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Families of periodic orbits around asteroids: From shape symmetry to asymmetry

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Abstract. In Karydis *et al.* (2021) we have introduced the method of shape continuation in order to obtain periodic orbits in the complex gravitational field of an irregularly-shaped asteroid starting from a symmetric simple model. What's more, we map the families of periodic orbits of the simple model to families of the real asteroid model. The introduction of asymmetries in a gravitational potential may significantly affect the dynamical properties of the families. In this paper, we discuss the effect of the asymmetries in the neighborhood of vertically critical orbits, where, in the symmetric model, bifurcations of 3D periodic orbit families occur. When asymmetries are introduced, we demonstrate that two possible continuation schemes can take place in general. Numerical simulations, using an ellipsoid and a mascon model of 433-Eros, verify the existence of these schemes.

Keywords. Asteroids, Orbital mechanics, Periodic orbits

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1. Introduction

Many space missions to small NEA have taken place recently or are planned in the coming years. Close proximity operations around such small bodies, which have irregular shape in general, demand sufficient knowledge of their gravitational field and their dynamics. In orbital mechanics, periodic orbits play an important role in understanding the dynamics and have been studied widely in celestial mechanics and especially in the three body problem. In addition, they can find direct applications in astrodynamics as parking orbits for a spacecraft or the unstable ones may be used for computing landing or escape paths (Scheeres (2012)). In such complex gravitational fields, which can be sufficiently modeled e.g. by polyhedrals or mascons (see Scheeres (2012)), the computation of periodic orbits is a challenge. The grid search method introduced by Yu & Baoyin (2012) has been proved very efficient and applied for various asteroids (e.g. Jiang *et al.* (2018)).

In Karydis *et al.* (2021), which will be referred in the following as 'Paper I', we approach the potential of an irregular body by starting from the symmetric potential of a simplified model (an ellipsoid), where the families of periodic orbits can be easily computed and show particular structures and types. Then, asymmetric terms are gradually introduced in the potential and periodic orbits are continued along this procedure, which is called *shape continuation* and ends when the 'real' potential of the target asteroid is adequately approximated. In this way, we assign families of the simplified model to families of the 'real' model and we can study the effect of the symmetric perturbations in the characteristic curves of the families and their stability. In the present study, we use a theoretical analysis and numerical simulations in order to show how families are affected by asymmetric forces when they are close to vertically critical orbits, where planar and 3D orbit families intersect in the symmetric model.

2. Description of the orbital mechanics

We consider the motion of a mass-less body in the gravitational field of an irregularly shaped asteroid which rotates with angular velocity ω. If the center of mass of the asteroid is considered as the origin point of a reference frame which rotates with the asteroid (i.e. it is a body-fixed frame), and $\mathbf{r}=(x,y,z)$ is the position vector of the mass-less body, its motion is described by the Hamiltonian

$$H(\mathbf{r},\mathbf{p})=\frac{1}{2}\mathbf{p}^2-\mathbf{p}(\omega\times\mathbf{r})+U(\mathbf{r}), \tag{2.1}$$

where the generalized momenta are given by $\mathbf{p}=\dot{\mathbf{r}}+\omega\times\mathbf{r}$ and U is the gravitational potential of the asteroid. If ω is constant, which is the case considered in this study, then H is also constant (H being the Jacobi integral or, simply, the energy).

Let $\mathbf{X}=(x,y,z,\dot{x},\dot{y},\dot{z})$ denote a phase space point and $\mathbf{X}=\mathbf{X}(t;\mathbf{X}_0)$ a trajectory with initial conditions $\mathbf{X}_0$. The system is autonomous and the condition $\mathbf{X}(T;\mathbf{X}_0)=\mathbf{X}_0$ implies a periodic orbit of period T. Supposing that the orbit intersects a Poincaré section, say $x=0$ with $\dot{x}>0$ and energy h, the orbits can be defined explicitly by a point in the $4D$ space of section, called Π_4, which is defined by vector $\mathbf{Y}=(y,z,\dot{y},\dot{z})$. Thus, the periodicity conditions are reduced to

$$\mathbf{Y}(t^*;X_0)=\mathbf{Y}_0, \tag{2.2}$$

where t^* is the time of the mth intersection of the orbit, with a section that satisfies (2.2) for the first time. In this case, t^* and period T coincide and m denotes the multiplicity of the section.

In general, in space Π_4, periodic orbits are isolated and analytically continued with respect to h, forming mono-parametric families (Meyer *et al.* (2009), Scheeres (2012)). In computations, we may consider a continuation with respect to any variable but it is more convenient to continue the families by using an extrapolation procedure and considering as parameter the length s of the characteristic curve of the family in Π_4 (see Paper I). In this way, the numerical continuation is still successful at energy extrema that may exist along the family.

Let ξ denote a variation vector that satisfies the system of linear variational equations of system (2.1), namely

$$\dot{\xi}=\mathbf{A}(t)\xi \quad\Rightarrow\quad \xi=\mathbf{\Phi}(t)\xi(0). \tag{2.3}$$

Matrix $\mathbf{A}$ is computed along a periodic orbit and thus, it is also periodic. $\Phi(t)$ is the fundamental matrix of solutions and the constant matrix $\mathbf{M}=\mathbf{\Phi}(T)$ is the monodromy matrix, which is symplectic. Therefore, two eigenvalues are equal to unit and the rest four form reciprocal pairs. If we remove the rows and columns that correspond to the variables which define the Poincaré section (e.g. x and $\dot{x}$) from $\mathbf{M}$, then we obtain the reduced monodromy matrix $\mathbf{M}'$ of size 4×4 and the unit eigevalues are removed. The periodic orbit is stable if the two reciprocal pairs of eigenvalues of $\mathbf{M}'$ lie on the complex unit circle. In computations, we use the Broucke's stability indicies b_1 and b_2, which are computed from the elements of $\mathbf{M}'$ and their stability implies that they are real and $|b_i|<2$ (Broucke (1969)).

When $\mathbf{M}'$ is computed for a planar orbit, then it is decomposed in two 2×2 submatrices, $\mathbf{M}_h$ and $\mathbf{M}_v$ that refer to horizontal stability (index b_1) and vertical stability (index b_2), respectively. If $b_2=2$ then, the planar orbit is called *vertically critical orbit*

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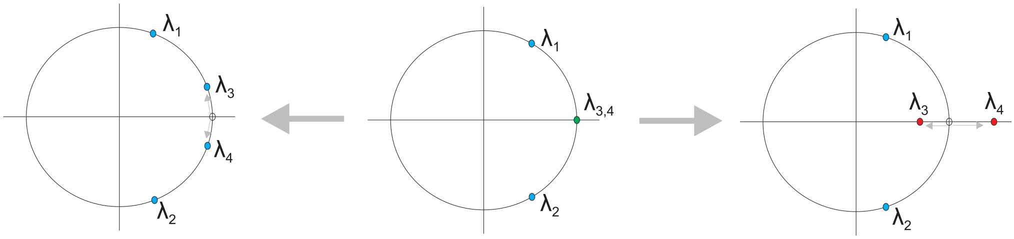

Figure 1. Distribution of eigenvalues for a v.c.o. of the symmetric model (center) and their displacement after introduction of asymmetry (*scheme I* in the left panel and *scheme II* in the right panel).

(v.c.o.) and signifies a bifurcation for another family of 3D periodic orbits (Hénon (1973)). We note that b_2 may also take the value of two when the planar orbit needs m times to complete a period (multiplicity). Then, if T is the period of the v.c.o., the 3D bifurcating orbit close to the v.c.o. will be of period mT.

3. Continuation near a v.c.o. : from a symmetric to an asymmetric model

Suppose that U_{ast} is a potential model of the asteroid provided by a 'real' model (e.g. by mascons or a polyhedral model). Let us define a mono-parametric set of potentials

$$U(\varepsilon) = U_0 + \varepsilon U_1, \quad 0 \leqslant \varepsilon \leqslant \varepsilon_0, \tag{3.1}$$

where U_0 is the symmetric potential of the ellipsoid that approximates the potential of the asteroid and U_1 includes the asymmetric part of the potential such that $U(\varepsilon_0) = U_{ast}$ with e_0 being sufficiently small.

Let F_p be a symmetric planar family of periodic orbits with a potential of U_0 and O a v.c.o. of F_p. We suppose that in the neighborhood of O the planar orbits of F_p are of the same horizontal stability type. In the present study, we consider that they are stable so, the eigenvalues of M_h are of the form λ_1, $\lambda_2 = e^{\pm i\phi}$, $\phi \in (\delta, \pi - \delta)$, $\delta > 0$. The eigenvalues of M_v are critical for O, i.e. $\lambda_{3,4} = 1$ when the appropriate multiplicity m is taken into account. The distribution of λ_i on the unit circle is shown in the middle panel of Fig. 1. Suppose we perform an analytic continuation of the v.c.o. O with respect to parameter ε. As ε increases smoothly towards value ε_0, the eigenvalues $\lambda_{1,2}$ should move smoothly on the unit circle due to the analyticity (see Meyer *et al.* (2009)) and if δ is sufficiently large, the eigenvalues do not reach the critical values ± 1 as $\varepsilon \to \varepsilon_0$. On the other hand, the critical eigenvalues $\lambda_{3,4}$, as ε varies, may move either on the unit circle or on the real axis. These cases are called *scheme I* and *scheme II*, respectively, and are presented in Fig. 1. Which one of the two schemes will take place, depends on the term U_1, which represents the asymmetric part of the asteroid's potential.

Applying analytic continuation to all orbits of F_p in the neighborhood of O,with respect to ε, we obtain the set of families $F(\varepsilon)$, with $F(0) = F_p$. All orbits of $F(\varepsilon)$ with $\varepsilon \neq 0$ are spatial and asymmetric and family $F_{ast} = F(\varepsilon_0)$ is the family of orbits of the real asteroid originating from the planar family of the ellipsoid. The initial orbit $O \in F_p$ is mapped to the orbit $O' \in F_{ast}$. When *scheme I* takes place, F_{ast} should consist of stable orbits at least near O'. Instead, in *scheme II* the orbit O' is unstable and there should exist a continuous segment on F_{ast} near O' consisting of unstable orbits.

Let us consider the family, $F_{3D}(0)$, of three dimensional orbits that bifurcates from O. Similarly to the planar family, analytic continuation with respect to ε can be also applied providing the set of families $F_{3D}(\varepsilon)$. All orbits should be asymmetric for $\varepsilon \neq 0$ and family $F_{3D}(\varepsilon_0)$ is the asteroid's family of periodic orbits associated to the family $F_{3D}(0)$ of the

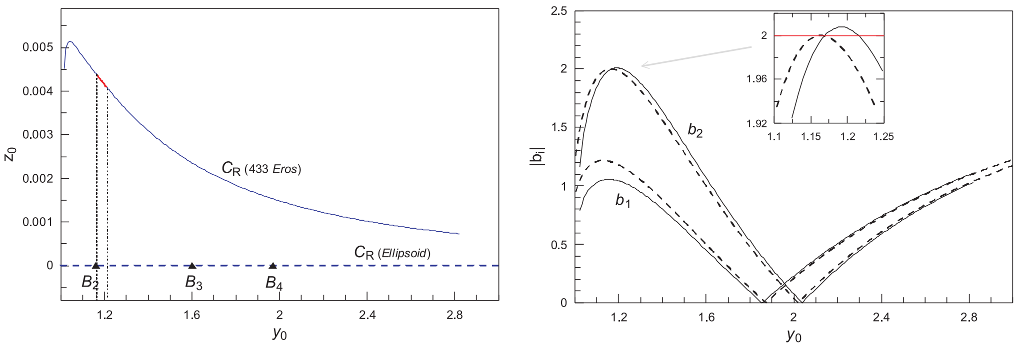

Figure 2. (left) The characteristic curve of the circular family C_R for the ellipsoid (dashed curve) and 433-Eros (solid curve) projected on the plane $y_0 - z_0$. The points B_m indicate the y_0-position of the v.c.o. with the subscript m being the multiplicity. The red segment indicates the part of the family with unstable orbits. (right) The variation of the stability indices b_1 and b_2 along the C_R-family of ellipsoid and Eros.

symmetric ellipsoid model. When *scheme I* takes place, the families F and F_{3D}, which for $\varepsilon = 0$ intersect at O, should be detached for $\varepsilon > 0$ since no bifurcation point exists on $F(\varepsilon)$ (whole family near O is stable). However, in *scheme II* the edges of the unstable segment formed on $F(\varepsilon_0)$ may be bifurcation points for the family $F_{3D}(\varepsilon_0)$. The above assumptions are verified by the numerical computations presented in the next section.

4. Numerical computations : The asteroid 433-Eros

In Paper I, we used the symmetric ellipsoid model (with normalized maximum semi-axis, $a = 1$, and angular velocity, $\omega = 1$) to initially approximate the potential of asteroid 433-Eros. Then, we applied shape-continuation to identify families of periodic orbits for the 'real' gravitational potential of 433-Eros, implemented with a sufficient number of mascons (Soldini *et al.* (2020)). In the ellipsoid model, we consider the family of planar ($z = 0$) circular retrograde orbits, C_R, which is fully stable. The family is also vertically stable but there are v.c.o. for higher period multiplicities ($m = 2, 3, 4, ..$). Their y_0-position (where y_0 is the approximate radius of the orbit) is shown in the left panel of Fig. 2. The right panel shows the stability indicies b_i along the family (dashed curves). The C_R is continued when asymmetric terms are added in the potential in order to simulate the potential of the asteroid. The computed family for 433-Eros consists of orbits which are no longer planar and symmetric but are almost circular. The family is presented in Fig. 2 with solid curves. The major part of C_R of Eros consists of stable orbits and this is also the case close to the radius of the v.c.o. B_3 and B_4. Therefore, such a situation implies *scheme I* for the 3D orbits emanating in the symmetric model from these v.c.o.. However, it is evident that the introduced asymmetries caused an unstable segment close to B_2 and this implies *scheme II*. It should be noted that this instability has been also mentioned in Ni *et al.* (2016) who used a polyhedral model for 433-Eros.

Scheme I is shown by considering the 3D family L_{24} of the ellipsoid, which bifurcates from the v.c.o. B_4. The family near B_4 is stable but becomes unstable when it becomes significantly inclined as shown in the left panel of Fig. 3. For the asymmetric potential of 433-Eros the family is represented by the characteristic curve in the right panel of Fig. 3. We can see that in the asymmetric asteroid case, family L_{24} does not intersect the planar family C_R and the two families are now separated. The stability type of orbits is not affected by the asymmetry for orbits close to the plane $z = 0$. However, a break of family L_{24} arises because of the irregular shape of Eros. After this break, the family continues with the unstable segment L'_{24}. Such family breaks are discussed also in Paper

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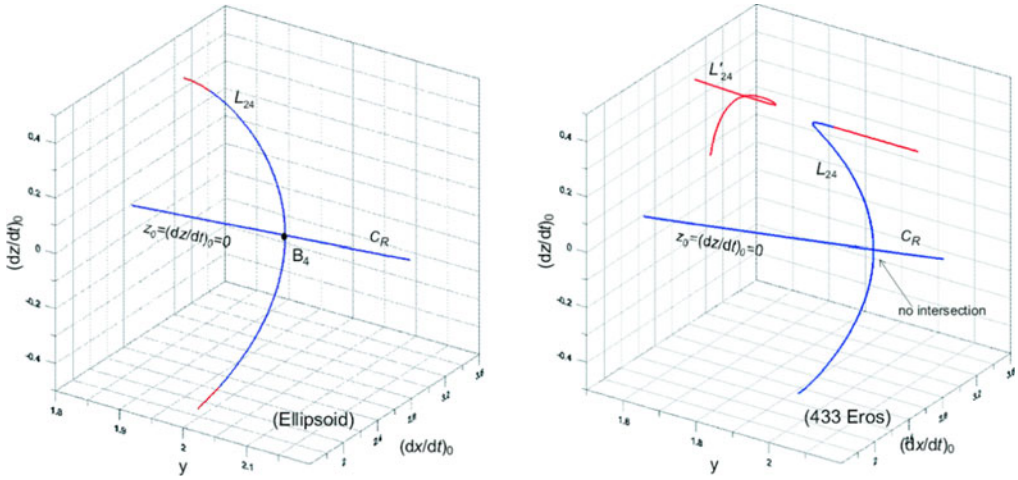

Figure 3. (left) The characteristic curves of the planar family C_R and the 3D family L_{24} of the ellipsoid. Blue (red) color indicates stability (instability). B_4 is the v.c.o. where the two families intersect. (right) The characteristic curves for the corresponding families of 433-Eros potential. The transition from the ellipsoid (left) to the mascon model of 433-Eros (right) indicates that *scheme I* takes place.

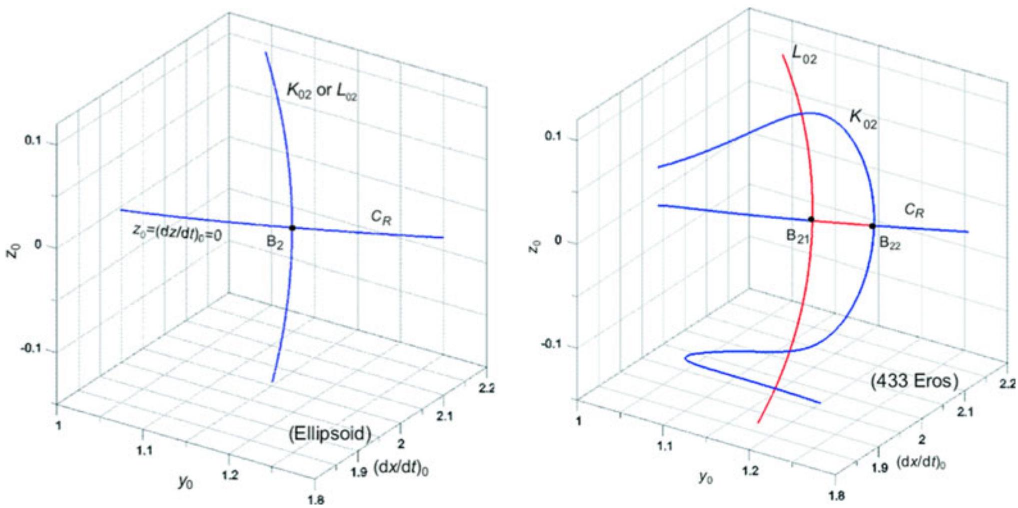

Figure 4. (left) The characteristic curves of the planar family C_R and the 3D family L_{02} (and its equivalent K_{02}) of the ellipsoid. Blue (red) color indicates stability (instability). B_2 is the vco where the two families intersect. (right) The characteristic curves for the corresponding families of 433-Eros potential. L_{02} and K_{02} are families of different orbits. The transition from the ellipsoid (left) to the mascon model of 433-Eros (right) indicates that *scheme II* takes place.

I. In the same paper, where family L_{13} is studied, *scheme I* also holds true, with a change of stability at $z \approx 0$.

Scheme II holds true for the case of v.c.o. B_2 of the ellipsoid from which the 3D families L_{02} and K_{02} originate (see Paper I). The two families are equivalent because they consist of the same doubly symmetric periodic orbits but their characteristic curves are presented in different spaces of initial conditions. In the left panel of Fig. 4, we present the initial conditions of the orbits in K_{02} family. As we have already mentioned, the C_R family of 433-Eros shows an unstable segment at B_2, defined by the points B_{21} and B_{22}. These

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points should be bifurcation points of other families. By computing the families K_{02} and L_{02} in the asymmetric potential of 433-Eros (see right panel of Fig. 4) we obtain that i) the two families are separated and they now consist of different asymmetric periodic orbits ii) the families pass from the points B_{21} and B_{22} and, therefore, the continuation *scheme II* is valid here. K_{02} consists of unstable orbits and L_{02} of stable ones (at least in the neighborhood of the bifurcation points). However, we cannot claim that the appearance of a stable and an unstable family is a general property for *scheme II*.

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