

**POCHHAMMER SYMBOL WITH NEGATIVE INDICES.
A NEW RULE FOR THE METHOD OF BRACKETS.**

IVAN GONZALEZ, LIN JIU, AND VICTOR H. MOLL

ABSTRACT. The method of brackets is a method of integration based upon a small number of heuristic rules. Some of these have been made rigorous. An example of an integral involving the Bessel function is used to motivate a new evaluation rule.

1. INTRODUCTION

The evaluation of definite integrals is connected, in a surprising manner, to many topics in Mathematics. The last author has described in [11] and [13] some of these connections. Many of these evaluations appear in the classical table [9] and proofs of these entries have appeared in a series of papers starting with [12] and the latest one is [1]. An interesting new method of integration, developed in [6] in the context of integrals coming from Feynman diagrams, is illustrated here in the evaluation of entry 6.671.7 in [9]:

$$(1.1) \quad I := \int_0^\infty J_0(ax) \sin(bx) dx = \begin{cases} 0, & \text{if } 0 < b < a, \\ 1/\sqrt{b^2 - a^2}, & \text{if } 0 < a < b. \end{cases}$$

This so-called *method of brackets* is based upon a small number of heuristic rules. These are described in the next section. Sections 3 and 4 present the evaluation of (1.1). The conflict between these two evaluations is resolved in Section 5 with the proposal of a new rule dealing with the extension of the Pochhammer symbol to negative integer indices.

2. THE METHOD OF BRACKETS

A method to evaluate integrals over the half line $[0, \infty)$, based on a small number of rules has been developed in [6, 7, 8]. This method of brackets is described next. The heuristic rules are currently being made rigorous in [2] and [10]. The reader will find in [3, 4, 5] a large collection of evaluations of definite integrals that illustrate the power and flexibility of this method.

For $a \in \mathbb{C}$, the symbol

$$(2.1) \quad \langle a \rangle \mapsto \int_0^\infty x^{a-1} dx$$

is the *bracket* associated to the (divergent) integral on the right. The symbol

$$(2.2) \quad \phi_n := \frac{(-1)^n}{\Gamma(n+1)}$$

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is called the *indicator* associated to the index n . The notation $\phi_{i_1 i_2 \dots i_r}$, or simply $\phi_{12 \dots r}$, denotes the product $\phi_{i_1} \phi_{i_2} \dots \phi_{i_r}$.

Rules for the production of bracket series

Rule P₁. If the function f is given by the formal power series

$$(2.3) \quad f(x) = \sum_{n=0}^{\infty} a_n x^{\alpha n + \beta - 1},$$

then the improper integral of f over the positive real line is formally written as the *bracket series*

$$(2.4) \quad \int_0^{\infty} f(x) dx = \sum_n a_n \langle \alpha n + \beta \rangle.$$

Rule P₂. For $\alpha \in \mathbb{C}$, the multinomial power $(a_1 + a_2 + \dots + a_r)^\alpha$ is assigned the r -dimension bracket series

$$(2.5) \quad \sum_{n_1} \sum_{n_2} \dots \sum_{n_r} \phi_{n_1 n_2 \dots n_r} a_1^{n_1} \dots a_r^{n_r} \frac{\langle -\alpha + n_1 + \dots + n_r \rangle}{\Gamma(-\alpha)}.$$

Rules for the evaluation of a bracket series

Rule E₁. The one-dimensional bracket series is assigned the value

$$(2.6) \quad \sum_n \phi_n f(n) \langle an + b \rangle = \frac{1}{|a|} f(n^*) \Gamma(-n^*),$$

where n^* is obtained from the vanishing of the bracket; that is, n^* solves $an + b = 0$.

The next rule provides a value for multi-dimensional bracket series where the number of sums is equal to the number of brackets.

Rule E₂. Assume the matrix $A = (a_{ij})$ is non-singular, then the assignment is

$$\begin{aligned} & \sum_{n_1} \dots \sum_{n_r} \phi_{n_1 \dots n_r} f(n_1, \dots, n_r) \langle a_{11}n_1 + \dots + a_{1r}n_r + c_1 \rangle \dots \langle a_{r1}n_1 + \dots + a_{rr}n_r + c_r \rangle \\ &= \frac{1}{|\det(A)|} f(n_1^*, \dots, n_r^*) \Gamma(-n_1^*) \dots \Gamma(-n_r^*) \end{aligned}$$

where $\{n_i^*\}$ is the (unique) solution of the linear system obtained from the vanishing of the brackets. There is no assignment if A is singular.

Rule E₃. Each representation of an integral by a bracket series has associated an *index of the representation* via

$$(2.7) \quad \text{index} = \text{number of sums} - \text{number of brackets}.$$

It is important to observe that the index is attached to a specific representation of the integral and not just to integral itself. The experience obtained by the authors using this method suggests that, among all representations of an integral as a bracket series, the one with *minimal index* should be chosen.

The value of a multi-dimensional bracket series of positive index is obtained by computing all the contributions of maximal rank by Rule E₂. These contributions to the integral appear as series in the free parameters. Series converging in a common region are added and divergent series are discarded.

Note 2.1. A systematic procedure in the simplification of the series obtained by this procedure has been used throughout the literature: express factorials in terms of the gamma function and the transform quotients of gamma terms into Pochhammer symbol, defined by

$$(2.8) \quad (a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}.$$

Any presence of a Pochhammer with a negative index k is transformed by the rule

$$(2.9) \quad (a)_{-k} = \frac{(-1)^k}{(1-a)_k}.$$

The example discussed in the next two section provides motivation for an additional evaluation rule for the method of brackets.

3. A FIRST EVALUATION OF ENTRY 6.671.7 IN GRADSHTEYN AND RYZHIK

The evaluation of (1.1) uses the series

$$(3.1) \quad J_0(ax) = \sum_{m=0}^{\infty} \phi_m \frac{a^{2m}}{\Gamma(m+1)2^{2m}} x^{2m}$$

and

$$(3.2) \quad \sin(bx) = \sum_{n=0}^{\infty} \phi_n \frac{\Gamma(n+1)}{\Gamma(2n+2)} b^{2n+1} x^{2n+1}.$$

Therefore the integral in (1.1) is given by

$$(3.3) \quad I = \sum_{m,n} \phi_{m,n} \frac{a^{2m} b^{2n+1} \Gamma(n+1)}{2^{2m} \Gamma(m+1) \Gamma(2n+2)} (2m+2n+2).$$

The duplication formula for the gamma function transforms this expression to

$$(3.4) \quad I = \frac{\sqrt{\pi}}{2} \sum_{m,n} \phi_{m,n} \frac{a^{2m} b^{2n+1}}{2^{2m+2n} \Gamma(m+1) \Gamma(n+3/2)} (2m+2n+2).$$

Eliminating the parameter n using Rule E_1 gives $n^* = -m - 1$ and produces

$$\begin{aligned} I &= \frac{\sqrt{\pi}}{b} \sum_{m=0}^{\infty} \phi_m \frac{1}{\Gamma(-m+\frac{1}{2})} \left(\frac{a}{b}\right)^{2m} \\ &= \frac{1}{b} \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{a^2}{b^2}\right)^m}{m! \left(\frac{1}{2}\right)_{-m}} \\ &= \frac{1}{b} \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)_m \frac{1}{m!} \left(\frac{a^2}{b^2}\right)^m \\ &= \frac{1}{\sqrt{b^2 - a^2}}. \end{aligned}$$

The condition $|b| > |a|$ is imposed to guarantee the convergence of the series on the third line of the previous argument.

The series obtained by eliminating the parameter m by $m^* = -n - 1$ vanishes because of the factor $\Gamma(m+1)$ in the denominator. The formula has been established.

4. AN ALTERNATIVE EVALUATION

A second evaluation of (1.1) begins with

$$(4.1) \quad \int_0^\infty J_0(ax) \sin(bx) dx = \sum_{m,n} \phi_{m,n} \frac{a^{2m} b^{2n+1}}{2^{2m} m! (2n+1)!} \langle 2m+2n+2 \rangle.$$

The evaluation of the bracket series is described next.

Case 1. Choose n as the free parameter. Then $m^* = -n - 1$ and the contribution to the integral is

$$(4.2) \quad I_1 = \frac{1}{2} \sum_{n=0}^\infty \phi_n \left(\frac{a}{2}\right)^{-2n-2} b^{2n+1} \frac{1}{\Gamma(-n)} \frac{n!}{(2n+1)!} \Gamma(n+1).$$

Each term in the sum vanishes because the gamma function has a pole at the negative integers.

Case 2. Choose m as a free parameter. Then $2m+2n+2=0$ gives $n^* = -m-1$. The contribution to the integral is

$$(4.3) \quad I_2 = \frac{1}{2} \sum_m \phi_m \left(\frac{a}{2}\right)^{2m} b^{-2m-1} \frac{1}{m!} \cdot \frac{n!}{(2n+1)!} \Big|_{n=-m-1} \Gamma(m+1).$$

Now write

$$(4.4) \quad \frac{n!}{(2n+1)!} = \frac{1}{(n+1)_{n+1}}$$

and (4.3) becomes

$$(4.5) \quad I_2 = \frac{1}{2b} \sum_{m=0}^\infty \frac{(-1)^m}{m!} \left(\frac{a}{2b}\right)^{2m} \frac{1}{(-m)_{-m}}.$$

Transforming the term $(-m)_{-m}$ by using

$$(4.6) \quad (x)_{-n} = \frac{(-1)^n}{(1-x)_n}$$

gives

$$(4.7) \quad (-m)_{-m} = \frac{(-1)^m}{(1+m)_m}.$$

Replacing in (4.5) produces

$$\begin{aligned} I_2 &= \frac{1}{2b} \sum_{k=0}^\infty \left(\frac{1}{2}\right)_k \frac{1}{k!} \left(\frac{a^2}{b^2}\right)^k \\ &= \frac{1}{2} \frac{1}{\sqrt{b^2 - a^2}}. \end{aligned}$$

The method of brackets produces *half of the expected answer*. Naturally, it is possible that the entry in [9] is erroneous (this happens once in a while). Some numerical computations and the evaluation in the previous section, should convince the reader that this is not the case. The source of the error is the use of (4.6) for the evaluation of the term $(-m)_{-m}$. A discussion is presented in Section 5.

5. EXTENSIONS OF THE POCHHAMMER SYMBOL

Rule E_1 of the method of brackets requires the evaluation of $f(n^*)$. In many instances, this involves the evaluation of the Pochhammer symbol $(x)_m$ for $m \notin \mathbb{N}$. In particular, the question of the value

$$(5.1) \quad (-m)_{-m} \quad \text{for } m \in \mathbb{N}$$

is at the core of the missing factor of 2 in Section 4.

The first extension of $(x)_m$ to negative values of n comes from the identity

$$(5.2) \quad (x)_{-m} = \frac{(-1)^m}{(1-x)_m}.$$

This is obtained from

$$(5.3) \quad (x)_{-m} = \frac{\Gamma(x-m)}{\Gamma(x)} = \frac{\Gamma(x-m)}{(x-1)(x-2)\cdots(x-m)\Gamma(x-m)}$$

and then changing the signs of each of the factors. This is valid as long as x is not a negative integer. The limiting value of the right-hand side in (5.2) as $x \rightarrow -km$, with $k \in \mathbb{N}$, is

$$(5.4) \quad (-km)_{-m} = \frac{(-1)^m (km)!}{((k+1)m)!}.$$

On the other hand, the limiting value of the left-hand side is

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} (-k(m+\varepsilon))_{-(m+\varepsilon)} &= \lim_{\varepsilon \rightarrow 0} \frac{\Gamma(-(k+1)m - (k+1)\varepsilon)}{\Gamma(-km - k\varepsilon)} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{\Gamma(-(k+1)\varepsilon) (-(k+1)\varepsilon)_{-(k+1)m}}{\Gamma(-k\varepsilon) (-k\varepsilon)_{-km}} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{\Gamma(-(k+1)\varepsilon)}{\Gamma(-k\varepsilon)} \frac{(-1)^{(k+1)m}}{(1+(k+1)\varepsilon)_{(k+1)m}} \cdot \frac{(1+k\varepsilon)_{km}}{(-1)^{km}} \\ &= \frac{(-1)^m (km)!}{((k+1)m)!} \cdot \frac{k}{k+1}. \end{aligned}$$

Therefore the function $(x)_{-m}$ is discontinuous at $x = -km$, with

$$(5.5) \quad \frac{\text{Direct } (-km)_{-m}}{\text{Limiting } (-km)_{-m}} = \frac{k+1}{k}.$$

For $k = 1$, this ratio becomes 2. This explains the missing $\frac{1}{2}$ in the calculation in Section 4. Therefore it is the discontinuity of (2.9) at negative integer values of the variables, what is responsible for the error in the evaluation of the integral (1.1).

This example suggest that the rules of the method of brackets should be supplemented with an additional one:

Rule E_4 . Let $k \in \mathbb{N}$ be fixed. In the evaluation of series, the rule

$$(5.6) \quad (-km)_{-m} = \frac{k}{k+1} \frac{(-1)^m (km)!}{((k+1)m)!}$$

must be used to eliminate Pochhammer symbols with negative index and negative integer base.

A variety of other examples confirm that this heuristic rule leads to correct evaluations.

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INSTITUTO DE FISICA Y ASTRONOMIA, UNIVERSIDAD DE VALPARAISO, CHILE
E-mail address: `ivan.gonzalez@uv.cl`

DEPARTMENT OF MATHEMATICS, TULANE UNIVERSITY, NEW ORLEANS, LA 70118
E-mail address: `ljiu@tulane.edu`

DEPARTMENT OF MATHEMATICS, TULANE UNIVERSITY, NEW ORLEANS, LA 70118
E-mail address: `vhm@tulane.edu`