# On the Nash Equilibrium in Game Theory A Learning Report for Optimization Method Class

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**Abstract**

The Nash equilibrium is the central pillar of modern non-cooperative game theory with a wide range of applications. Based on the references, I will demonstrate my learning process about Nash equilibrium. This is followed by a proof of the existence of Nash equilibrium and some examples. In this report, I also include a review of graph theory and linear programming in professor Zhang's class.

**Keywords:** Nash Equilibrium; Kakutani Fixed Point Thm; Course Review

## 1 Introduction

Formally, in a non-cooperative game with normal-form $G = \langle N, \{S_i\}_{i=1}^n, \{v_i\}_{i=1}^n \rangle$, the mixed-strategy profile $\sigma^* = (\sigma_1^*, \sigma_2^*, \ldots, \sigma_n^*)$ is a *Nash equilibrium* if for each player, $\sigma_i^*$ is a best response to $\sigma_{-i}^*$. That is,

$$\forall i \in N, v_i \left(\sigma_i^*, \sigma_{-i}^*\right) \geq \left(\sigma_i, \sigma_{-i}^*\right) \; \forall \sigma_i \in \Delta S_i$$

Intuitively, when individual rational player cannot benefit more by changing strategy with the other players' strategies remaining statu quo, the current strategy profile and the corresponding payoffs consitutues a Nash equilibrum(NE).

Aware of the vital importance of Nash equilibrium and impressed by the character of John Forbes Nash Jr in 'A beautiful mind', I have been looking forward to delving into Nash equilibrium. Starting from what is covered in class, I finished reading chapters 1-6 of Steven Tadelis's *Game Theory An Introduction* [1]. Also, some references to

Fudenberg's[2], as well as open MIT course [3] and open Yale course[4], help shed light on the structure of game theory.

In section 2, I explore the history of Nash equilibrium and quote some applications. From what I have learned about the minmax equilibrium of zero-sum finite game in class, I will introduce some core concepts and definitions in order to step into NE. However, due to the technical difficulty, I'd like to restrict our attention in static games with complete information. Also, rationality and common knowledge of rationality should be guaranteed.

In section 3, with my understanding and experience about NE, I will give some analysis of NE in the aspects of existence, uniqueness, and invariance/stability, just like every equilibrium in other solution concepts. In the mean time, intuitively, I try to make some comparisons of equilibria in other subjects like Static Mechanics, PDE and Stochastic Process.

In section 4, I head back to the John Nash's famous existence theorem. Equipped with the knowledge from Real Analysis[6] and Functional Analysis[7], after a sketchy proof of *Kakutani's fixed point thm*, I will try to show the proof of *Nash's Existence Theorem* in details and carefully verify the specific conditions.

In the following section, from the perpective of NE, I present the typical example *Prisoner's Dilemma* to illustrate the critical concepts argued above. Then, I introduce *Matching Pennies* to give some flavor to the computation of NE. In the last example, I lay out some further ideas about relationship between pure-strategy and mixed-strategy NE.

In the section of course review, after recalling what is covered about graph theory and linear programming in class, I give some reflections and puts forth several questions. In the graph theory, minimum spnning tree, shortest path problem, max-flow min-cut theorem and min-cost max-flow problem are revisited. As for linear programming, I will conduct the simplex algorithm step by step in a specific example. Duality, important in many ways, will be given further consideration.

Finally, I will draw some conclusions and write further notes in which I will analyze both the advantages and disadvantages of this article, followed by my sincere acknowledgements.

**Remark** Instead of being a research paper, this article should serve as a learning report which maybe and reasonally includes some parts far from being outright original. The author would like to avoid the originality problem and claim that every word is written after aborbing the related knowledge throughly.

# Contents

# 2 Background of Nash Equilibrium

## 2.1 History

In this subsection, I will introduce the brief history of game theory while trying to make connections to NE and the contents of optimization class. Following the timeline, I include some of my own opinions and reflections.

Early before the rise of modern game theory marked by the contributions of John von Neumann and Oskar Morgenstern, the first known discussion of game theory occurred in a letter about *Waldegrave problem* in which James Waldegrave provides a solution to a two-person version of the card game *le Her*, in 1713.

In his 1838 *"Research into the Mathematical Principles of Theory Wealth"*, Antonie Augustin Cournot considered a duopoly and presents a solution that is a restricted version of the Nash equilibrium. In the chapter 4 of Tadelis's[1], it introduces the example of *Cournot Duopoly* to illustrate a the concept of **Iterated Elimination of Strictly Dominated Pure Strategies (IESDS)**. By IESDS, we can rule out the strictly dominated strategies iteratedly to reach a smaller set of strategy profiles called IESDS equilibrium in the solution concept IESDS. I think it worth mentioning that IESDS is based on the **rationality** of the players and the **common knowledge** of rationality among the players in the game, because a rational player will never choose to ignore the possible situations without assuring that his opponents won't take the corresponding strategies. With the background omitted, the procedure of IESDS in *Cournot Duopoly* is presented in Figure 1.

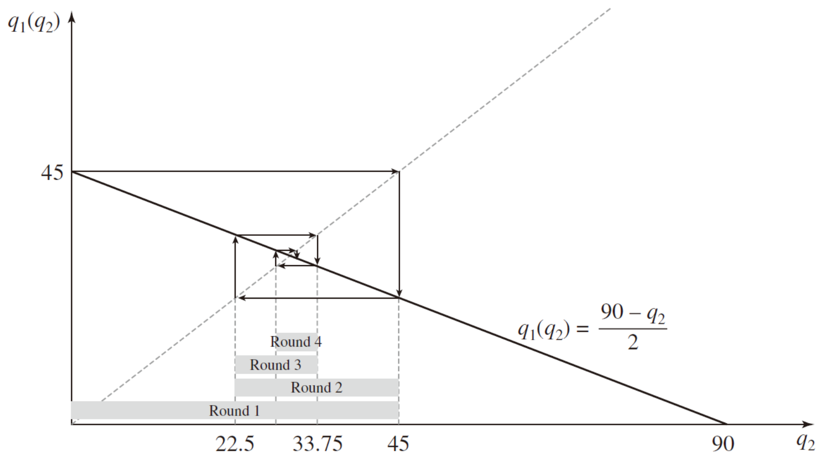


Figure 1: IESDS convergence in the Cournot game

**Question** What is Nash equilibrium? Or by what kind of solution concept the NE is predicted? How to analyze and predict the decisions of players?

In 1913, Ernst Zermelo published *On an Application of Set Theory to the Theory of the Game of Chess*. It proved that optimal chess strategy is strictly determined, which paved the way for more generay theorems.

**Remark** I believe most students are familiar with *Zermelo-Fraenkel axiomatic set theory*. Along with *the choice axiom*, the ZF-C axiomatic set theory is the base of most modern mathematical disciplines. To illustrate, many theorems like *Hahn-Banach Theorem* or *Tychonoff Product Theorem*, as crutial as cornerstones, are derived from *the choice axiom* or its equivalents. Hence the application of set theory implies the essential role of game theory in some sense.

As time went by, the modern game-theoretic concept of Nash equilibrium is instead defined in terms of mixed strategies, where players choose a probability distribution over possible actions. The concept of the mixed-strategy Nash equilibrium was introduced by John von Neumann and Oskar Morgenstern in their 1944 book *The Theory of Games and Economic Behavior*. However, their analysis was restricted to the special case of zero-sum games. They showed that a mixed-strategy Nash equilibrium will exist for any zero-sum game with a finite set of actions.

**Course Review** In the beginning of optimization method class, professor Zhang introduced the standard notations in game theory and the simplified history, especially the origin of modern game theory.

Following John von Neumann's steps, prof. Zhang focused on the zero-sum games and expressed the equilibrium in the solution concept of *minmax principle*. Afterwards, professor Zhang proved the saddle point condition in both pure-strategy and mixed-strategy game, which serve as a useful theoretical principle to decide whether a specific strategy profile is a equilibrium. In my opinion, one of the obvious advantages of zero-sum game is that, since condition of zero-sum can reduce the 'degree of freedom' by 1 and lay a retriction of payoffs, game theorists are allowed to replace payoff matrix with 'gain matrix'.

To achieve higher practivity, we learned several equivalent theorems of saddle point principle, one of which is of linear-programming version. From this LP-version theorem plus the *strict duality theorem* in LP, professor Zhang proved the existence of the equilibrium in mixed-strategy game, which was obtained by the great mathematican John von Neumann sixty years ago! Personally, I really admire John von Neumann for his ground-breaking contributions to so many areas like PDE, topology, ergodic theory, computer science. . .

As for the computation of solution in mixed-strategy game, we mainly learned three kinds of methods. The first one is graphical method, which can not be effectively applied if there are more than three strategies. The second method is derived from linear-equation-version of the aforementioned principle. However, it requires the chosen probability vector $\overrightarrow{X} > \overrightarrow{0}$ in the usual partial order of $R^n$. The last method is LP. Among all the algoritms to solve LP probelms *Interior-point method* and *Simplex method* are famous. The *Simplex Algorithm* will be conducted in section 6.2. By the way, our famous HUST alumnus, professor Yinyu Ye from Stanford U, as well as a *John von Neumann Theory Prize* laureate, works on mathematical optimization, especially *Interior-point method.*

**Remark** In the above review, it is not difficult to perceive the two places where LP is used, leading to the review of LP in section 6.2. Now that the main content of game theory in our class locates here, where is our target–the Nash equilibrium?

The contribution of Nash in his 1951 article *'Non-Cooperative Games'* was to define a mixed-strategy Nash equilibrium for any game with a finite set of actions and prove that at least one (mixed-strategy) Nash equilibrium must exist in such a game. The key to Nash's ability to prove existence far more generally than von Neumann lay in his definition of equilibrium. According to Nash, "an equilibrium point is an n-tuple such that each player's mixed strategy maximizes his payoff if the strategies of the others are held fixed. Thus each player's strategy is optimal against those of the others." Just putting the problem in this framework allowed Nash to employ the *Kakutani fixed-point theorem* in his 1950 paper, and a variant upon it in his 1951 paper used the *Brouwer fixed-point theorem* to prove that there had to exist at least one mixed strategy profile that mapped back into itself for finite-player (not necessarily zero-sum) games, namely, a strategy profile that did not call for a shift in strategies that could improve payoffs.

| | | Player2 | | |
|---|---|---|---|---|
| | | L | C | R |
| | U | 7, 7 | 4, 2 | $\overline{1,8}$ |
| Player1 | M | 2, 4 | $\overline{\underline{\mathbf{5,\ 5}}}$ | $\underline{2,3}$ |
| | D | $\underline{8,1}$ | $\overline{3,2}$ | 0, 0 |

Table 1: From von Neumann's equilibrium to NE

**Remark** There it is! In Table 1, the bold strategy $(M, C)$ is decided by best-response correspondence, a generation of *minmax principle* of zero-sum game. Nash equilibrium is a generation of the equilibrium introduced in our class which captures the essential idea of a equilibrium. **Non-cooperative** means self-enforcing where individual player seeks to maximize his own profits with horizon "restricted", matching the selfishness in human nature. In the author's opinion, the sadness of non-cooperative can be illustrated in *Tragedy of the Commons*. However, for the purpose of academical research, selfish incentives is indispensable in prediction. On the other hand, in cooperative game, a third party can enact laws to "control" the players' actions. The *Prisoner's Dilemma* in section 5.1 will illustrate this scenario. The meaning of non-cooperative can also be felt in the definition of NE where individual cannot improve payoff as long as the others refuse to change their strategies. Last but not least, the existence of NE and the *Kakutani fixed-point theorem* will be revisted in section 4.

Since the development of the Nash equilibrium concept, game theorists have discovered that it makes misleading predictions (or fails to make a unique prediction) in certain circumstances. They have proposed many related solution concepts (also called 'refinements' of Nash equilibria) designed to overcome perceived flaws in the Nash concept. One particularly important issue is that some Nash equilibria may be based on threats that are not 'credible'. In 1965 Reinhard Selten proposed subgame perfect equilibrium as a refinement that eliminates equilibria which depend on non-credible threats. However, subsequent refinements and extensions of the Nash equilibrium concept share the main insight on which Nash's concept rests: **all equilibrium concepts analyze what choices will be made when each player takes into account the decision-making of others.**

**Remark** *Ultimatum Game* can shed light on the subgame perfect equilibrium and non-credible threats, but it costs a huge amount of efforts to delve further into the applications and modern developments of NE. With a wide range of application, game theory has become a heated research area in many disciplines nowadays while our attention will be restricted in static games with complete information in the following sections.

## 2.2 Applications

As is mentioned for many times, both from the nature and principle of NE, Nash equilibrium has gained a huge variety of applications, resulting in NE's critical role in game theory as well as in the modern society.

> Game theorists use the Nash equilibrium concept to analyze the outcome of the strategic interaction of several decision makers. In other words, it provides a way of predicting what will happen if several people or several institutions are making decisions at the same time, and if the outcome for each of them depends on the decisions of the others. The simple insight underlying John Nash's idea is that one cannot predict the result of the choices of multiple

decision makers if one analyzes those decisions in isolation. Instead, one must ask what each player would do, taking into account the decision-making of the others.

Nash equilibrium has been used to analyze hostile situations like war and arms races, and also how conflict may be mitigated by repeated interaction. It has also been used to study to what extent people with different preferences can cooperate, and whether they will take risks to achieve a cooperative outcome. It has been used to study the adoption of technical standards, and also the occurrence of bank runs and currency crises. Other applications include traffic flow, how to organize auctions, the outcome of efforts exerted by multiple parties in the education process, regulatory legislation such as environmental regulations, natural resource management, analysing strategies in marketing, and even penalty kicks in football.

**Reflection** With the assitance of references, surely I have read and understood many typical examples of aforementioned applications at an introductory level. As for our daily life, I do think out a game when I am typing this line. Imagine that it is raining outside, while, in the dormitory, both my roommate and I don't want to go to West One Dinning Hall for lunch at 12 am. Also, if only one person choose to go, he shoud buy a lunch for the other one. If both go outside, they will get wet in the rain and become unhappy. Finally, if both choose to stay, we will be hungry, which is more serious than getting wet. Moreover, my roomate is more lazy than me. The payoff matrix is listed below.

| | | Roommate | |
|---|---|---|---|
| | | Go | Stay |
| Me | Go | -1, -2 | -4, 1 |
| | Stay | 0, -5 | **-3, -3** |

Table 2: Payoff matrix of *HUST Lunch Dilemma*

**Analysis** With the aforementioned IESDS, both of us will prefer to stay in the dormitory and starve, leading to a Nash equilibrium! However, following our instinct, this equilibrium is not *Pareto Opitimal*. In mathematics, *Pareto Opitimal* is a maximial element in the poset consisting of payoff vectors. Further analysis lies in section 5.1 where there is a game similiar to *HUST Lunch Dilemma*. Of course, this example is really primitive and fails to demonstrate the vital importance of NE in enconomics or political science, because, in the practice, except the classic examples like *Bertrand Duopoly of Prize Competetion* or *Political Ideology and Electoral Competition*, I think the dynamic game of incomplete information often arises and dominates.

# 3 Evaluating Nash Equilibrium

## 3.1 Existence

Nash has proved the existence for any non-cooperative mixed-strategy finite game with fixed point theorems. I leave this job in the next section. What I should mention is that the condition of mixed-strategy implies the convexity and allows richer space of strategis and beliefs–possible strategies player believe his opponents could choose, thus ensuring the existence.

1. Firstly, convexity condition is essential in most fixed point theorems including Brower's, Kakutani's and Schauder's.

2. Secondly, convexity is important in applied mathematics including convex analysis, linear programming and Supported Vector Machine(SVM) in classfying problem of mathematical modeling.

3. Last but not least, I believe the essence and nature of convexity should be found in pure mathematics. Locally convex topological space is one of the most important object in Functional Analysis, for example, the *Hyperplane Seperation Theorem* and *the Krein-Milman Theorem* considering the extreme points in convex compact set. In some sense, I think the extreme points are like the basic feasible solutions in LP.

However, much more is waiting for me to find out about convexity. Moreover, the mixed-strategy resembles the uncertain world by allowing players to choose strategies stochastically. As for finding a NE, mixed-strategy shifts the discrete situation into a continuous one which provides the tools in calculus like differential.

On the other hand, the conditions where the existence is not guaranteed are not rare. Nash equilibrium need not exist if the set of choices is infinite and noncompact. A simple example is when two players simultaneously name a natural number with the player naming the larger number wins. If the two firms have different marginal prices, the *Bertrand Duopoly* about price competition can also serve as an excellent instance failing to have a NE.

## 3.2 Uniqueness

If a non-cooperative game has a unique NE, surely the corresponding strategy profile will be adopted, hence we can predict the strategies precisely. However, NE may have its companions if there exists one.

With the background information omitted for convinience, the following example has three Nash equilibria. The payoff matrix is listed below.

| | | Player2 | | |
|---|---|---|---|---|
| | | A | B | C |
| Player1 | A | 0, 0 | **20, 30** | 5, 10 |
| | B | **37, 22** | 0, 0 | 5, 15 |
| | C | 8, 4 | 13, 2 | **9, 10** |

Table 3: Game with three NE

It is not difficult to discern the three Nash equilibria bold in this pure-strategy game. However, the values of the entries of payoff matrix are designed by myself deliberately. In the section 5.3, a more fruitful example is evaluated by generating a mixed-strategy NE from two obvious pure-strategy Nash equilibria.

## 3.3 Stability/Invariance

Just like many other equilibria, the stablity of Nash equilibrium should also enjoy considerable attention.

**Importance** Stablility intuitively means that a small perturbance won't affect the equilibrium too much or even results in nothing. In PDE, we do not want the little inaccuracy of parameters, inital condition, boundary condition or maybe 'right-hand function' to cause huge variation of the solution. On the other hand, we concern about different kinds of stability like Lyapunov Stability or Global Stability of equilibrium points. The motivation is that people cannot measure the condition precisely because the whole existing measure system is 'discrete' and approximation is usually the best choice, attaching vital importance to stability.

For the similiar reason, stability is crucial in practical applications of Nash equilibria, since the mixed strategy of each player is not perfectly known, but has to be inferred from statistical distribution of their actions in the game. Be aware that the Nash equilibrium defines stability only in terms of unilateral deviations due to the self-enforement/non-cooperative in the definition of NE.

**Remark** As for the small perturbance, a fluctuation of the number of players or strategies shouldn't be considered minute while it is likely to cause a completely different NE. Take the extreme case of two-player game with two strategies as example, no one can deny the great influence caused by the change in the size of the game. However, a recall of professor Zhang's lecture reveals that the solution in class is indeed stable with minute fluctuation of the values of payoff function. Since most payoff matrix/gain matrice given in class consist of exclusively integer entries, a small change, say 0.1, to payoff won't affect the players' choice of strategy profile. But it is rare to witness this scenario in mixed-strategy games because the NE in mixed-stratey is essentially the precise proportion of the pure-strategy.

# 4 Existence of Nash Equilibrium

**Theorem 1** ***Nash's Existence Theorem*** *Any n-player normal-form game with finite strategy sets $S_i$ for all players has a Nash equilibrium in mixed strategis.*

The necessarity of mixed-strategy has been intuitively discussed in section 3.1. To begin with, I present the abstract *Kakutani's fixed point theorem* in the chapter 5 of Fuctional Analysis[7], leaving out the version in Real Analysis [6] (Because in chapter 22 of [6], *Kakutani's fixed point theorem* is proved in order to obtain the existence of Haar measure that I think deviates from what we aquire. )

**Theorem 2** ***Abstract Kakutani's Fixed Point Theorem*** *Suppose that*

*(a) $K$ is a nonempty compact convex set in a locally convex space $X$;*

*(b) $G$ is an equicontinuous group of affine maps taking $K$ in to $K$.*

*Then $G$ has a common fixed point in $K$.*

**Sketch of Proof.** Proof by contradiction.

On the one hand, firstly, use *Hausdorff Maximality Theorem* or *Zorn's Lemma* to contruct a minimal compact convext set $Q$ invariant under maps of $G$, then apply the equilcontinuity to get two points $x$ and $y$ whose image can't be too close under any map T in G.

On the other hand, by the *Krein-Milman Theorem* and the property of extreme point $p$, we get $x^* = y^*$ in the closure of G-orbit of x and y respectly, leading to a contradiction. ■

Back to NE, we define the collection of best-response correspondences, $BR \equiv BR_1 \times BR_2 \times \cdots \times BR_n$ maps $\Delta S = \Delta S_1 \times \Delta S_2 \times \cdots \times \Delta S_n$, the set of profiles of mixed strategies, onto itself. That is, $BR : \Delta S \rightrightarrows \Delta S$ takes every element $\sigma \in \Delta S$ and converts it into a subset $BR(\sigma) \subset \Delta S$. The following fact comes as a direct consequence:

**Fact** A mixed-strategy profile $\sigma^* \in \Delta S$ is a Nash equilibrium iff it is a fixed point of $BR$, that is $\sigma^* \in BR(\sigma^*)$.

The biggest difficulty to apply the theorem 2 is that $BR$ is the multivalued/set-valued function instead of a function/map. To solve this, I indeed try to use the concepts like equivalent class, but in vail. To construct endomorphisms, I thougt out $F_i(\sigma) := (BR_i(\sigma_{-i}), \sigma_{-i})$ which still fails to be a function let alone the surgectivity and injectivity.

Hence, the biggest problem is how to deal with the set-valued function/ correspondence. The following variant of *Kakutani's fixed point thm* is proposed for the situation.

**Theorem 3** ***Kakutani's Fixed Point Theorem for correspondence*** *A correspondence $C : X \rightrightarrows X$ has a fixed point $x \in C(x)$ if four condition are satisfied:*

1. *$X$ is a non-empty, compact, and convex subset of $R^n$;*
2. *$C(x)$ is non-empty for all x;*
3. *$C(x)$ is convex for all x;*
4. *$C$ has a closed graph.*

Now, we verify the four conditions of theorem 3:

1. $BR : \Delta S \rightrightarrows \Delta S$ operates on $\Delta S$. If the size of $S_i$ is $m_i$, then X is the direct product of $n$ unit cubes in $R^{m_i}$, which is non-empty, convex, and compact subset of $R^{\sum_{i=1}^{n} m_i}$.

2. Since $v_i(\sigma_{-i})$ is a continuous linear function on unit cube in $R^{m_i}$, the best-reponse $BR_i(\sigma_{-i})$ is non-empty by the *extreme value theorem.*

3. Since each set $BR_i(\sigma_{-i})$ is convex, the direct product, $BR(\sigma)$ is convex.

4. In my opinion, the last condition is the greatest strength of theorem 3. Though the *closed graph theorem* doesn't suit here due to the multivaluedness, the fourth condition implies the continuity of $BR$ in some sense. Consider a sequence of mixed-strategy profiles $\{(\xi_i^n, \xi_{-i}^n)\}_{n=1}^{\infty}$ and a best-response sequence $\{(\sigma_i^n, \sigma_{-i}^n)\}_{n=1}^{\infty}$ where $(\sigma_i^n, \sigma_{-i}^n) \in BR(\xi_i^n, \xi_{-i}^n)$. The reason why we can do this is that by *Bolzano-Weierstrass theorem*, we can always find a convergent subsequence. So we let $\lim_{n\to\infty}(\xi_i^n, \xi_{-i}^n) = (\xi_i, \xi_{-i})$ and $\lim_{n\to\infty}(\sigma_i^n, \sigma_{-i}^n) = (\sigma_i, \sigma_{-i})$.

   To conclude that $BR(X)$ is closed, what's left is to verify $(\sigma_i, \sigma_{-i}) \in BR(\xi_i, \xi_{-i})$. By the continuity and the inequality $v_i(\sigma_i^n, \xi_{-i}^n) \geq (s_i, \xi_{-i}^n), \forall s_i \in \Delta S_i$ , we have $v_i(\sigma_i, \xi_{-i}) \geq (s_i, \xi_{-i}), \forall s_i \in \Delta S_i$. This implies that $\sigma_i \in BR_i(\xi_{-i})$. Since i is arbitrary, we prove that $(\sigma_i, \sigma_{-i}) \in BR(\xi_i, \xi_{-i})$.

By theorem 3 and the above detailed verifications, indeed, we prove the *Nash's Existence theorem.* However, as a student majoring in mathematics, I should try to find some materials to study the properties of multivalued functions, thus making up the gap between theorem 2 and theorem 3.

# 5 Typical Examples

In this section, three classic games with discrete strategy set are introduced to demonstrate the important results about NE. Among the games with interval strategy set, interesting instances won't be included, for example, *Tragedy of Commons* for computation, *Cournot Duopoly* for the uniqueness, *Bertrand Duopoly* for elaborate analysis of the existence, and *Electoral Competition* for application in political science.

## 5.1 Prisoner's Dilemma

**Description of background** The Prisoners Dilemma is perhaps the best-known example in game theory, and it often serves as a parable for many different applications in economics and political science. It is a static game of complete information that represents a situation consisting of two players who are suspects in a serious crime, say, armed robbery. The police has evidence of only petty theft, and to nail the suspects for the armed robbery they need testimony from at least one of the suspects. The police decides to be clever, separating the two suspects at the police station and questioning each in a different room. Each suspect is offered a deal that reduces the sentence he will get if he confesses, or finks **(F)**, on his partner in crime. The alternative is for the suspect to say nothing to the investigators, or remain mum **(M)**, so that they do not get the incriminating testimony from him.

The payoff of each suspect is determined as follows: If both choose mum, then both get **2** years in prison because the evidence can support only the charge of petty theft. If, say, player1 mums while player2 finks, then player1 gets **5** years in prison while player2 gets only **1** year in prison for being the sole cooperator. The reverse outcome occurs if player1 finks while player2 mums. Finally, if both fink then both get only **4** years in prison. Because it is reasonable to assume that more time in prison is worse, we use the payoff representation that equates each year in prison with a value of **-1**. Its payoff matrix is represented below:

| | | Prisoner 2 | |
|---|---|---|---|
| | | M | F |
| Prisoner 1 | M | -2, -2 | -5, -1 |
| | F | -1, -5 | **-4,-4** |

Table 4: Payoff matrix of *Prisoner's Dilemma*

For what I understand, the importance of *Prisoner's Dilemma* lies in the following aspects.

1. It can clearly illustrated the concept of strict dominance. In the game, for any player, the stategy **F** is always better with the opponent's strategy fixed. Based on strictly dominated strategy, IESDS can be introduced as the first solution concept.

2. As a sister solution concept, the *rationalizability* can arise from concept of the best-response. In the game, strategy **M** is never a best response to any belief, thus being eliminated in the process of obtaining *rationalizable strategies.*

3. Both of aforementioned two solution concepts require **rationality** and **common knowledge of rationality**, which are also the bases of NE.

4. Revisiting the *Perato optimality*, while strategy $(F, F)$ is the unique Nash equilibrium, we can find it is Perato dominated by stratety $(M, M)$.

From points 1-3, I can draw a conclution as a demo. In set the of all strategies,

$$\{s | \text{s is a NE}\} \subseteq \{s | \text{s is a } \textit{rationalizable solution}\} \subseteq \{s | \text{s survives IESDS}\}$$

As for the point 4, the failure of *Pareto optimality* implies that the players would benefit from modifying the environment in which they find themselves to create other enforcement mechanismsfor example, punishing those who fink.

To explicitly see how this can work, if the pain from punishment is equivalent to z, then we have to subtract z units of payoff for each player who finks. The modified Prisoner's Dilemma is represented by the following matrix

| | | Prisoner 2 | |
|---|---|---|---|
| | | M | F |
| Prisoner 1 | M | -2, -2 | -5, -1-z |
| | F | -1-z, -5 | **-4-z,-4-z** |

Table 5: *Modified Prisoner's Dilemma*

If z is strictly greater than 1, then this punishment will be enough to flip our predicted equilibrium outcome of the game, because then M becomes the strict dominant strategy, as well as NE and Pareto optimality. From this, the flavor of self-enforcement/non-cooperative is obvious. And *institutional design/cooperative contract* can guide players to a best situation, leading to the necessarity of managements from government.

## 5.2 Matching Pennies

In the matching pennies game, two players put two coins on the table simultaneously. If both heads or tails, player 2 wins while player 1 wins if the sides of coins are different. To compute the mixed-strategy Nash equilibrium, assign player 1 the probability $p$ of playing H and $(1-p)$ of playing T, and assign player 2 the probability $q$ of playing H and $(1-q)$ of playing T. The similar games include *rock-paper-scissors* and *Penalty Goal.*

$$v_1(H, q) = (-1)\, q + (+1)(1-q) = 1 - 2q$$

|  |  | Player 2 |  |
|---|---|---|---|
|  |  | H | T |
| Player 1 | H | -1, +1 | +1, -1 |
|  | T | +1, -1 | -1, +1 |

Table 6: Payoff Matrix of *Matching Pennies*

$$v_1(T,q) = (+1)q + (-1)(1-q) = 2q - 1$$

$$v_1(T,q) = v_1(H,q) \implies 1 - 2q = 2q - 1 \implies q = \frac{1}{2}$$

In the same way, $p = \dfrac{1}{2}$. Thus a mixed-strategy Nash equilibrium, in this game, is for each player to randomly choose H or T with $p = \dfrac{1}{2}$ and $q = \dfrac{1}{2}$. The above computation use a important proposition about NE. That is,

**Proposition 1** *If $\sigma^*$ is a Nash equilibrium, and both $s_i$ and $s_i'$ are in the support of $\sigma_i^*$, then*

$$v_i\left(s_i, \sigma_{-i}^*\right) = v_i\left(s_i', \sigma_{-i}^*\right) = v_i\left(\sigma_i^*, \sigma_{-i}^*\right)$$

Intuitively, the proposition implies the logic that if a player is mixing several strategies then he must be indifferent between them. This logic has been used implicitly in the verification of the convexity of $BR_i(\sigma_{-i})$ in section 4.

## 5.3 Multiple Equilibria

Consider the flollowing game:

|  |  | Player 2 |  |
|---|---|---|---|
|  |  | C | R |
| Player 1 | M | 0, 0 | **3, 5** |
|  | D | **4, 4** | 0, 3 |

Table 7: Payoff Matrix of *Matching Pennies*

It is easy to check that (M, R) and (D, C) are both pure-strategy Nash equilibria. It turns out that in $2 \times 2$ matrix games like this one, when there are two distinct pure-strategy Nash equilibria then there will almost always be a third one in mixed strategies.

For this game, a computaion similar to *Matching Pennies* yields the third Nash equilibrium:$\left(\sigma_i^*, \sigma_{-i}^*\right) = \left(\left(\frac{1}{6}, \frac{5}{6}\right), \left(\frac{3}{7}, \frac{4}{7}\right)\right)$.

Using the payoff functions $v_1(M,q)$ and $v_1(D,q)$ , we have

$$BR_1(q) = \begin{cases} p = 1 & \text{if} q \leq \frac{3}{7} \\ p \in [0,1] & \text{if} q = \frac{3}{7} \\ p = 0 & \text{if} q \geq \frac{3}{7}. \end{cases}$$

Similarly, the best response correspondence for player 2 is

$$BR_2(p) = \begin{cases} q = 1 & \text{if} p \leq \frac{1}{6} \\ q \in [0,1] & \text{if} p = \frac{1}{6} \\ q = 0 & \text{if} p \geq \frac{1}{6}. \end{cases}$$

Now, we draw the two best-response correspondences as they appear in Figure 2.

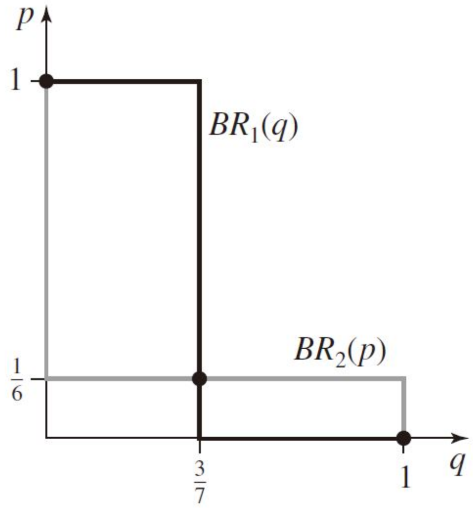


Figure 2: Best-response correspondences and Nash equilibria

Notice that all three Nash equilibria are revealed in Figure 2: $(p, q) \in \{(1, 0) \left(\frac{1}{6}, \frac{3}{7}\right) (0, 1)\}$. From the definition of NE in introduction section 1, we can conceive the NE are the intersection of the subsets constructed from the best-response correspondences of all the players in the set of all the possible strategy profiles.

# 6 Course Review

## 6.1 Graph Theory

Starting from the basic notations in graph theory,

**MST** We learned about the minimum spanning tree along with two classical algorithms both of which are greedy!

**SPP** Then we moved to another classical problem of shortest path along with *Dijkstra Algorithm* solving the problem with non-negative edge weights and *Bellman-Ford Algorithm* solving the problem in which edge weights allowed to be negative. With Matlab, I constructed typical examples for each of these two algoritms. The examples are attached as appendix 6.2.2 in the end.

**MFMC** Max-flow min-cut problem is another important problem in class. Professor Zhang proved the *Max-flow min-cut theorem*– **The maximum value of an s-t flow is equal to the minimum capacity over all s-t cut**, based on which the *Ford–Fulkerson algorithm* is introduced in class.

Residual graph $G_f\,(V, E_f)$ is introduced to find a argumenting path in the process of *Ford–Fulkerson algorithm*. A path from s to t in $G_f$ can be understood as a augmenting chain in the original graph $G\,(V, E)$ and the minimal weight on the path is the amount of flow you can increase through the corresponding augmenting chain. By iteration, you will reach a situation where you can not find a path from s(source) to t(sink) in $G_f$ any more.

Further reflection: Since the *Max-flow min-cut theorem* only ensure that if the algorithm terminate, the flow is maximal. But whether the algorithm will terminate in the end? As a question should be asked to every algorithm, what is the computation complexity of *Ford–Fulkerson algorithm*?

**MCMF** Min-cost max-flow problem is the final problem discussed in class, which is to find the maximum flow with the minimum cost. The first step in each iteration is to find the shortest path from s to t in the graph $G_e\,(V, E_e)$ with extra edges. The auxiliary graph $G_e$ allows to reduce the flow in some edge. Then flow $f_n$ is modified based on the corresponding shortest path in $G_e$ to increase the value of the flow $f_{n+1}$ of next iteration with minimum cost per unit flow.

Notice that the algorithm used to find shortest path should not be *Dijkstra Algorithm*, because weights in $G_e\,(V, E_e)$ can be negative.

**Remark** A quick reference to [8] shows that *max-flow min-cut theorem* is a special case of *strict dual theorem* in LP between the primal max-flow problem and the dual min-cut problem, strengthening the necessarity of the review of linear programming.

Graph theory itself is a heated research area nowadays. Apart from the wide applications, to my knowlege, my roommate Yuyao Zhai will aquire for a Phd setting graph theory as his researh interest. Also, I read a paper about fractal geometry[9] in which the techniques in graph theory are surprisingly used to calculate Hausdorff dimension of Bernoulli Convolution. Although, in that paper, it concentrates more on constructing a graph instead of studying a existed graph, the power and unbiquity of graph theory is remarkable. Of course, besides what is covered in class, a large amount of knowledge about graph theory is waiting for me!

## 6.2 Linear Programming

### 6.2.1 Simplex Method

For the convenience, the backgroud of the linear programming problem is omitted. However, one should not consider it as a consequence of the author's downplaying the role of modeling for specific problems, on the contrary, the ability to contruct a reasonable mathematical model is the first crucial step to solve a problem. As a famous saying from computer science goes, 'Garbage in, garbage out.' If we made a flawed assumption, the result of our model will be totally wrong.

The first step of LP is using slack variables, surplus variables and restricted variable subsitution to translate the original problem into a standard form. However, as a key to start *simplex algorithm*, the initial basic feasible solution is important, which is also emphasized by professor Zhang. Hence, in the following example, as is suggested by professor Zhang, artificial variables are used to solve this problem.

Consider the following linear program:

$$\begin{aligned} \text{Minimize:} \quad Z &= -2x - 3y - 4z \\ \text{Subject to:} \quad 3x + 2y + z &= 10 \\ 2x + 5y + 3z &= 15 \\ x, y, z &\geq 0 \end{aligned}$$

In the following, LP will be manipulated in tableau form. This tableau is slightly different from the class, but essentially the same.

**Step1** Translate the above LP into tableau form,

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 0 \\ 0 & 3 & 2 & 1 & 10 \\ 0 & 2 & 5 & 3 & 15 \end{bmatrix}$$

**Step2** Introduce artificial variables $u$ and $v$ and objective function $W = u + v$, giving a new tabeau,

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 2 & 3 & 4 & 0 & 0 & 0 \\ 0 & 0 & 3 & 2 & 1 & 1 & 0 & 10 \\ 0 & 0 & 2 & 5 & 3 & 0 & 1 & 15 \end{bmatrix}$$

Note that the equation defining the original objective function is retained in anticipation of Phase II of the *Two Phases Method* introduced in class.

**Step3** After pricing out by adding the last two rows to the top row, this becomes

$$\begin{bmatrix} 1 & 0 & 5 & 7 & 4 & 0 & 0 & 25 \\ 0 & 1 & 2 & 3 & 4 & 0 & 0 & 0 \\ 0 & 0 & 3 & 2 & 1 & 1 & 0 & 10 \\ 0 & 0 & 2 & 5 & 3 & 0 & 1 & 15 \end{bmatrix}$$

Then we start the pivot operation. The entering variable should be chosen in the free variables whose objective row is positive in a minimize problem. After selecting column 5 as pivot column(entering variable), the pivot row 4(leaving variable) should be selected based on the *minimum ratio test* suggested in class. The updated tableau is

$$\begin{bmatrix} 3 & 0 & 7 & 1 & 0 & 0 & -4 & 15 \\ 0 & 3 & -2 & -11 & 0 & 0 & -4 & -60 \\ 0 & 0 & 7 & 1 & 0 & 3 & -1 & 15 \\ 0 & 0 & 2 & 5 & 3 & 0 & 1 & 15 \end{bmatrix}$$

**Step4** Now, the column 4 and column 5 become the basic variables, so repeat the actions in Step3. Now select column 3 as a pivot column, for which row 3 must be the pivot row, to get

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 7 & 0 & -25 & 0 & 2 & -10 & -130 \\ 0 & 0 & 7 & 1 & 0 & 3 & -1 & 15 \\ 0 & 0 & 0 & 11 & 7 & -2 & 3 & 25 \end{bmatrix}$$

**Step5** The artificial variables are now 0 and they may be dropped giving a canonical tableau equivalent to the original problem:

$$\begin{bmatrix} 7 & 0 & -25 & 0 & -130 \\ 0 & 7 & 1 & 0 & 15 \\ 0 & 0 & 11 & 7 & 25 \end{bmatrix}$$

Since positive entries do not exist in objective row of all the variables, this is, fortuitously, already optimal and the optimum value for the original linear program is 130/7.

**Analysis** During the above steps of *simplex algorithm*, it explicitly announces the advantage of *Two Phases Methods* for providing a systematic way to transfer a standard form into canonical tableau. Hence it avoids the tricky ways to achieve this goal in various situations. Due to the applicability to computer, the author prefers to use artificial variables.

For the existence of solution proved in class, aided with knowledge in Real Analysis[6], I think the '$\leq$' of constraints plus the linearity of the map $A$ implies the convexity. As $R^n$ is a Banach space thus locally convex, if the feasible region is compact, all the conditions needed in *Krein-Milman theorem* are satisfied. There should exist extreme points in feasible region, and the region is the convex hull of extreme points, namely $\overline{co}\{$Basic feasible solutions$\}$. I think this result is translated into another version as 'basic theorems in LP' in class. Surely this observation is not rigid and lacks further consideration. However, as a student, if the abstract theory can be applied here, the range of similar optimization problems we can deal with will essentially widen, just like how many other applied mathematics developed.

#### 6.2.2 About the Dual

Inspired by Sir Michael Francis Atiyah who treats mathematics as science of analogy, I have been trying to distill the similarity from my limited mathematical knowledge. The word 'dual' appears several times hitherto in my journey of mathematics exploration. Along with other related concepts like bidual, reflexivity, duality in category theory... **dual** seems to inpire me to avoid thinking the problem isolatedly by taking the functionals into consideration, which might help produce more fruitful results. If we ignore the restrictions and step even further, many concepts or theories in mathematics come in pairs like *A and co-A*, *local B and global B*...Why? I don't know, or even I should not ask this question and just focus on learning.

Back to LP, the standard form of duality in LP is presented below.

$$\textbf{Primal}\text{ problem:}\quad c^T x \text{ subject to } Ax \leq b, x \geq 0; \tag{1}$$

$$\textbf{Dual}\text{ problem:}\quad b^T y \text{ subject to } A^T y \geq c, y \geq 0; \tag{2}$$

**Weak duality:**

**Proof.** Last week in class, with a recall of Functional Analysis [7], the *weak duality* can be obtained in a simple way.

Considering two Banach spaces $X$ and $Y$, for convenience, I use $\langle x, x^* \rangle$ to denote that the continous linear functional $x^*$ in $X^*$, dual of $X$, applies to a point $x$ in $X$. The same goes for $Y$.

By the *Riesz–Fisher Theorem*, $R^n$ as a Hilbert space is reflexive. Hence the $\langle x, x^* \rangle$ could be understood as inner product in corresponding space for simplicity. Also, besides being a mapping from $Y^*$ to $X^*$, $A^T$ becomes the conjugate of $A$ from $Y$ to $X$.

$x$, as a functional, is positive means $\forall \alpha \geq 0 \in X, \langle \alpha, x \rangle \geq 0$. Also $A^T y - c \geq 0 \in X$ implies,

$$\langle A^T y - c, x \rangle \geq 0 \implies \langle A^T y, x \rangle \geq \langle c, x \rangle$$

Similarly,

$$\langle b - Ax, y \rangle \geq 0 \implies \langle b, y \rangle \geq \langle Ax, y \rangle$$

By the definition of conjugate,

$$\langle A^T y, x \rangle = \langle Ax, y \rangle \tag{3}$$

Finally,

$$\langle b, y \rangle \geq \langle Ax, y \rangle = \langle A^T y, x \rangle \geq \langle c, x \rangle$$

■

**Analysis** One should argue that the direct use of inner product will get the same result faster. It cannot be denied, but I strongly contend that we should be aware that the inner products in the equation (3) are in different Hilbert spaces $X$ and $Y$.

**Strong duality** Strong duality indicates the gap between $c^T x_s$ of primal problem and $b^T y_s$ of dual problem is zero. In the class of game theory, professor Zhang used strong duality to prove the existence of von Neumann's equilibrium in finite zero-sum game with mixed-strategy and improved *simplex algorithm* in the last two classes.

In a word, according to the contents of former sections, linear programming is a crucial tool in both game theory and graph theory, as well as a key theme in operation research. In my opinion, LP deserves my further efforts.

# Further Notes

- **Advantages** Through classic examples and original reflections, I manage to present what I have learned about Nash equilibrium in a natural way. Moreover, by reviewing course and making connections, I obtain a deeper understanding of what is covered in class.

- **Disadvantages** Examples used in article are simple and contain little computation. Unfamiliar with the correct tone in formal academic writing, I surely have made many mistakes and typos. Most of my arguments are intuitive and lack rigid inductions.

- **Prospective** Apart from improving the knowledge structure about operation research, I will try to become more proficient in LaTeX. Plenty preparation should be down to include more mathemetical proof in the article.

# Acknowledgements

The author would like to thank professor Zhang for her heuristic and comprehensive class which lays out a basic ground of operation research for my future study. Also, I really appreciate this opportunity to implement my first academic writing in English, as well as the support from my roommate Yuyao Zhai who strengthened my confidence to make the first step. And thanks my friend Jingyuan Liu for being willing to scrutinize this article. Finally, since optimization method class might be the last undergraduate course I can attend, I would like to express my sincere gratitude to all the instructors in School of Mathematics at HUST who help me grow into who I am.


# References


[1] Steven Tadelis. Game theory: an introduction, Princeton University Press, USA, 2013. ISBN 978-0-691-12908-2 (hbk. : alk. paper), HB144.T33

[2] Fudenberg, Drew & Tirole, Jean. (2018). Game theory/ Drew Fudenberg, Jean Tirole. SERBIULA (sistema Librum 2.0).

[3] Mihai Manea. 14.126 Game Theory. Spring 2016. Massachusetts Institute of Technology: MIT OpenCourseWare, https://ocw.mit.edu. License: Creative Commons BY-NC-SA.

[4] Ben Polak. Game Theory. Fall 2007. Yale University: Open Yale Courses. https://oyc.yale.edu/economics/econ-159 Course Number: ECON 159

[5] Cabrales, Antonio. (2018). Nash and game theory.

[6] Royden, H. L., and Patrick Fitzpatrick. Real Analysis. Pearson, 2018.

[7] Rudin, Walter. Functional Analysis. McGraw-Hill, 2006.

[8] Bondy, J.A, and U.S.R Murty. Graph Theory. 1st ed., ser. 0072-5285, Springer-Verlag London, 2008. Graduate Texts in Mathematics. GTM 244

[9] Shigeki Akiyama, De-Jun Feng, Tom Kempton, Tomas Persson; On the Hausdorff Dimension of Bernoulli Convolutions, International Mathematics Research Notices, , rny209, https://doi.org/10.1093/imrn/rny209

[10] Wikipedia contributors. "Nash equilibrium." Wikipedia, The Free Encyclopedia. Wikipedia, The Free Encyclopedia, 2 Nov. 2018. Web. 11 Nov. 2018.

[11] Wikipedia contributors. "Ultimatum game." Wikipedia, The Free Encyclopedia. Wikipedia, The Free Encyclopedia, 25 Jul. 2018. Web. 11 Nov. 2018.

[12] Wikipedia contributors. ”Max-flow min-cut theorem.” Wikipedia, The Free Encyclopedia. Wikipedia, The Free Encyclopedia, 31 Oct. 2018. Web. 11 Nov. 2018.

[13] Wikipedia contributors. ”Linear programming.” Wikipedia, The Free Encyclopedia. Wikipedia, The Free Encyclopedia, 30 Oct. 2018. Web. 11 Nov. 2018.

# Appendix

# Homework of graph theory

Weekly Homework 1 Math Class: Optimization Methods
Zhou FENG
U201510104 November 14, 2018

# Exercise 1

Use the Dijkstra algorithm and the successive approximation algorithm taught in class to find the minimal distances in a graph.

## Dijkstra Algorithm

### Discription of problem

The target undirected graph $G_1 = (V, E, W)$ is showed below as an image plotted by matlab and an adjacent matrix.

| | v1 | v2 | v3 | v4 | v5 | v6 | v7 | v8 | v9 |
|---|---|---|---|---|---|---|---|---|---|
| v1 | 0 | 6 | 3 | Inf | Inf | Inf | Inf | Inf | Inf |
| v2 | 6 | 0 | 2 | Inf | 3 | 2 | Inf | Inf | Inf |
| v3 | 3 | 2 | 0 | 1 | Inf | 3 | Inf | Inf | Inf |
| v4 | Inf | Inf | 1 | 0 | 5 | 7 | 3 | Inf | Inf |
| v5 | Inf | 3 | Inf | 5 | 0 | 4 | 3 | Inf | 2 |
| v6 | Inf | 2 | 3 | 7 | 4 | 0 | Inf | 6 | Inf |
| v7 | Inf | Inf | Inf | 3 | 3 | Inf | 0 | 5 | Inf |
| v8 | Inf | Inf | Inf | Inf | Inf | 6 | 5 | 0 | 2 |
| v9 | Inf | Inf | Inf | Inf | 2 | Inf | Inf | 2 | 0 |

Table 8: Adjacent Matrix of $G_1$

Then apply the Dijkstra algorithm to find the minimal distance between $V_1$ and other vertices.

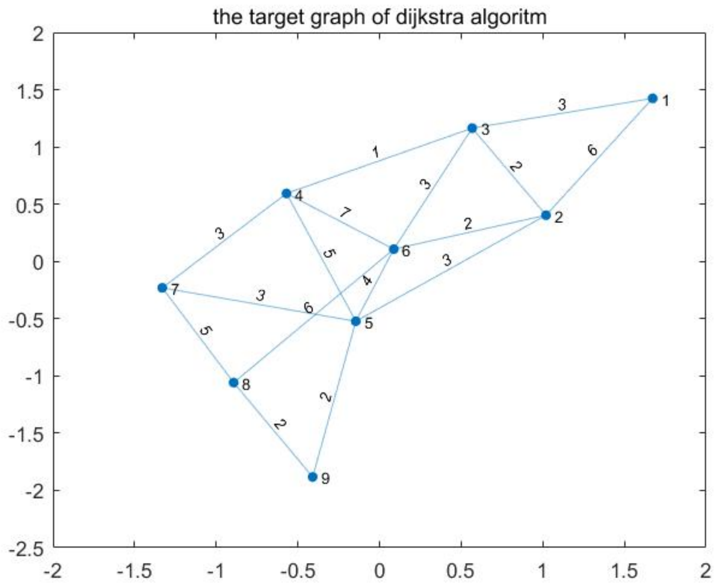


Figure 3: The plotted $G_1$

### Solution

Since the weight attached to each edge of $G_1$ is nonnegative, the principle that the Dijkstra algorithm is based on is assued. Hence the Dijkstra algorithm can be applied properly. The final result is listed below.

| | Minimal Distance | Path |
|---|---|---|
| v2 | 5 | 1 3 2 |
| v3 | 3 | 1 3 |
| v4 | 4 | 1 3 4 |
| v5 | 8 | 1 3 2 5 |
| v6 | 6 | 1 3 6 |
| v7 | 7 | 1 3 4 7 |
| v8 | 12 | 1 3 6 8 |
| v9 | 10 | 1 3 2 5 9 |

Table 9: The result of Dijkstra algorithm

## Bellman-Ford Algorithm

### Discription of problem

The target directed graph $G_2 = (V, E, W)$ is showed below as an image plotted by matlab and an adjacent matrix.

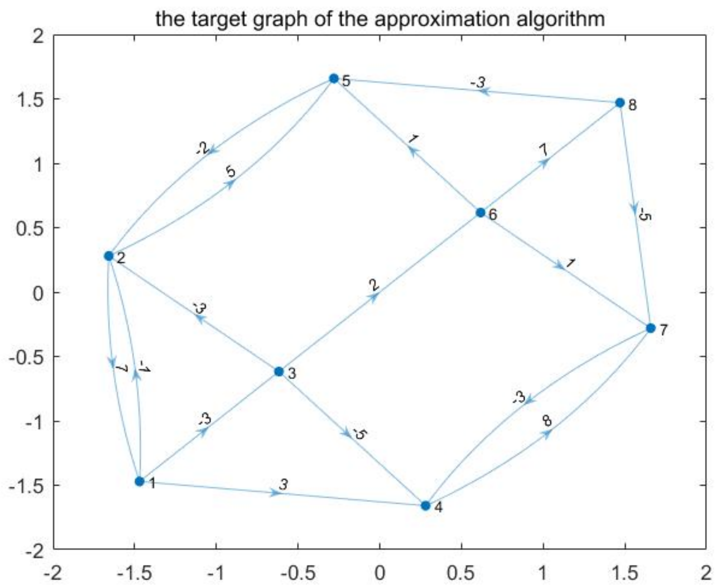


Figure 4: The plotted $G_2$

Then apply the successive approximation algorithm to find the minimal distance between $V_1$ and other vertices in $G_2$.

**Solution**

The result is listed below.

| Vertices | v1 | v2 | v3 | v4 | v5 | v6 | v7 | v8 | t=1 | t=2 | t=3 | t=4 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| v1 | 0 | -1 | -3 | 3 | Inf | Inf | Inf | Inf | 0 | 0 | 0 | 0 |
| v2 | 7 | 0 | Inf | Inf | 5 | Inf | Inf | Inf | -1 | -6 | -6 | -6 |
| v3 | Inf | -3 | 0 | -5 | Inf | 2 | Inf | Inf | -3 | -3 | -3 | -3 |
| v4 | Inf | Inf | Inf | 0 | Inf | Inf | 8 | Inf | 3 | -8 | -8 | -8 |
| v5 | Inf | -2 | Inf | Inf | 0 | Inf | Inf | Inf | Inf | 4 | -1 | -1 |
| v6 | Inf | Inf | Inf | Inf | 1 | 0 | 1 | 7 | Inf | -1 | -1 | -1 |
| v7 | Inf | Inf | Inf | -3 | Inf | Inf | 0 | Inf | Inf | 11 | 0 | 0 |
| v8 | Inf | Inf | Inf | Inf | -3 | Inf | -5 | 0 | Inf | Inf | 6 | 6 |

Table 10: The result of the algorithm

# Matlab Code

## function: main

## Contents

- set the target graph of successive approximation method

- plot the graph
- the dijkstra algorithm
- set the target graph of successive approximation method
- plot the graph
- the successive approxiamtion algoritm

```
clc,clear
```

## set the target graph of successive approximation method

```
%(mainly the adjacent matrix)
M=[1 6 3 0 0 0 0 0 0;
0 1 2 0 3 2 0 0 0;
0 0 1 1 0 3 0 0 0;
0 0 0 1 5 7 3 0 0;
0 0 0 0 1 4 3 0 2;
0 0 0 0 0 1 0 6 0 ;
0 0 0 0 0 0 1 5 0;
0 0 0 0 0 0 0 1 2;
0 0 0 0 0 0 0 0 1];
```

## plot the graph

```
figure(1)
Grph= graph(M,'upper','OmitSelfLoops');
plot(Grph,'EdgeLabel',Grph.Edges.Weight)
title('the target graph of dijkstra algoritm')
```

## the dijkstra algorithm

```
D=M+M';
D(find(D==0))=inf;
D=D-diag(diag(D));
for i=2:9
[mydistance mypath]=mydijkstra(D,1,i);
end
```

## set the target graph of successive approximation method

```
%(mainly the adjacent matrix)
M=[1 -1 -3 3 0 0 0 0;
```

```
7 1 0 0 5 0 0 0;
0 -3 1 -5 0 2 0 0;
0 0 0 1 0 0 8 0;
0 -2 0 0 1 0 0 0;
0 0 0 0 1 1 1 7;
0 0 0 -3 0 0 1 0;
0 0 0 0 -3 0 -5 1];
```

## plot the graph

```
figure(2)
Grph= digraph(M,'OmitSelfLoops');
plot(Grph,'EdgeLabel',Grph.Edges.Weight)
title('the target graph of the approximation algorithm')
```

## the successive approxiamtion algoritm

```
S=M;
S(find(S==0))=inf;
S=S-diag(diag(S));
stepmat=mystepsapprox(S,1,8)
```

```
stepmat =


0     0     0     0
-1    -6    -6    -6
-3    -3    -3    -3
3    -8    -8    -8
Inf     4    -1    -1
Inf    -1    -1    -1
Inf    11     0     0
Inf   Inf     6     6
```

## function: mydijkstra

```
function [mydistance mypath]=mydijkstra(a,sb,db)
% input:aadjacent matrix
% sbinitial point, dbterminal point
% out putmydistanceminmun distance, mypathminmum path
n=size(a,1); visited(1:n)=0;
distance(1:n)=inf; % store the distance
```

```
distance(sb)=0; parent(1:n)=0;
for i=1:n-1
temp=distance;
id1=find(visited==1); %find the marked point
temp(id1)=inf; %put the distance to the marked point to infinity
[t, u] = min(temp); %find the point with minmum marked number
visited(u) = 1; %remember the marked point
id2=find(visited==0); %find the unmarked point
for v = id2
if a(u, v) + distance(u) < distance(v)
distance(v) = distance(u) + a(u, v); %change the number of the point
parent(v) = u;
end
end
end
mypath=[];
if parent(db)~= 0 %if there exists the path
t = db; mypath = [db];
while t~= sb
p=parent(t);
mypath=[p mypath];
t=p;
end
end
mydistance=distance(db);
return
end
```

### function: mystepsapprox

```
function stepmat = mystepsapprox( ad, spoint, tpoint )
% input:adadjacent matrix
% spointinitial point, tpointterminal point
% outputstepmat- the matrix in the procedure to find the minimal distance
gdegree=length(ad(1,:));
stepmat=ad(spoint,:)';
step=1;
while 1
tempvector=inf*ones(gdegree,1);
```

```
canreach=find(stepmat(:,step)<inf);
%templength=length(canreach);
for i=1:gdegree
canback=find(ad(:,i)<inf);
findex=intersect(canreach,canback);
if ~isempty(findex)
flagvector=stepmat(findex,step)+ad(findex,i);
tempvector(i)=min(flagvector);
else
tempvector(i)=inf;
end
end
stepmat=[stepmat tempvector];
if norm(stepmat(:,step)-tempvector)==0
break
end
step=step+1;
end
```