q$, and $0 \le \alpha < p-q$. Suppose that ϕ is a univalent function in the unit disk $\mathbb{D}$ with $\phi(0) = 1$, and satisfies the condition (4).*

For $l, s \in \mathbb{N}_0$, with $l \le s+1$, we define the following two subclasses of $\mathcal{A}(p)$, that are:

$$S^{\lambda,\gamma}_{p,q,\alpha;l,s}[a_1;\phi] := S^{\lambda,\gamma}_{p,q,\alpha;l,s}[a_1,\dots,a_l;b_1,\dots,b_s;\phi] = \left\{f \in \mathcal{A}(p) : H_{p,l,s}(a_1)f \in S^{\lambda,\gamma}_{p,q,\alpha}(\phi)\right\},$$

and

$$C^{\lambda,\gamma}_{p,q,\alpha;l,s}[a_1;\phi] := C^{\lambda,\gamma}_{p,q,\alpha;l,s}[a_1,\dots,a_l;b_1,\dots,b_s;\phi] = \left\{f \in \mathcal{A}(p) : H_{p,l,s}(a_1)f \in C^{\lambda,\gamma}_{p,q,\alpha}(\phi)\right\}.$$

In particular, for $\gamma = 1$, we denote

$$S^{\lambda}_{p,q,\alpha;l,s}[a_1;\phi] := S^{\lambda,1}_{p,q,\alpha;l,s}[a_1;\phi] \quad \text{and} \quad C^{\lambda}_{p,q,\alpha;l,s}[a_1;\phi] := C^{\lambda,1}_{p,q,\alpha;l,s}[a_1;\phi].$$

Moreover, by choosing appropriately the parameters in the above defined two classes of functions we obtain the following special cases introduced and studied by different authors:

(i) For $p = 1$, $\lambda = q = \alpha = 0$ and $\phi(z) = \dfrac{1+Az}{1+Bz}$, with $-1 \le B < A \le 1$, we obtain

$$S^{0}_{1,0,0;l,s}\left(a_1;\frac{1+Az}{1+Bz}\right) =: S^{*}_{l,s}[a_1;A,B] \quad \text{and} \quad C^{0}_{1,0,0;l,s}\left(a_1;\frac{1+Az}{1+Bz}\right) =: K_{l,s}[a_1;A,B]$$

(see Aouf and Seoudy [21]);

(ii) For $p = 1$, $q = \alpha = 0$ and $\phi(z) = \dfrac{1+Az}{1+Bz}$, with $-1 \le B < A \le 1$, we have

$$S^{\lambda}_{1,0,0;l,s}\left(a_1;\frac{1+Az}{1+Bz}\right) =: S^{\lambda}_{l,s}[a_1;A,B] \quad \text{and} \quad C^{\lambda}_{1,0,0;l,s}\left(a_1;\frac{1+Az}{1+Bz}\right) =: C^{\lambda}_{l,s}[a_1;A,B]$$

(see Seoudy [22]);

(iii) For $l = 2$, $s = 1$, $a_1 = a$, $a_2 = 1$ and $b_1 = c$, with $a > 0$, $c > 0$, we get

$$S^{\lambda,\gamma}_{p,q,\alpha}[a,c;\phi] := \left\{f \in \mathcal{A}(p) : L_p(a,c)f \in S^{\lambda,\gamma}_{p,q,\alpha}(\phi)\right\}$$

and

$$C^{\lambda,\gamma}_{p,q,\alpha}[a,c;\phi] := \left\{f \in \mathcal{A}(p) : L_p(a,c)f \in C^{\lambda,\gamma}_{p,q,\alpha}(\phi)\right\},$$

where the operator $L_p(a,c)$ was introduced by Saitoh [12];

(iv) For $l = 2$, $s = 1$, $a_1 = \nu + p$ with $\nu > -p$, $a_2 = 1$ and $b_1 = \nu + p + 1$ we obtain

$$S^{\lambda,\gamma}_{p,q,\alpha}[\nu+p,1;\nu+p+1;\phi] := \left\{f \in \mathcal{A}(p) : J_{\nu,p}f \in S^{\lambda,\gamma}_{p,q,\alpha}(\phi)\right\}$$

and

$$C^{\lambda,\gamma}_{p,q,\alpha}[\nu+p,1;\nu+p+1;\phi] := \left\{f \in \mathcal{A}(p) : J_{\nu,p}f \in C^{\lambda,\gamma}_{p,q,\alpha}(\phi)\right\},$$

where $J_{\nu,p}$ is *the generalized Bernardi-Libera-Livingston operator* (see [7]);

(v) For $l = 2$, $s = 1$, $a_1 = \mu + p$, with $\mu > -p$, $a_2 = b_1 = 1$ we have

$$S^{\lambda,\gamma}_{p,q,\alpha}[\mu + p, 1; 1; \phi] := \left\{ f \in \mathcal{A}(p) : D^{\mu+p-1} f \in S^{\lambda,\gamma}_{p,q,\alpha}(\phi) \right\}$$

and

$$C^{\lambda,\gamma}_{p,q,\alpha}[\mu + p, 1; 1; \phi] := \left\{ f \in \mathcal{A}(p) : D^{\mu+p-1} f \in C^{\lambda,\gamma}_{p,q,\alpha}(\phi) \right\},$$

where $D^{\mu+p-1}$ is the $(\mu + p - 1)$-th order *Ruscheweyh derivative* (see Goel and Sohi [9]);

(vi) For $l = 2$, $s = 1$, $a_1 = p + 1$, $a_2 = 1$ and $b_1 = n + p$, with $n \in \mathbb{Z}$, $n > -p$, we get

$$S^{\lambda,\gamma}_{p,q,\alpha}[p + 1, 1; n + p; \phi] := \left\{ f \in \mathcal{A}(p) : I_{n,p} f \in S^{\lambda,\gamma}_{p,q,\alpha}(\phi) \right\}$$

and

$$C^{\lambda,\gamma}_{p,q,\alpha}[p + 1, 1; n + p; \phi] := \left\{ f \in \mathcal{A}(p) : I_{n,p} f \in C^{\lambda,\gamma}_{p,q,\alpha}(\phi) \right\},$$

where the operator $I_{n,p}$ was defined by Liu and Noor [10];

(vii) For $l = 2$, $s = 1$, $a_1 = \lambda + p$ with $\lambda > -p$, $a_2 = c$ and $b_1 = a$, with $a, c \in \mathbb{R} \setminus \mathbb{Z}^-_0$, we obtain

$$S^{\lambda,\gamma}_{p,q,\alpha}[\lambda + p, c; a; \phi] := \left\{ f \in \mathcal{A}(p) : I^{\lambda}_p(a, c) f \in S^{\lambda,\gamma}_{p,q,\alpha}(\phi) \right\}$$

and

$$C^{\lambda,\gamma}_{p,q,\alpha}[\lambda + p, c; a; \phi] := \left\{ f \in \mathcal{A}(p) : I^{\lambda}_p(a, c) f \in C^{\lambda,\gamma}_{p,q,\alpha}(\phi) \right\},$$

where the operator $I^{\lambda}_p(a, c)$ was considered by Cho et al. [8];

(viii) For $l = 2$, $s = 1$, $a_1 = p + 1$, $a_2 = 1$ and $b_1 = 1 + p - \lambda$ we have

$$S^{\lambda,\gamma}_{p,q,\alpha}[p + 1, 1; 1 + p - \lambda; \phi] := \left\{ f \in \mathcal{A}(p) : \Omega^{(\lambda,p)}_z f \in S^{\lambda,\gamma}_{p,q,\alpha}(\phi) \right\}$$

and

$$C^{\lambda,\gamma}_{p,q,\alpha}[p + 1, 1; 1 + p - \lambda; \phi] := \left\{ f \in \mathcal{A}(p) : \Omega^{(\lambda,p)}_z f \in C^{\lambda,\gamma}_{p,q,\alpha}(\phi) \right\},$$

where the operator $\Omega^{(\lambda,p)}_z$ was introduced by Srivastava and Aouf [13] for $0 \leq \lambda < 1$, and was investigated by Patel and Mishra [11] for $\lambda < p + 1$, $p \in \mathbb{N}$.

2. Convolution and Inclusion Properties

Our first result represent a necessary and sufficient condition, in term of convolution product, for a function $g \in \mathcal{A}(p)$ to belongs to class $S^{\lambda}_{p,q,\alpha}(\phi)$.

Theorem 1. *Let $f \in \mathcal{A}(p)$, such that $\dfrac{f^{(q)}(z)}{z^{p-q}} \neq 0$ for all $z \in \mathbb{D}$ and $\gamma \in \mathbb{C}$. We will define the function $g \in \mathcal{A}(p)$, such that $g^{(q)}(z) := \delta(p, q) z^{p-q} \left[\dfrac{f^{(q)}(z)}{\delta(p, q) z^{p-q}} \right]^{\gamma}$, where the power function is considered to the main branch, i.e., $\left. \left[\dfrac{f^{(q)}(z)}{\delta(p, q) z^{p-q}} \right]^{\gamma} \right|_{z=0} = 1$.*

Then, $g \in S^{\lambda}_{p,q,\alpha}(\phi)$ if and only if

$$\frac{1}{z^{p-q}} \left[f^{(q)}(z) * \frac{z^{p-q} - C z^{p-q+1}}{(1 - z)^2} \right] \neq 0, \text{ for all } x \in \mathbb{C} : |x| = 1, \text{ for all } z \in \mathbb{D}, \tag{7}$$

where

$$C = C_{\gamma,q,\alpha}(x) := \frac{(p - q - \alpha)(1 - \phi(x)) \cos \lambda - \gamma e^{i\lambda}}{(p - q - \alpha)(1 - \phi(x)) \cos \lambda}. \tag{8}$$

Proof. For any function $f \in \mathcal{A}(p)$ we can easily verify that

$$f^{(q)}(z) = f^{(q)}(z) * \frac{z^{p-q}}{1-z} \quad \text{and} \quad zf^{(1+q)}(z) = f^{(q)}(z) * \left[\frac{z^{p-q+1}}{(1-z)^2} + \frac{(p-q)z^{p-q}}{1-z}\right],\ z \in \mathbb{D}, \tag{9}$$

and from the definition formula of the function g we get

$$\gamma e^{i\lambda}\frac{zf^{(1+q)}(z)}{f^{(q)}(z)} + (p-q)(1-\gamma)e^{i\lambda} = e^{i\lambda}\frac{zg^{(1+q)}(z)}{g^{(q)}(z)},\ z \in \mathbb{D}.$$

First, according to the above identity we have that $g \in S^{\lambda}_{p,q,\alpha}(\phi)$ is equivalent to

$$\gamma e^{i\lambda}\frac{zf^{(1+q)}(z)}{f^{(q)}(z)} + (p-q)(1-\gamma)e^{i\lambda} \prec \phi(z)(p-q-\alpha)\cos\lambda + \alpha\cos\lambda + i(p-q)\sin\lambda, \tag{10}$$

and using the definition of the subordination, from (10) it follows that

$$\gamma e^{i\lambda}\frac{zf^{(1+q)}(z)}{f^{(q)}(z)} + (p-q)(1-\gamma)e^{i\lambda} \neq \phi(x)(p-q-\alpha)\cos\lambda + \alpha\cos\lambda + i(p-q)\sin\lambda,$$
$$\text{for all } x \in \mathbb{C} : |x| = 1,\ \text{for all } z \in \mathbb{D}. \tag{11}$$

From the convolution relations (9) a simple computation shows that (11) could be rewritten in the following form

$$\frac{1}{z^{p-q}}\left\{f^{(q)}(z) * \left[\gamma e^{i\lambda}\left(\frac{z^{p-q+1}}{(1-z)^2} + \frac{(p-q)z^{p-q}}{1-z}\right)\right.\right.$$
$$\left.\left. + \Big((p-q-\alpha)\cos\lambda - \phi(x)(p-q-\alpha)\cos\lambda - \gamma(p-q)e^{i\lambda}\Big)\frac{z^{p-q}}{1-z}\right]\right\} \neq 0,$$
$$\text{for all } x \in \mathbb{C} : |x| = 1,\ \text{for all } z \in \mathbb{D},$$

which is equivalent to (7), where C given by (8).

Reversely, like it was shown in the first part of the proof, the assumption (7) is equivalent to (11). Denoting

$$\Phi(z) := \gamma e^{i\lambda}\frac{zf^{(1+q)}(z)}{f^{(q)}(z)} + (p-q)(1-\gamma)e^{i\lambda}$$

and

$$\Psi(z) := \phi(z)(p-q-\alpha)\cos\lambda + \alpha\cos\lambda + i(p-q)\sin\lambda,$$

these functions are analytic in $\mathbb{D}$ and Ψ is univalent since ϕ is univalent in $\mathbb{D}$. The relation (11) shows that $\Phi(\mathbb{D}) \cap \Psi(\partial\mathbb{D}) = \emptyset$ and therefore the simply-connected domain $\Phi(\mathbb{D})$ is included in a connected component of $\mathbb{C} \setminus \Psi(\partial\mathbb{D})$. From here, and using the fact that $\Phi(0) = \Psi(0)$ together with the univalence of the function Ψ, it follows that $\Phi(z) \prec \Psi(z)$ which represents in fact the subordination (10), i.e., $g \in S^{\lambda}_{p,q,\alpha}(\phi)$. $\square$

The next result that is also a necessary and sufficient condition for a function $g \in \mathcal{A}(p)$ to belongs to class $S^{\lambda}_{p,q,\alpha}(\phi)$ generalizes some other previously results like we mention after its proof.

Theorem 2. *Let $g \in \mathcal{A}(p)$, such that $\dfrac{g^{(q)}(z)}{z^{p-q}} \neq 0$ for all $z \in \mathbb{D}$. Then, $g \in S^{\lambda,\gamma}_{p,q,\alpha}(\phi)$ with $\gamma \in \mathbb{C}^*$ if and only if*

$$\frac{1}{z^{p-q}}\left[g^{(q)}(z) * \frac{z^{p-q} - Dz^{p-q+1}}{(1-z)^2}\right] \neq 0,\ \text{for all } x \in \mathbb{C} : |x| = 1,\ \text{for all } z \in \mathbb{D}, \tag{12}$$

where

$$D = D_{\gamma,q,\alpha}(x) := \frac{(p-q-\alpha)(1-\phi(x))\gamma\cos\lambda - e^{i\lambda}}{(p-q-\alpha)(1-\phi(x))\gamma\cos\lambda}. \tag{13}$$

Proof. First, from Definition 4 we have that $g \in S^{\lambda,\gamma}_{p,q,\alpha}(\phi)$ if there exists a function $f \in S^{\lambda}_{p,q,\alpha}(\phi)$ such that $g^{(q)}(z) = \delta(p,q)z^{p-q}\left[\dfrac{f^{(q)}(z)}{\delta(p,q)z^{p-q}}\right]^{\gamma}$, where $\left.\left[\dfrac{f^{(q)}(z)}{\delta(p,q)z^{p-q}}\right]^{\gamma}\right|_{z=0} = 1.$

Differentiating the above definition formula we get

$$\frac{zg^{(q+1)}(z)}{g^{(q)}(z)} = \gamma\frac{zf^{(1+q)}(z)}{f^{(q)}(z)} + (p-q)(1-\gamma),\ z \in \mathbb{D},$$

and using the fact that $g \in S^{\lambda,\gamma}_{p,q,\alpha}(\phi)$ if and only if $f \in S^{\lambda}_{p,q,\alpha}(\phi)$ it follows that $g \in S^{\lambda,\gamma}_{p,q,\alpha}(\phi)$ is equivalent to

$$\begin{aligned} &e^{i\lambda}\frac{zg^{(1+q)}(z)}{g^{(q)}(z)} - e^{i\lambda}(p-q)(1-\gamma) = \gamma e^{i\lambda}\frac{zf^{(1+q)}(z)}{f^{(q)}(z)} \\ &\prec \phi(z)(p-q-\alpha)\gamma\cos\lambda + \alpha\gamma\cos\lambda + i\gamma(p-q)\sin\lambda. \end{aligned}$$

Since the right-hand side of the above subordination is a univalent function in $\mathbb{D}$ it follows that

$$\begin{aligned} &\gamma e^{i\lambda}\frac{zg^{(1+q)}(z)}{g^{(q)}(z)} - e^{i\lambda}(p-q)(1-\gamma) \neq \phi(x)(p-q-\alpha)\gamma\cos\lambda \\ &+\alpha\gamma\cos\lambda + i\gamma(p-q)\sin\lambda,\ \text{for all } x \in \mathbb{C} : |x| = 1,\ \text{for all } z \in \mathbb{D}. \end{aligned} \tag{14}$$

Finally, from the convolution relations (9), a simple computation shows that (14) is equivalent to (12) where D is given by (13).

Reversely, since $\dfrac{g^{(q)}(z)}{z^{p-q}} \neq 0$ for all $z \in \mathbb{D}$, it follows that the function $f \in \mathcal{A}(p)$ given by

$$f^{(q)}(z) := \delta(p,q)z^{p-q}\left[\frac{g^{(q)}(z)}{\delta(p,q)z^{p-q}}\right]^{1/\gamma},\ z \in \mathbb{D},$$

where $\left.\left[\dfrac{g^{(q)}(z)}{\delta(p,q)z^{p-q}}\right]^{1/\gamma}\right|_{z=0} = 1$ is correctly defined. The above definition relation is equivalent to

$$g^{(q)}(z) = \delta(p,q)z^{p-q}\left[\frac{f^{(q)}(z)}{\delta(p,q)z^{p-q}}\right]^{\gamma},\ z \in \mathbb{D},$$

where $\left.\left[\dfrac{f^{(q)}(z)}{\delta(p,q)z^{p-q}}\right]^{\gamma}\right|_{z=0} = 1$ and by similar computations like in the first part of the proof we deduce that $g \in S^{\lambda,\gamma}_{p,q,\alpha}(\phi)$ is equivalent to (14). Now, denoting

$$\Phi(z) := e^{i\lambda}\frac{zg^{(1+q)}(z)}{g^{(q)}(z)} - e^{i\lambda}(p-q)(1-\gamma)$$

and

$$\Psi(z) := \phi(z)(p-q-\alpha)\gamma\cos\lambda + \alpha\gamma\cos\lambda + i\gamma(p-q)\sin\lambda,$$

and using the same arguments as in the second part of the proof of Theorem 1 we deduce that $\Phi(z) \prec \Psi(z)$, that is $g \in S^{\lambda,\gamma}_{p,q,\alpha}(\phi)$. □

Remark 3. *(i) For the special case $q = \alpha = 0$ Theorems 1 and 2 reduces to the convolution results obtained by Sarkar et al. [2] (Theorems 2.1 and 2.2);*

(ii) Putting $p = 1$, $q = \alpha = 0$, $\phi(x) = \dfrac{1+Ax}{1+Bx}$, with $-1 \le B < A \le 1$, and writing $-\overline{x}$ as x in Theorems 1 and 2 we obtain the convolution results of Ahuja [3] (Theorems 2.1 and 2.2);

(iii) For $\gamma = 1$ we have $f = g$, where g is defined like in Theorem 1 and $C_{1,q,\alpha} = D_{1,q,\alpha}$. Moreover, for this special case the additional assumptions $\dfrac{f^{(q)}(z)}{z^{p-q}} \neq 0$ and $\dfrac{g^{(q)}(z)}{z^{p-q}} \neq 0$ for all $z \in \mathbb{D}$ of these theorems are not necessary, hence both of the results of Theorems 1 and 2 coincide with the following corollary:

Corollary 1. *If $f \in \mathcal{A}(p)$, then $f \in S^{\lambda}_{p,q,\alpha}(\phi)$ if and only if (7) holds for*

$$C = C_{q,\alpha}(x) := C_{1,q,\alpha}(x) = \frac{(p-q-\alpha)(1-\phi(x))\cos\lambda - e^{i\lambda}}{(p-q-\alpha)(1-\phi(x))\cos\lambda}. \tag{15}$$

Example 1. *Let consider in Theorems 1 and 2 the function*

$$\phi(z) = \frac{1+Az}{1+Bz},\ -1 \le B < A \le 1,$$

and $q := p-1$. Then, the assumption (4) is equivalent to

$$\frac{1-A}{1-B} > 1 - \frac{1}{1-\alpha},$$

which holds for all $0 \le \alpha < p-q = 1$, and from the above mentioned theorems we obtain the next particular cases, respectively:

1. Let $f \in \mathcal{A}(p)$ such that $\dfrac{f^{(p-1)}(z)}{z} \neq 0$ for all $z \in \mathbb{D}$, and $\gamma \in \mathbb{C}$. We will define the function $g \in \mathcal{A}(p)$ such that $g^{(p-1)}(z) := p!z\left[\dfrac{f^{(p-1)}(z)}{p!z}\right]^{\gamma}$, where the power function is considered to the main branch.

Then, $g \in S^{\lambda}_{p,p-1,\alpha}(\phi)$ if and only if

$$\frac{1}{z}\left[f^{(p-1)}(z) * \frac{z-Cz^2}{(1-z)^2}\right] \neq 0, \text{ for all } x \in \mathbb{C} : |x| = 1, \text{ for all } z \in \mathbb{D},$$

where

$$C = C_{\gamma,p-1,\alpha}(x) := \frac{(1-\alpha)(1-\phi(x))\cos\lambda - \gamma e^{i\lambda}}{(1-\alpha)(1-\phi(x))\cos\lambda}.$$

2. Let $g \in \mathcal{A}(p)$, such that $\dfrac{g^{(p-1)}(z)}{z} \neq 0$ for all $z \in \mathbb{D}$. Then, $g \in S^{\lambda,\gamma}_{p,p-1,\alpha}(\phi)$ with $\gamma \in \mathbb{C}^*$, if and only if

$$\frac{1}{z}\left[g^{(p-1)}(z) * \frac{z-Dz^2}{(1-z)^2}\right] \neq 0, \text{ for all } x \in \mathbb{C} : |x| = 1, \text{ for all } z \in \mathbb{D},$$

where

$$D = D_{\gamma,q,\alpha}(x) := \frac{(1-\alpha)(1-\phi(x))\gamma\cos\lambda - e^{i\lambda}}{(1-\alpha)(1-\phi(x))\gamma\cos\lambda}.$$

The next theorem is a necessary and sufficient condition for a function $g \in \mathcal{A}(p)$ to belongs to class $C^{\lambda}_{p,q,\alpha}(\phi)$ and it extends a few previously results obtained by different authors.

Theorem 3. *If $f \in \mathcal{A}(p)$, then $f \in C^{\lambda}_{p,q,\alpha}(\phi)$ if and only if*

$$\frac{1}{z^{p-q}}\left[f^{(q)}(z) * \mathcal{M}\right] \neq 0, \text{ for all } x \in \mathbb{C} : |x| = 1, \text{ for all } z \in \mathbb{D},$$

where

$$\mathcal{M} := \frac{(p-q)z^{p-q} - \left[(p-q) + C_{q,\alpha}(p-q+1) - 2\right]z^{p-q-1} + C_{q,\alpha}(p-q-1)z^{p-q+2}}{(1-z)^3}$$

and $C_{q,\alpha} := C_{1,q,\alpha}$ is given by (15).

Proof. If we let

$$G_{p,q}(z) = \frac{z^{p-q} - C_{q,\alpha}z^{p-q+1}}{(1-z)^2}, \ z \in \mathbb{D},$$

then

$$zG'_{p,q}(z) = \frac{(p-q)z^{p-q} - \left[(p-q) + C_{q,\alpha}(p-q+1) - 2\right]z^{p-q+1} + C_{q,\alpha}(p-q-1)z^{p-q+2}}{(1-z)^3}.$$

From (5) and according to the Corollary 1 we deduce that $F \in S^{\lambda}_{p,q,\alpha}(\phi)$ if and only if

$$\frac{1}{z^{p-q}}\left[F^{(q)}(z) * G_{p,q}(z)\right] \neq 0, \text{ for all } x \in \mathbb{C} : |x| = 1, \text{ for all } z \in \mathbb{D}. \tag{16}$$

Using (5) in (16) we obtain that $f \in C^{\lambda}_{p,q,\alpha}(\phi)$ if and only if

$$\frac{1}{z^{p-q}}\left[\frac{zf^{(1+q)}(z)}{p-q} * G_{p,q}(z)\right] \neq 0, \text{ for all } x \in \mathbb{C} : |x| = 1, \text{ for all } z \in \mathbb{D}. \tag{17}$$

Since the identity

$$\frac{zf^{(1+q)}(z)}{p-q} * G_{p,q}(z) = f^{(q)}(z) * \frac{zG'_{p,q}(z)}{p-q}, \ z \in \mathbb{D},$$

holds, then (17) is equivalent to

$$\frac{1}{z^{p-q}}\left[f^{(q)}(z) * \frac{zG'_{p,q}(z)}{p-q}\right] \neq 0, \text{ for all } x \in \mathbb{C} : |x| = 1, \text{ for all } z \in \mathbb{D},$$

which completes our proof. □

Remark 4. *(i) Putting $p = 1$, $q = \alpha = \lambda = 0$ and $\phi(x) = \dfrac{1+Ax}{1+Bx}$, with $-1 \leq B < A \leq 1$, in Corollary 1 and Theorem 3 we obtain the convolution results due to Padmanabhan and Ganesan [4] (Theorems 1 and 2);*

(ii) Putting $p = 1$, $q = \alpha = 0$ and $\phi(x) = \dfrac{1+Ax}{1+Bx}$, with $-1 \leq B < A \leq 1$, in Corollary 1 and Theorem 3 we obtain the convolution results due to Padmanabhan and Ganesan [4] (Theorems 3 and 4).

The following result is a necessary and sufficient condition for a function $g \in \mathcal{A}(p)$ to belongs to class $C^{\lambda}_{p,q,\alpha}(\phi)$ and for the special case $\gamma = 1$ it coincide with the previous theorem.

Theorem 4. *Let $f \in \mathcal{A}(p)$ such that $\dfrac{f^{(1+q)}(z)}{z^{p-q-1}} \neq 0$ for all $z \in \mathbb{D}$. Then, $f \in C^{\lambda,\gamma}_{p,q,\alpha}(\phi)$, with $\gamma \in \mathbb{C}^*$, if and only if*

$$\frac{1}{z^{p-q}}\left[f^{(q)}(z) * \mathcal{N}\right] \neq 0, \text{ for all } x \in \mathbb{C} : |x| = 1, \text{ for all } z \in \mathbb{D},$$

where

$$\mathcal{N} := \frac{(p-q)z^{p-q} - [(p-q) + D(p-q+1) - 2]z^{p-q+1} + D(p-q-1)z^{p-q+2}}{(1-z)^3}$$

and $D = D_{\gamma,q,\alpha}$ is given by (13).

Proof. Letting

$$H_{p,q}(z) = \frac{z^{p-q} - Dz^{p-q+1}}{(1-z)^2}, \; z \in \mathbb{D},$$

we have

$$zH'_{p,q}(z) = \frac{(p-q)z^{p-q} - [(p-q) + D(p-q+1) - 2]z^{p-q+1} + D(p-q-1)z^{p-q+2}}{(1-z)^3}.$$

According to the Definition 4 we have

$$f \in C^{\lambda,\gamma}_{p,q,\alpha}(\phi) \text{ if and only if } F \in S^{\lambda,\gamma}_{p,q,\alpha}(\phi), \tag{18}$$

$$\text{where} \quad F \in \mathcal{A}(p), \quad \text{such that} \quad F^{(q)}(z) = \frac{zf^{(1+q)}(z)}{p-q}, \; z \in \mathbb{D},$$

and from Theorem 2 we deduce that $F \in S^{\lambda,\gamma}_{p,q,\alpha}(\phi)$ if and only if

$$\frac{1}{z^{p-q}}\left[F^{(q)}(z) * H_{p,q}(z)\right] \neq 0, \text{ for all } x \in \mathbb{C} : |x| = 1, \text{ for all } z \in \mathbb{D}. \tag{19}$$

Thus, using (18) in (19) we obtain that $f \in C^{\lambda}_{p,q,\alpha}(\phi)$ if and only if

$$\frac{1}{z^{p-q}}\left[\frac{zf^{(1+q)}(z)}{p-q} * H_{p,q}(z)\right] \neq 0, \text{ for all } x \in \mathbb{C} : |x| = 1, \text{ for all } z \in \mathbb{D}. \tag{20}$$

From the identity

$$\frac{zf^{(1+q)}(z)}{p-q} * H_{p,q}(z) = f^{(q)}(z) * \frac{zH'_{p,q}(z)}{p-q}, \; z \in \mathbb{D},$$

it follows that (20) is equivalent to

$$\frac{1}{z^{p-q}}\left[f^{(q)}(z) * \frac{zH'_{p,q}(z)}{p-q}\right] \neq 0, \text{ for all } x \in \mathbb{C} : |x| = 1, \text{ for all } z \in \mathbb{D},$$

and the proof of our theorem is complete. □

Example 2. *Let the function*

$$\phi(z) = \frac{1+Az}{1+Bz}, 1 \leq B < A \leq 1,$$

and $q := p-1$. From the Theorems 3 and 4 we obtain the next two special cases, respectively:

1. If $f \in \mathcal{A}(p)$, then $f \in C^{\lambda}_{p,p-1,\alpha}(\phi)$ if and only if

$$\frac{1}{z}\left[f^{(p-1)}(z) * \mathcal{M}\right] \neq 0, \text{ for all } x \in \mathbb{C} : |x| = 1, \text{ for all } z \in \mathbb{D},$$

where

$$\mathcal{M} := \frac{z + 1 - 2C_{p-1,\alpha}}{(1-z)^3}$$

and $C_{p-1,\alpha} := C_{1,p-1,\alpha}$ is given by (15).

2. Let $f \in \mathcal{A}(p)$, such that $f^{(p)}(z) \neq 0$ for all $z \in \mathbb{D}$. Then, $f \in C^{\lambda,\gamma}_{p,p-1,\alpha}(\phi)$ with $\gamma \in \mathbb{C}^*$, if and only if

$$\frac{1}{z}\left[f^{(p-1)}(z) * \mathcal{N}\right] \neq 0, \text{ for all } x \in \mathbb{C} : |x| = 1, \text{ for all } z \in \mathbb{D},$$

where

$$\mathcal{N} := \frac{z - (2D-1)z^2}{(1-z)^3}$$

and $D = D_{\gamma,p-1,\alpha}$ is given by (13).

Remark 5. *Note that for the special case $\gamma = 1$ we have $C_{1,q,\alpha} = D_{1,q,\alpha}$, and using the same reasons like in Remark 3 (iii) we deduce that the results of Theorems 3 and 4 coincide.*

The following result represents a necessary and sufficient condition for a function $g \in \mathcal{A}(p)$ to belongs to class $S^{\lambda,\gamma}_{p,q,\alpha;l,s}[a_1;\phi]$ connected with the Dziok and Srivastava convolution operator [1].

Theorem 5. *Let $f \in \mathcal{A}(p)$ of the form (1) such that $\dfrac{\left(H_{p,l,s}(a_1)f(z)\right)^{(q)}}{z^{p-q}} \neq 0$ for all $z \in \mathbb{D}$. Then, $f \in S^{\lambda,\gamma}_{p,q,\alpha;l,s}[a_1;\phi]$ with $\gamma \in \mathbb{C}^*$, if and only if*

$$1 + \sum_{k=p+1}^{+\infty} \frac{(p-q-\alpha)(1-\phi(x))\gamma\cos\lambda + (k-p)e^{i\lambda}}{(p-q-\alpha)(1-\phi(x))\gamma\cos\lambda} \cdot \frac{\delta(k,p)}{\delta(p,q)}\Gamma_{k-p}[a_1;b_1]a_k z^{k-p} \neq 0,$$
$$\text{for all } x \in \mathbb{C} : |x| = 1, \text{ for all } z \in \mathbb{D}. \quad (21)$$

Proof. According to Theorem 2 we have that $f \in S^{\lambda,\gamma}_{p,q,\alpha;l,s}[a_1;\phi]$ if and only if

$$\frac{1}{z^{p-q}}\left[\left(H_{p,l,s}(a_1)f(z)\right)^{(q)} * \frac{z^{p-q} - Dz^{p-q+1}}{(1-z)^2}\right] \neq 0, \text{ for all } x \in \mathbb{C} : |x| = 1, \text{ for all } z \in \mathbb{D},$$

where D is given by (13), which is equivalent to

$$\frac{1}{z^{p-q}}\left\{\left[\delta(p,q)z^{p-q} + \sum_{k=p+1}^{+\infty}\delta(k,p)\Gamma_{k-p}[a_1;b_1]a_k z^{k-q}\right] * \frac{z^{p-q} - Dz^{p-q+1}}{(1-z)^2}\right\} \neq 0,$$
$$\text{for all } x \in \mathbb{C} : |x| = 1, \text{ for all } z \in \mathbb{D}. \quad (22)$$

Since

$$\frac{z^{p-q}}{(1-z)^2} = z^{p-q} + \sum_{k=p+1}^{+\infty}(k-p+1)z^{k-p}, \; z \in \mathbb{D},$$
$$\frac{z^{p-q+1}}{(1-z)^2} = z^{p-q+1} + \sum_{k=p+1}^{+\infty}(k-p+1)z^{k-q+1} = \sum_{k=p+1}^{+\infty}(k-p)z^{k-q}, \; z \in \mathbb{D},$$

after a simple computation we have obtain that (22) is equivalent to

$$1+\sum_{r=p+1}^{+\infty}[1+(1-D)(k-p)]\cdot\frac{\delta(k,q)}{\delta(p,q)}\Gamma_{k-p}[a_1;b_1]a_k z^{k-p}\neq 0,$$

for all $x\in\mathbb{C}:|x|=1$ and for all $z\in\mathbb{D}$, which simplifies to (21), and the proof of the theorem is completed. □

A similar result with the previous theorem deals with a sufficient condition for a function $g\in\mathcal{A}(p)$ to belongs to class $C^{\lambda,\gamma}_{p,q,\alpha;l,s}[a_1;\phi]$ define by using the Dziok and Srivastava convolution operator.

Theorem 6. *Let $f\in\mathcal{A}(p)$, such that $\dfrac{\left(H_{p,q,s}(a_1)f(z)\right)^{(1+q)}}{z^{p-q-1}}\neq 0$ for all $z\in\mathbb{D}$. Then, $f\in C^{\lambda,\gamma}_{p,q,\alpha;l,s}[a_1;\phi]$ with $\gamma\in\mathbb{C}^*$, if and only if*

$$\begin{aligned}1+\sum_{k=p+1}^{+\infty}\frac{k-q}{p-q}\cdot\frac{(p-q-\alpha)(1-\phi(x))\gamma\cos\lambda+(k-p)e^{i\lambda}}{(p-q-\alpha)(1-\phi(x))\gamma\cos\lambda}\\ \cdot\frac{\delta(k,q)}{\delta(p,q)}\Gamma_{k-p}[a_1;b_1]a_k z^{k-p}\neq 0,\ \textit{for all } x\in\mathbb{C}:|x|=1,\ \textit{for all } z\in\mathbb{D}.\end{aligned}\tag{23}$$

Proof. From Theorem 4 we have that $f\in C^{\lambda,\gamma}_{p,q,\alpha;l,s}[a_1;\phi]$ if and only if

$$\begin{aligned}&\frac{1}{z^{p-q}}\Bigg[\left(H_{p,q,s}(a_1)f(z)\right)^{(q)}\\ &*\frac{(p-q)z^{p-q}-[(p-q)+D(p-q+1)-2]z^{p-q+1}+D(p-q-1)z^{p-q+2}}{(1-z)^3}\Bigg]\neq 0,\\ &\qquad\text{for all } x\in\mathbb{C}:|x|=1,\ \text{for all } z\in\mathbb{D},\end{aligned}\tag{24}$$

where D is given by (13). It can be easily shown that

$$\frac{z^{p-q}}{(1-z)^3}=z^{p-q}+\sum_{k=p+1}^{+\infty}\frac{(k-p+1)(k-p+2)}{2}z^{k-q},\ z\in\mathbb{D},$$
$$\frac{z^{p-q+1}}{(1-z)^3}=\sum_{k=p+1}^{+\infty}\frac{(k-p)(k-p+1)}{2}z^{k-q},\ z\in\mathbb{D},$$

and

$$\frac{z^{p-q+2}}{(1-z)^3}=\sum_{k=p+2}^{+\infty}\frac{(k-p-1)(k-p)}{2}z^{k-q},\ z\in\mathbb{D}.$$

Using these identities and after some simple computations we deduce that (24) is equivalent to

$$\begin{aligned}\frac{1}{z^{p-q}}\left[\delta(p,q)z^{p-q}+\sum_{k=p+1}^{+\infty}\frac{k-q}{p-q}[1+(1-D)(k-p)]\delta(k,q)\Gamma_{k-p}[a_1;b_1]a_k z^{k-q}\right]\neq 0,\\ \text{for all } x\in\mathbb{C}:|x|=1,\ \text{for all } z\in\mathbb{D},\end{aligned}$$

which reduces to (23). □

Example 3. *If we consider the function*

$$\phi(z)=\frac{1+Az}{1+Bz}, 1\le B<A\le 1,$$

and $q:=p-1$, then from the Theorems 5 and 6 we obtain the next result, respectively:

1. Let $f\in\mathcal{A}(p)$ of the form (1), such that $\left(H_{p,l,s}(a_1)f(z)\right)^{(p-1)}\neq 0$ for all $z\in\mathbb{D}$. Then, $f\in S^{\lambda,\gamma}_{p,p-1,\alpha;l,s}[a_1;\phi]$ with $\gamma\in\mathbb{C}^*$, if and only if

$$1+\sum_{k=p+1}^{+\infty}\frac{(1-\alpha)(1-\phi(x))\gamma\cos\lambda+(k-p)e^{i\lambda}}{(1-\alpha)(1-\phi(x))\gamma\cos\lambda}\cdot\frac{\delta(k,p)}{p!}\Gamma_{k-p}[a_1;b_1]a_k z^{k-p}\neq 0,$$
$$\text{for all } x\in\mathbb{C}:|x|=1,\ \text{for all } z\in\mathbb{D}.$$

2. Let $f\in\mathcal{A}(p)$, such that $\left(H_{p,p-1,s}(a_1)f(z)\right)^{(p)}\neq 0$ for all $z\in\mathbb{D}$. Then, $f\in C^{\lambda,\gamma}_{p,p-1,\alpha;l,s}[a_1;\phi]$ with $\gamma\in\mathbb{C}^*$, if and only if

$$1+\sum_{k=p+1}^{+\infty}(k-p+1)\cdot\frac{(1-\alpha)(1-\phi(x))\gamma\cos\lambda+(k-p)e^{i\lambda}}{(1-\alpha)(1-\phi(x))\gamma\cos\lambda}$$
$$\cdot\frac{\delta(k,p-1)}{p!}\Gamma_{k-p}[a_1;b_1]a_k z^{k-p}\neq 0,\ \text{for all } x\in\mathbb{C}:|x|=1,\ \text{for all } z\in\mathbb{D}.$$

Now we will give an inclusion relation for the classes $S^{\lambda}_{p,q,\alpha;l,s}[a_1;\phi]$ and to prove this result we shall require the following lemma:

Lemma 2 (see [23]). *Let ϕ be convex (univalent) in $\mathbb{D}$, with $\operatorname{Re}[\beta\phi(z)+\gamma]>0$, $z\in\mathbb{D}$. If θ is analytic in $\mathbb{D}$ with $\theta(0)=\phi(0)$, then*

$$\theta(z)+\frac{z\theta'(z)}{\beta\theta(z)+\gamma}\prec\phi(z)\ \text{implies}\ \theta(z)\prec\phi(z).$$

The above subordination is the well-known *Briot–Bouquet* type differential subordination and it allows us to find a simple sufficient condition such that the inclusion $S^{\lambda}_{p,q,\alpha;l,s}[a_1+1;\phi]\subset S^{\lambda}_{p,q,\alpha;l,s}[a_1,\phi]$ holds.

Theorem 7. *Suppose that the function ϕ is convex (univalent) in $\mathbb{D}$ and satisfies the inequality*

$$\operatorname{Re}\left(e^{-i\lambda}\phi(z)\right)>\cos\lambda-\frac{\operatorname{Re}a_1}{(p-q-\alpha)\cos\lambda},\ z\in\mathbb{D}. \tag{25}$$

If $f\in S^{\lambda}_{p,q,\alpha;l,s}[a_1+1;\phi]$, such that $\dfrac{\left(H_{p,l,s}(a_1)f(z)\right)^{(q)}}{z^{p-q}}\neq 0$ for all $z\in\mathbb{D}$, then $f\in S^{\lambda}_{p,q,\alpha;l,s}[a_1,\phi]$.

Proof. Suppose that $f\in S^{\lambda}_{p,q,\alpha;l,s}[a_1+1;\phi]$ and let us define the function θ by

$$\theta(z)=\frac{1}{(p-q-\alpha)\cos\lambda}\left[\frac{e^{i\lambda}z\left(H_{p,l,s}(a_1)f(z)\right)^{(1+q)}}{\left(H_{p,l,s}(a_1)f(z)\right)^{(q)}}-\alpha\cos\lambda-i(p-q)\sin\lambda\right],\ z\in\mathbb{D}. \tag{26}$$

Then, θ is analytic in $\mathbb{D}$ and from (3) we obtain

$$z\left(H_{p,l,s}(a_1)f(z)\right)^{(1+q)}=a_1\left(H_{p,l,s}(a_1+1)f(z)\right)^{(q)}-(a_1-p+q)\left(H_{p,l,s}(a_1)f(z)\right)^{(q)},\ z\in\mathbb{D}.$$

Therefore, (26) can be written as

$$\theta(z)\cos\lambda+\frac{a_1-p+q}{p-q-\alpha}e^{i\lambda}+\frac{\alpha\cos\lambda}{p-q-\alpha}+\frac{i(p-q)\sin\lambda}{p-q-\alpha}$$
$$=\frac{e^{i\lambda}a_1\left(H_{p,l,s}(a_1+1)f(z)\right)^{(q)}}{(p-q-\alpha)\left(H_{p,l,s}(a_1)f(z)\right)^{(q)}},\ z\in\mathbb{D},$$

and differentiating logarithmically the above identity, from $f\in S^{\lambda}_{p,q,\alpha;l,s}[a_1+1;\phi]$ and by using (26) we obtain

$$\theta(z)+\frac{z\theta'(z)}{[(p-q-\alpha)\theta(z)\cos\lambda+\alpha\cos\lambda+i(p-q)\sin\lambda]e^{-i\lambda}+(a_1-p+q)}\prec\phi(z). \quad (27)$$

Since

$$\operatorname{Re}\left\{[(p-q-\alpha)\phi(z)\cos\lambda+\alpha\cos\lambda+i(p-q)\sin\lambda]e^{-i\lambda}+(a_1-p+q)\right\}>0,\ z\in\mathbb{D},$$

is equivalent to (25), according to Lemma 2 the subordination (27) implies $\theta(z)\prec\phi(z)$ which proves that $f\in S^{\lambda}_{p,q,\lambda;l,s}[a_1;\phi]$. $\square$

Example 4. *1. For $\lambda=0$ the above theorem reduces to the next result:*
Suppose that the function ϕ is convex (univalent) in $\mathbb{D}$ and satisfies the inequality

$$\operatorname{Re}\phi(z)>1-\frac{\operatorname{Re}a_1}{p-q-\alpha},\ z\in\mathbb{D}.$$

If $f\in S^{0}_{p,q,\alpha;l,s}[a_1+1;\phi]$ such that $\dfrac{\left(H_{p,l,s}(a_1)f(z)\right)^{(q)}}{z^{p-q}}\neq 0$ for all $z\in\mathbb{D}$, then $f\in S^{0}_{p,q,\alpha;l,s}[a_1,\phi]$.
2. For

$$\phi(z)=\frac{1+Az}{1+Bz},1\le B<A\le 1,$$

and $q:=p-1$, the above special case becomes:
Suppose that $-1\le B<A\le 1$ and satisfies the inequality

$$\frac{1-A}{1-B}>1-\frac{\operatorname{Re}a_1}{1-\alpha},\ z\in\mathbb{D}.$$

If $f\in S^{0}_{p,p-1,\alpha;l,s}[a_1+1;\phi]$ such that $\dfrac{\left(H_{p,l,s}(a_1)f(z)\right)^{(p-1)}}{z}\neq 0$ for all $z\in\mathbb{D}$, then $f\in S^{0}_{p,p-1,\alpha;l,s}[a_1,\phi]$.

Remark 6. *For special choices for a_i $i=1,\dots,l$ and b_j $j=1,\dots,s$, where $l,s\in\mathbb{N}_0$, we can obtain the corresponding results for different linear operators which are defined in the introduction.*

3. Concluding Remarks

Using higher order derivatives we defined the classes $S^{\lambda}_{p,q,\alpha}(\phi)$ and $S^{\lambda,\gamma}_{p,q,\alpha}(\phi)$ that for special choices of ϕ generalize some classes previously studied by Aouf [14], Libera [17], Srivastava et al. [16], and Aouf [15,18]. In the first two theorems we determine, in term of convolution product, necessary and sufficient conditions for a function to belong to these classes, respectively. For special choices of the parameters these results extend those of Sarkar et al. [2] and of Ahuja [3].

For the other new subclasses $C^{\lambda}_{p,q,\alpha}(\phi)$ and $C^{\lambda,\gamma}_{p,q,\alpha}(\phi)$, the Theorems 3 and 4 give us necessary and sufficient condition for a function to be in these classes, respectively, extending for particular cases of the parameters and of the function ϕ some results of Padmanabhan and Ganesan [4].

The next two theorems deal with necessary and sufficient conditions for a multivalent function to be in the new defined classes $S^{\lambda,\gamma}_{p,q,\alpha;l,s}[a_1;\phi]$ and $C^{\lambda,\gamma}_{p,q,\alpha;l,s}[a_1;\phi]$ introduced by using higher order derivatives and the well-known Dziok-Srivastava linear operator, that generalize the previous ones, while the last result gives us a sufficient condition such the inclusion $S^{\lambda}_{p,q,\alpha;l,s}[a_1+1;\phi] \subset S^{\lambda}_{p,q,\alpha;l,s}[a_1,\phi]$ holds.

The results we obtain are new and could help the researchers in the field of *Geometric Function Theory* to obtain other new results in this field, or for used them in some appropriate particular cases for different studies.

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