A note on the equivalence of some metric and cone metric fixed point results

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In the present work, using Minkowski functionals in topological vector spaces, we establish the equivalence between some fixed point results in metric and in (topological vector space) cone metric spaces. Thus, a lot of results in the cone metric setting can be directly obtained from their metric counterparts. In particular, a common fixed point theorem for \( f \)-quasicontractions is obtained. Our approach is even easier than that of Du [Wei-Shih Du, A note on cone metric fixed point theory and its equivalence, Nonlinear Anal. 72 (2010) 2259–2261] where similar conclusions were obtained using scalarization functions.

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1. Introduction

Ordered normed spaces and cones have applications in applied mathematics, for instance, in using Newton’s approximation method [1–4] and in optimization theory [5]. \( K \)-metric and \( K \)-normed spaces were introduced in the mid-20th century ([2]; see also [3,6]) by using an ordered Banach space instead of the set of real numbers, as the codomain for a metric. Huang and Zhang [7] re-introduced such spaces under the name of cone metric spaces, and went further, defining convergent and Cauchy sequences in the terms of interior points of the underlying cone. These and other authors (see, for example, [8–19]) proved some fixed point and common fixed point theorems for contractive-type mappings in cone metric spaces, in most cases adapting known results from metric to the cone metric setting.

In a recent paper [20] Du has shown that some of these results can be obtained in an easier way. Namely a kind of equivalence has been established between metric and cone metric results (or topological vector cone metric results), using the so-called nonlinear scalarization function. This function was introduced in [21] and applied to the study of vector quasi-equilibrium problem. Note that the proof of Theorem 2.2(iii) in [20] is incomplete since only one implication is proved in both (i) and (ii) and it is not clear why equivalences hold true.

In the present work we show that the same can be obtained even more easily using Minkowski functionals in topological vector spaces. On the way, we fill the aforementioned gap in the proof of [20, Theorem 2.2]. Thus, a lot of fixed point results from the metric setting can be proved in the cone metric (or tvs-cone metric) setting. In particular, a common fixed point theorem for \( f \)-quasicontractions in cone metric spaces is obtained.

Note that another approach for obtaining cone metric versions of the known fixed point results in the case of solid and normal cones was presented in [22].
2. Preliminaries

Let $E$ be a Hausdorff topological vector space (tvs for short) with the zero vector $\theta$. A proper nonempty and closed subset $K$ of $E$ is called a (convex) cone if $K + K \subset K$. If $K \subset K$ for $\lambda \geq 0$ and $K \cap (-K) = \{\theta\}$. We shall always assume that the cone $K$ has a nonempty interior int $K$ (such cones are called solid).

Each cone $K$ induces a partial order $\leq$ on $E$ by $x \leq y \iff y - x \in K$. $x < y$ will stand for $x \leq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{int} K$. The pair $(E, K)$ is an ordered topological vector space.

For a pair of elements $x$, $y$ in $E$ such that $x \leq y$, put

$$[x, y] = \{z \in E : x \leq z \leq y\}.$$

The sets of the form $[x, y]$ are called order-intervals. It is easily verified that order-intervals are convex. A subset $A$ of $E$ is said to be order-convex if $[x, y] \subset A$ whenever $x, y \in A$ and $x \leq y$.

Ordered topological vector space $(E, K)$ is order-convex if it has a base of neighborhoods of $\theta$ consisting of order-convex subsets. In this case the cone $K$ is said to be normal. In the case of a normed space this condition means that the unit ball is order-convex, which is equivalent to the condition that there is a number $k$ such that $x, y \in E$ and $\theta \leq x \leq y$ implies that $\|x\| \leq k\|y\|$. The smallest constant $k$ satisfying the last inequality is called the normal constant of $K$.

Note the following properties of bounded sets.

If the cone $K$ is solid, then each topologically bounded subset of $(E, K)$ is also order-bounded, i.e., it is contained in a set of the form $[-c, c]$ for some $c \in \text{int} K$.

If the cone $K$ is normal, then each order-bounded subset of $(E, K)$ is topologically bounded. Hence, if the cone is both solid and normal these two properties of subsets of $E$ coincide. Moreover, a proof of the following assertion can be found, e.g., in [3].

**Theorem 2.1.** If the underlying cone of an ordered tvs is solid and normal, then such a tvs must be an ordered normed space.

Thus, proper generalizations when passing from norm-valued cone metric spaces of [7] to tvs-cone metric spaces can be obtained only in the case of nonnormal cones.

Recall (see, e.g., [23]) that if $V$ is an absolutely convex and absorbing subset of a tvs $E$, its Minkowski functional is defined by

$$E \ni x \mapsto q_V(x) = \inf\{\lambda > 0 : x \in \lambda V\}.$$  

It is a semi-norm on $E$ (i.e., $q_V(x + y) \leq q_V(x) + q_V(y)$ for $x, y \in E$ and $q_V(\lambda x) = |\lambda|q_V(x)$ for $x \in E$, $\lambda$ a scalar) and $V \subset W$ implies that $q_W(x) \leq q_V(x)$ for $x \in E$. If $V$ is an absolutely convex neighborhood of $\theta$ in $E$, then $q_V$ is continuous and

$$\{x \in E : q_V(x) < 1\} = \text{int } V \subset V \subset \text{int } V = \{x \in E : q_V(x) \leq 1\}.$$

Let now $(E, K)$ be an ordered tvs and let $e \in \text{int} K$. Then $[-e, e] = (K - e) \cap (e - K) = \{z \in E : -e \leq z \leq e\}$ is an absolutely convex neighborhood of $\theta$; its Minkowski functional $q_{[-e, e]}$ will be denoted by $q_e$. Observe that $\text{int }[-e, e] = (\text{int } K - e) \cap (e - \text{int } K)$. If the cone $K$ is solid and normal, $q_e$ is a norm in $E$. Moreover, it is an increasing function on $K$.

Indeed, let $\theta \leq x_1 \leq x_2$; then $\{\lambda : x_1 \in \lambda[-e, e]\} \supseteq \{\lambda : x_2 \in \lambda[-e, e]\}$ and it follows that $q_e(x_1) \leq q_e(x_2)$.

Following [7,20,24] we give the following:

**Definition 2.2.** Let $X$ be a nonempty set and $(E, K)$ an ordered tvs. A function $d : X \times X \rightarrow E$ is called a tvs-cone metric and $(X, d)$ is called a tvs-cone metric space if the following conditions hold:

1. $\theta \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$ for all $x, y \in X$;
3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Let $x \in X$ and $\{x_n\}$ be a sequence in $X$. Then it is said that:

1. $\{x_n\}$ tvs-cone converges to $x$ if for every $c \in E$ with $\theta \ll c$ there exists a natural number $n_0$ such that $d(x_n, x) \ll c$ for all $n > n_0$; we denote it by $d - \lim_{n \rightarrow \infty} x_n = x$ or $x_n \xrightarrow{d} x$ as $n \rightarrow \infty$;
2. $\{x_n\}$ is a tvs-cone Cauchy sequence if for every $c \in E$ with $\theta \ll c$ there exists a natural number $n_0$ such that $d(x_m, x_n) \ll c$ for all $m, n > n_0$;
3. $(X, d)$ is tvs-cone complete if every tvs-Cauchy sequence is tvs-convergent in $X$. 

3. Results

In the rest of the work $K$ will always be a solid cone in a tvs $E$.

**Theorem 3.1.** Let $(X, d)$ be a tvs-cone metric space and let $e \in \text{int } K$. Let $q_e$ be the corresponding Minkowski functional of $[-e, e]$. Then $d_q = q_e \circ d$ is a (real-valued) metric on $X$. Moreover, for arbitrary $x, y, x_1, y_1 \in X$, $d(x, y) \leq d(x_1, y_1)$ implies that $d_q(x, y) \leq d_q(x_1, y_1)$.

**Proof.** Since $d_q = q_e \circ d$ is a semi-norm and $d$ is a tvs-cone metric, we have only to prove that $d_q(x, y) = 0$ implies that $x = y$. Let $q_e \circ d(x, y) = 0$. Then $\inf \{ \lambda > 0 : d(x, y) \in \lambda[-e, e] \} = 0$ and there exists a sequence of positive scalars $\lambda_n \to 0$ such that $d(x, y) \in \lambda_n[-e, e]$. Suppose that $x \neq y$. Then, since $\theta < d(x, y) \leq \lambda_n e$ we have that for each $c \in K$ there exists $n_0$ such that $d(x, y) < c$ for $n \geq n_0$. Indeed, since $\lambda_n e \to \theta$ in $E$, it follows that $\lambda_n e \ll c$ for $n \geq n_0$. Using the property that $((a \leq b \text{ and } b \ll c) \Rightarrow a \ll c)$ we obtain that $d(x, y) \ll c$. Since $c$ is an arbitrary interior point of the cone $K$ it follows that $d(x, y) = 0$. A contradiction.

The last assertion is a consequence of the monotonicity of the function $q_e$. □

**Theorem 3.2.** Let $(X, d)$, $e$, $q_e$ and $d_q$ be as in the previous theorem and let $x \in X$ and $\{x_n\}$ be a sequence in $X$. Then:

1° $X \ni x_n \xrightarrow{d} x$ $\iff$ $x_n \xrightarrow{d_q} x$.

2° $\{x_n\}$ is a $d$-Cauchy sequence if and only if it is a $d_q$-Cauchy sequence.

3° $(X, d)$ is complete if and only if $(X, d_q)$ is complete.

**Proof.** 1° Let $\epsilon > 0$; observe that $\epsilon \cdot e \in \text{int } K$ because $e \in \text{int } K$. Now $x_n \xrightarrow{d} x$ means that there exists $k \in \mathbb{N}$ such that $d(x_n, x) < \epsilon \cdot e$ holds for $n > k$. It follows that $q_e(d(x_n, x)) < \epsilon$, i.e., $d_q(x_n, x) < \epsilon$ for $n > k$. Hence, $x_n \xrightarrow{d_q} x$.

Conversely, let $x_n \xrightarrow{d_q} x$, and fix $c \in K$. Then for each $\epsilon \in (0, 1)$ there exists $k_1 \in \mathbb{N}$ such that $d_q(x_n, x) < \epsilon$ for $n > k_1$. Hence, $q_e(d(x_n, x)) < \epsilon$ and $d(x_n, x) \ll \epsilon \cdot e$. Consider the function $f : (0, 1) \to E$ defined by $f(\epsilon) = \epsilon \cdot e$ which is continuous and tends to $\theta$ when $\epsilon \to 0^+$. Then $\lim_{\epsilon \to 0^+} \epsilon d_q(x_n) = \theta$ for any sequence of reals $\{\epsilon_n\}$ tending to $0^+$ when $n \to \infty$. In particular, it holds for $\epsilon_n = \frac{1}{n}$. We will use the following property:

If $\theta \leq a_n$ and $a_n \to \theta$ in $E$ then for each $\epsilon \ll c$ there exists $k_3 \in \mathbb{N}$ such that $a_n \ll c$ for each $n > k_3$.

In our case, we obtain that $d(x_n, x) \ll c$ for $n > \max\{k_1, k_2\}$, which means that $x_n \xrightarrow{d} x$ in the tvs-cone metric $d$.

2° The proof is similar to that for 1°.

3° follows from 1° and 2°. □

As a consequence, topologies induced on $X$ by $d$ and $d_q$ are equivalent, i.e., these spaces have the same collections of closed and open sets respectively, and the same continuous functions (provided the underlying cone is solid).

**Remark 3.3.** For the same purpose, a nonlinear scalarization function was used in [20]. It was introduced in [21] in the following way:

$$\xi_e(y) = \inf \{ r \in \mathbb{R} : y \leq re \}, \quad \text{for } y \in X,$$

where $e \in \text{int } K$ is fixed. It was proved in [20] that $d_q = \xi_e \circ d$ is a metric on $X$. It was also proved that

$$x_n \xrightarrow{d} x \implies x_n \xrightarrow{d_q} x.$$

Using the same approach as in the proof of **Theorem 3.2** one can show that also

$$x_n \xrightarrow{d_q} x \iff x_n \xrightarrow{d} x.$$

Then it follows that also

$$x_n \xrightarrow{d_q} x \iff x_n \xrightarrow{d} x.$$

The same remark can be made for Cauchy sequences and the completeness property of the spaces $(X, d)$, $(X, d_q)$ and $(X, d_q)$.

In the Fixed Point Theory (starting with the basic Banach Contraction Principle), contractive or nonexpansive conditions for a self-map are usually given in the form of an inequality. Hence, using properties of the Minkowski functional $q_e$ (in particular its monotonicity), most of the corresponding results obtained in the metric setting remain valid in the (tvs-)cone metric setting (provided the underlying cone is solid). Thus, most of the cone metric results obtained recently can be proved in an easier way.

For example, fixed point and common fixed point results from [7-11] are direct consequences of **Theorems 3.1** and **3.2**, as well as the known corresponding results from the metric setting. We will prove a result on quasicontractions in tvs-cone metric spaces.
Definition 3.4 ([13,14]). Let $(X, d)$ be a cone metric space, and let $f, g : X \to X$. Then, $g$ is called a quasicontraction (resp. an $f$-quasicontraction) if for some constant $\lambda \in [0, 1)$ and for all $x, y \in X$, there exists
\[ u \in C(x, y) = \{d(x, y), d(x, gx), d(x, gy), d(y, gy), d(x, y)\}. \]
(resp. $u \in C(f; x, y) = \{d(fx, fy), d(fx, gx), d(fx, gy), d(fy, gy), d(fx, gx)\}$).

such that
\[ d(gx, gy) \leq \lambda \cdot u. \]

Theorem 3.5. (a) Let $(X, d)$ be a complete cone metric space, and let us have $f, g : X \to X$, $f$ commutes with $g, f$ or $g$ is continuous, $g$ is an $f$-quasicontraction and $g(X) \subset f(X)$. Then $f$ and $g$ have a unique common fixed point in $X$.

(b) Let $(X, d)$ be a complete cone metric space, and let the mapping $g : X \to X$ be a quasicontraction. Then $g$ has a unique fixed point in $X$ and for any $x \in X$, the iterative sequence $(g^n x)$ converges to the fixed point.

In the case of cone metric spaces with normal cones result (a) was obtained in [13] and result (b) in [14]. The normality condition in (b) was removed in [15,16]. We will prove these results using our Theorems 3.1 and 3.2, and hence showing that they are equivalent to the respective metric results from [25,26].

Proof. (a) The condition that $d(gx, gy) \leq \lambda \cdot u$, for some $u \in C(f; x, y)$, implies, by Theorem 3.1, that
\[ d_q(gx, gy) \leq \lambda \cdot \max\{d_q(fx, fy), d_q(fx, gx), d_q(fx, gy), d_q(fy, gy), d_q(fx, gx)\}. \]

In other words, contractivity condition from [25] is fulfilled. Theorem 3.2 implies that the metric space $(X, d_q)$ is complete and it follows that mappings $f$ and $g$ have a unique common fixed point.

The result (b) is obtained putting $f = Id$ or directly, as for (a), using the fixed point result from [26].

We state also a fixed point result in ordered cone metric spaces.

Corollary 3.6 ([17–19]). Let $(X, \preceq)$ be a partially ordered set and let $d$ be a cone metric on $X$ such that $(X, d)$ is a complete cone metric space. Let $f : X \to X$ be a continuous and nondecreasing map w.r.t. $\preceq$. Suppose that the following conditions hold:

(i) there exists $k \in (0, 1)$ such that $d(fx, fy) \leq kd(x, y)$ for all $x, y \in X$ with $x \preceq y$;

(ii) there exists $x_0 \in X$ such that $x_0 \preceq fx_0$.

Then $f$ has a fixed point $x^* \in X$.

The cone metric version of this result was first obtained in [17] (in the case of a normal cone) and then in [18,19], without using the normality condition.

Proof. Using the metric version of the given assertion from [27], and Theorems 3.1 and 3.2, the result follows. \(\square\)

4. Conclusion

Let $(X, d)$ be a tvs-cone metric space over a solid cone $K$ and let $e$ be an interior point of $K$. Using the Minkowski functional $q$ of the set $[-e, e]$ we obtain the metric $d_q = q_{[-e,e]} \circ d$ and prove that it generates the same topology on $X$ as the cone metric $d$. In this way we can investigate properties of tvs-cone metric spaces applying corresponding results from the metric setting.

As a further example we state the following:

Theorem 4.1. Let $(X, d)$ be a tvs-cone metric space over a solid cone and let $(X, d_q)$ be the corresponding metric space. Then:

(a) $(X, d)$ is paracompact.

(b) $(X, d)$ is connected (separable, compact) if and only if $(X, d_q)$ is such, respectively.

(c) $(X, d)$ is Hausdorff; hence the limit of each convergent sequence is unique.

(d) A function $f : (X, d) \to (X, d)$ is continuous if and only if it is sequentially continuous.

Note that the result (a) was recently proved in [28] for the case where the cone $K$ is normal.

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References


