The plastic number and its generalised polynomial

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Abstract

We study the zeroes of the polynomial $X^n - \sum_{j=0}^{n-2} X^j$ and prove that its unique positive root converges to the golden ratio $\phi = \frac{1+\sqrt{5}}{2}$.

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1 Introduction

The recurrence $F_n = F_{n-1} + F_{n-2}$, with initial values $F_0 = 0$ and $F_1 = 1$ yields the celebrated Fibonacci numbers. It is well known that

$$F_n = \frac{\phi^n - (1-\phi)^n}{\sqrt{5}}$$

where $\phi = \frac{1+\sqrt{5}}{2}$ is the positive root of the characteristic polynomial $X^2 - X - 1$, known as *golden ratio*.

One can readily generalise the recurrence and define the $k \geq 2$ order Fibonacci sequence $F_n = F_{n-1} + \ldots + F_{n-k}$, with initial conditions $F_0 = \ldots = F_{k-2} = 0$ and $F_{k-1} = 1$. The characteristic polynomial of this recurrence is $X^k - X^{k-1} - \ldots - X - 1$. Its zeroes are much studied in literature: we refer to [3], [4], [5], [8] and [9], where it is proved that the unique positive root tends to 2, as $k \to \infty$. Series representations for this root are derived in [1] by Lagrange inversion theorem.

In this note, we turn our attention to the positive zero of the polynomial $X^3 - X - 1$, known as *plastic number*, which will throughout be denoted by ρ and is equal to $\sqrt[3]{1 + \sqrt[3]{1 + \sqrt[3]{1 \ldots}}}$ [6]. The plastic number was introduced by van der Laan [2]. The recurrence relation is $a_n = a_{n-2} + a_{n-3}$, with initial conditions $a_0 = a_1 = a_2 = 1$ and defines the integer sequence, known as Padovan sequence [7]. Although the bibliography regarding the analysis of Fibonacci numbers is quite extensive, it seems not to be this case regarding the plastic number. In the next section, we will examine a generalisation of the Padovan sequence and its associated characteristic polynomial.

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2 The Generalised sequence

Consider the recurrence

$$a_n = \sum_{l=2}^{k} a_{n-l}$$

for $k \geq 3$ and initial conditions $a_0 = \ldots = a_{k-1} = 1$. For $k = 3$, we obtain as a special case the Padovan sequence. A lemma follows regarding the roots of its characteristic polynomial.

Lemma 2.1. *The polynomial $\mathcal{F}_k(X) = X^k - X^{k-2} - \ldots - X - 1$ has k simple roots. Its real roots are the positive λ_k; λ_k and -1 when k is even, along with the $2\lfloor \frac{k-1}{2} \rfloor$ complex roots $\mu_1, \ldots, \mu_{\lfloor \frac{k-1}{2} \rfloor}$ with their conjugates $\overline{\mu}_1, \ldots, \overline{\mu}_{\lfloor \frac{k-1}{2} \rfloor}$.*

Proof. It can be easily seen that neither 0 nor 1 are roots of $\mathcal{F}_k(X)$. Following [4] and [5], it is convenient to work with the polynomial

$$(X-1)\mathcal{F}_k(X) = X^{k+1} - X^k - X^{k-1} + 1. \tag{1}$$

Differentiating Eq. (1), we obtain

$$((X-1)\mathcal{F}_k(X))' = (k+1)X^k - kX^{k-1} - (k-1)X^{k-2}. \tag{2}$$

Eq. (2) is 0, at $X = 0$ or at the roots of the quadratic polynomial:

$$(k+1)X^2 - kX - (k-1). \tag{3}$$

Its discriminant can be easily computed to $\Delta = 5k^2 - 4 > 0$, for all $k \geq 3$ and the two real roots of polynomial of Eq. (3) are

$$\beta_{1,2}(k) = \frac{k \pm \sqrt{5k^2 - 4}}{2(k+1)}. \tag{4}$$

For any root r of $(X-1)\mathcal{F}_k(X)$, holds that

$$r^{k+1} - r^k = r^{k-1} - 1,$$

which by the binomial theorem is not valid for $r = \beta_{1,2}(k)$. Therefore the polynomial $(X-1)\mathcal{F}_k(X)$ has $(k+1)$ simple roots.

We identify the real roots by elementary means. Note that $\mathcal{F}_k(1) = 2 - k < 0$ and $\mathcal{F}_k(\phi) = \phi$ and applying Descartes' rule of signs to Eq. (1), there is a unique positive root λ_k in $(1, \phi)$ and for k even, the unique negative root of the polynomial is -1. Further, the polynomial of Eq. (3) is positive and increasing for $X > \frac{k+\sqrt{5k^2-4}}{2(k+1)}$ and $(X-1)\mathcal{F}_k(X)$ is positive and increasing for $X > \lambda_k$ and negative for $1 < X < \lambda_k$. □

A direct consequence of Lemma 2.1 is

Corollary 2.2. *The polynomial $\mathcal{F}_k(X)$ is irreducible on the field of rational numbers $\mathbb{Q}$ if and only if k is odd.*

Further, it is easy to prove that all complex zeroes of the polynomial are inside the unit circle. The next Lemma is from Miles [4] and Miller [5].

Lemma 2.3 (Miles [4], Miller [5]). *For all the complex zeroes μ of the polynomial $\mathcal{F}_k(X)$, it holds that $|\mu| < 1$.*

Proof. Assume that there exists a complex μ (and hence $\overline{\mu}$), with $1 < |\mu| < \lambda_k$. We have that $(\mu - 1)\mathcal{F}_k(\mu) = 0$ and

$$|\mu^{k+1}| = |\mu^k + \mu^{k-1} - 1|. \tag{5}$$

Applying the triangle inequality to Eq. (5), we deduce that

$$(|\mu| - 1)\mathcal{F}_k(|\mu|) > 0,$$

which contradicts Lemma 2.1. Assuming now that $|\mu| > \lambda_k$, we have

$$|\mu^k| = \left|\sum_{j=0}^{k-2} \mu^j\right| \leq \sum_{j=0}^{k-2} |\mu^j|,$$

which is equivalent to $\mathcal{F}_k(|\mu|) \leq 0$ and again we arrive in contradiction. Finally, by the same reasoning it can be easily proved that there is no complex zero μ, with either $|\mu| = \lambda_k$ or $|\mu| = 1$. □

Lemma 2.3 implies that the solution of the generalised recurrence can be approximated by

$$a_n \approx C\lambda_k^n, \tag{6}$$

with negligible error term. In Eq. (6), C is a constant to be determined by the solution of a linear system of the initial conditions.

We now consider more carefully Eq. (4)

$$\beta_{1,2}(k) = \frac{k \pm \sqrt{5k^2 - 4}}{2(k+1)}.$$

Observe that $\beta_1(k) = \frac{k+\sqrt{5k^2-4}}{2(k+1)}$ is increasing and bounded sequence. Furthermore,

$$\lim_{k\to\infty} \frac{k + \sqrt{5k^2 - 4}}{2(k+1)} = \frac{1}{2} + \sqrt{\frac{5}{4}} = \phi. \tag{7}$$

Also, $\beta_2(k) = \frac{k-\sqrt{5k^2-4}}{2(k+1)}$ is decreasing and bounded and

$$\lim_{k\to\infty} \frac{k - \sqrt{5k^2 - 4}}{2(k+1)} = \frac{1}{2} - \sqrt{\frac{5}{4}} = 1 - \phi. \tag{8}$$

From Eq. (7) and (8), we deduce that two of the critical points of Eq. (1), (recall that these are 0 with multiplicity $(k-2)$, $\beta_1(k)$ and $\beta_2(k)$), converge to ϕ and $1-\phi$. An elementary calculation can show that $\beta_1(k)$ are points of local minima of the function $(X-1)\mathcal{F}_k(X)$ to the interval $(1, \lambda_k)$ and $\beta_1(k) < \lambda_k < \phi$ for all $k \geq 3$, so $\lim_{k\to\infty} \lambda_k = \phi$ and more precisely $\lambda_k \in [\rho, \phi)$.

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