Adaptive output feedback consensus tracking for heterogeneous multi-agent systems with unknown dynamics under directed graphs

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ABSTRACT
This paper considers the distributed consensus tracking problem for the linear multi-agent systems with unknown dynamics under directed graphs. Based on the output information among the agents, distributed adaptive consensus tracking protocols together with two observers, consisting of a local observer and an adaptive estimator, are designed to guarantee that all the signals in the closed-loop dynamics are uniformly ultimately bounded and the tracking errors converge to a small neighborhood around the origin. Moreover, the consensus protocols are in fully distributed fashion in the sense that the coupling gains in the controllers are independent without requiring the global knowledge that the eigenvalues of the Laplacian matrix associated with the whole communication graph. Finally, a simulation example is provided to verify the theoretical results.

1. Introduction

In recent years, coordination control of multi-agent systems has gained considerable interest partially due to its potential applications in many areas such as formation flying of unmanned air vehicles, cooperative surveillance, cooperative manipulation of multiple robots, distributed sensor networks [1–4], etc. In the previous literature, consensus control problem is a fundamental issue since the state agreement is often the premise for many coordination behaviors. The key objective of consensus control is to design distributed protocols, meaning that only the local state or output information is required, to achieve certain global task.

In the pioneering work [5], a framework of consensus control for the first-order integrators with different topologies and time delays has been established. Since then, the consensus control has been studied intensively in different directions such as quantized consensus [6,7], finite-time consensus [8,9], consensus with input or communication time delays [10,11], consensus with different dynamics [12–16]. In terms of the number of leaders, the above research works roughly fall into two classes, that is, leader-following consensus (or consensus tracking) with a reference to determine the final agreement value, and leaderless consensus whose agreement value depends on the initial states of the agents. Thus, the consensus tracking control has the advantages to determine the consensus value in advance, especially in some tasks, for example, to avoid hazardous obstacle.

In this paper, we consider the distributed consensus control problem for the general linear multi-agent systems with unknown dynamics in the agents. Related works on adaptive consensus control with unknown dynamics include [17–26], where the unknown nonlinear dynamics in the leader or the followers are linearly parameterized to design the consensus protocols. In [17], the authors considered the leaderless consensus problem for the single and double integrator-type systems with unknown dynamics under the undirected communication topology. Then in [18], the consensus tracking problem for the single integrator systems was solved with the directed graphs, but the controller required the relative velocity and position simultaneously. Under a strongly-connected communication graph, [19–21] studied the consensus tracking problem in a similar idea for the first-, second- and high-order integrator-type systems, respectively. In [22], adaptive consensus protocols were proposed for the single-integrator systems by exactly linearly parameterizing the unknown dynamics in the leader and followers. Among the aforementioned works, one common feature is that the consensus protocols are designed based on the state information which is not always available in practice. Moreover, another feature is that the dynamics of the agents focus on the first-order, second-order or high-order integrator-type systems which are special cases of the more general linear multi-agent systems. In [23,24], the output feedback
consensus protocols were designed, respectively, for the double integrators and linear multi-agent systems with unknown dynamics. However, one limitation in [23,24] is that the control gains in the consensus protocols depend on the eigenvalues of the Laplacian matrix associated with the whole communication topology which is actually the global information of the network. Besides, such a limitation also exists in the controllers in [19–21]. By virtue of the adaptive coupling gain technique in [27–29], the state-feedback and output-feedback consensus tracking protocols have been designed in [25,26], respectively, without knowing the Laplacian matrices’ eigenvalues. However, both of the protocols are only applicable to the undirected communication topology. Recently, adaptive consensus protocols have been designed under directed graphs for the second-order integrator-type systems [30] and general linear multi-agent systems [31], but both of them are state-feedback controllers. In [32], the authors proposed a new sequential observer approach to deal with the output feedback consensus for the linear multi-agent systems on directed graphs, but the dynamics focus on the nominal linear systems without considering the heterogeneous unknown dynamics in the followers.

Motivated by the aforementioned literature, in this paper, we aim to design fully distributed adaptive output-feedback consensus protocols for the multi-agent systems with unknown dynamics under the directed communication topology. Based on the output information among the agents, the observer-based adaptive protocols are designed for the leader-following consensus under the directed graph with a spanning tree. The main contributions of this paper are fourfold. Firstly, the protocols are designed for the general linear multi-agent systems which include the single, double, and high-order integrator-type systems [17–21] as its special cases. Secondly, compared with [17–21,25], in which the protocols are designed based on the state information among the agents, the observer-based adaptive controllers in this paper are developed by using the output information which is more practical. Moreover, the control gains in the protocols are independent of eigenvalues of the Laplacian matrix associated with the information-exchange graph, thus the protocols are in fully distributed fashion in the sense that only the local information is required without knowing global knowledge of the entire graph a priori. In [23,24], although the output-feedback protocols are developed, the parameters in the controllers are determined by the eigenvalues of the Laplacian matrix. Finally, the protocols in the present paper are applicable to more general directed graphs.

The paper is organized as follows. In Section 2, some preliminaries and the consensus problem are introduced. In Section 3, observer-based protocols are developed for the consensus tracking problem. A simulation example is given in Section 4 and the concluding remarks are provided in Section 5.

2. Preliminaries and problem statement

2.1. Notations

In this paper, let $I_n$ and $1$ denote the identity matrix of dimension $n$ and a column vector with all entries equal to one, respectively. Let $R^{n \times m}$ represent a set of $n \times m$ real matrices, and $O_{n \times m}$ denote the matrices with all zeros. Given a real vector $x \in \mathbb{R}^n$, $||x||$ is the Euclidean norm of $x$, and for a matrix $A$, $||A||_F$ denotes the Frobenius norm that is defined by $||A||_F = \sqrt{\text{tr}(A^T A)}$, where $\text{tr}(\cdot)$ denotes the trace of a matrix. For a matrix $P$, $\lambda_{\text{min}}(P)$, $\lambda_{\text{max}}(P)$ represent its minimum and maximum eigenvalue, respectively, and $\sigma_{\text{min}}(P)$, $\sigma_{\text{max}}(P)$ denote its minimum and maximum singular value, respectively. Given two matrices $X$ and $Y$, $X \otimes Y$ denotes the Kronecker product of the matrices with the following properties that $X(Y \otimes Y) = (\lambda_iX)\sigma_i(Y)$ and $(X \otimes Y) = [\sigma_i(X)\sigma_i(Y)]$, diag($A$) denotes a block-diagonal matrix with $A_{ii}$, $i = 1, \ldots, N$, on the diagonal. Given two symmetric real matrices $A$ and $B$, $A > B$ denotes that $A - B$ is positive definite.

2.2. Graph theory

In this paper, the communication topology among the agents is denoted by the directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$, where $\mathcal{V} = \{1, \ldots, N\}$ is the node set, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the edge set consisting of ordered pair of distinct nodes and $A = [a_{ij}] \in \mathbb{R}^{N \times N}$ is the adjacency matrix associated with the communication graph. In the directed graph $\mathcal{G}$, $(i, j) \in \mathcal{E}$ means that the $j$th agent has access to $i$th agent’s information, but not vice versa. A directed path from node $i_1$ to node $i_n$ is a sequence of edges of the form $(i_1, i_2), \ldots, (i_{n-1}, i_n)$. The communication graph $\mathcal{G}$ is strongly connected if there is a directed path from every node to the other node. A directed graph that contains a spanning tree is that there is a node called the root that has no parent node, and the root has a directed path to every other node of the graph. For the adjacent matrix $A = [a_{ij}]$, $a_{ij} = 1$ if $(j, i) \in \mathcal{E}$, otherwise is zero. It is assumed that there is no self-edge, that is, $a_{ii} = 0$ for all nodes. The Laplacian matrix $L = [L_{ij}]$ associated with the graph $\mathcal{G}$ is defined as $L_{ij} = \sum_{k=0}^{N} a_{ik} - a_{ij}$, $i \neq j$.

**Lemma 1** ([33]). The Laplacian matrix $L$ associated with graph $\mathcal{G}$ has the property that it has a simple zero eigenvalue with vector $1$ as a corresponding right eigenvector and all other eigenvalues have positive real parts if and only if $\mathcal{G}$ contains a directed spanning tree.

**Assumption 1.** The graph $\mathcal{G}$ contains a directed spanning tree with the leader as the root node.

Because the leader has no neighbors, the Laplacian matrix $L$ of $\mathcal{G}$ has the following structure

$$L = \begin{bmatrix} 0 & 0_{1 \times N} \\ L_2 & L_1 \end{bmatrix},$$

where $L_1 \in \mathbb{R}^{N \times N}$ and $L_2 \in \mathbb{R}^{N \times 1}$. Under Assumption 1, zero is a simple eigenvalue of the Laplacian matrix $L$ by virtue of Lemma 1. Thus it is obvious that $L_1$ is a nonsingular $M$-matrix which has the following property:

**Lemma 2** ([31,34]). For the nonsingular $M$-matrix $L_1$, there exists a diagonal matrix $G = \text{diag}(g_1, \ldots, g_N)$ with $g_j > 0$, $i = 1, \ldots, N$ such that $G L_1 + L_1^T G = \lambda_1^* > 0$. And the positive definite matrix $G$ can be given with $[g_1, \ldots, g_N]^T = (\lambda_1^*)^{-1}$.

**Lemma 3** ([27,35]). For a system $\dot{x} = g(x, t)$, in which $g(\cdot)$ is locally Lipschitz in $x$ and piecewise continuous in $t$, and it is assumed that there exists a continuously differentiable function $V(x, t) \geq 0$ such that the derivative along the trajectory of the system,$$
K_1(\|x\|) \leq V(x, t) \leq K_2(\|x\|),$$

$$\dot{V}(x, t) \leq K_3(\|x\|) + \epsilon$$

where the constant $\epsilon > 0$, $K_1$, $K_2$ belong to class $\mathcal{K}_\infty$ functions, and $K_3$ belongs to class $\mathcal{K}$ function. Then, the solution $x(t)$ of the system is uniformly ultimately bounded.

**Lemma 4.** For the matrices $W$, $\tilde{W}$, $\hat{W} \in \mathbb{R}^{m \times n}$ with the relation that $\hat{W} = W - \tilde{W}$, then

$$\text{tr}(\tilde{W}^T \hat{W}) = \frac{1}{2} \text{tr}(\tilde{W}^T \tilde{W} + \hat{W}^T \hat{W} - \hat{W}^T W).$$

**Lemma 5** (Young’s Inequality [36]). For nonnegative real numbers $a$, $b$, if $p$, $q$ are real numbers that satisfy $\frac{1}{p} + \frac{1}{q} = 1$, then $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$. 
2.3. Problem statement

Consider a group of $N+1$ agents, consisting of $N$ followers and one leader labeled by 0. The dynamics of the $i$th follower, $(i = 1, 2, \ldots, N)$ are modeled as

\[
\dot{x}_i = A x_i + B [u_i + f_i (x_i)],
\]
\[
y_i = C x_i, \quad i = 1, \ldots, N, \tag{2}
\]

where $x_i \in \mathbb{R}^n$ denotes the state, $u_i \in \mathbb{R}^m$ is the control input, $y_i \in \mathbb{R}^m$ is the measured output, $f_i (x_i)$ is an unknown smooth nonlinear function, and $A, B, C$ are constant matrices with compatible dimensions. Also, the leader’s dynamics are modeled as follows

\[
\dot{x}_0 = A x_0 + B f_0 (x_0),
\]
\[
y_0 = C x_0, \tag{3}
\]

where $x_0 \in \mathbb{R}^n$ denotes the state of the leader, $y_0 \in \mathbb{R}^m$ is the leader’s measured output, and $f_0 (x_0)$ is the unknown control input which is bounded with $\|f (x_0)\|_\infty \leq r_0$.

**Assumption 2.** For the multi-agent systems, the matched unknown dynamics $f_i (x_i)$ in the follower can be linearly parameterized as

\[
f_i (x_i) = W_i \phi_i (x_i) + \epsilon_i (x_i), \tag{4}
\]

where $W_i \in \mathbb{R}^{m \times p}$ is an unknown constant matrix to be adaptive estimated that satisfies $\|W_i\| \leq M_W$, $\phi_i (x_i) \in \mathbb{R}^p$ is the basis function vector with $\|\phi_i\| \leq M_\phi$, and $\epsilon_i (x_i)$ is the residual error vector to be arbitrary small bounded with $\|\epsilon_i\| \leq M_\epsilon$.

The control objective of tracking problem in this paper is to design distributed consensus protocols with output information for each follower such that the state of each follower converges to that of the leader, i.e., $x_i (t) - x_0 (t) \to 0$ as $t \to \infty$ for any initial condition of $x_i (0), i = 1, \ldots, N$, with bounded errors.

**Remark 1.** The unknown dynamics in the followers can be seen as nonlinear functions which can be approximated by the neural network that has the linear-in-the-weight property. Besides, the parameterized expression is widely adopted in many literature such as [37,38] in classical adaptive control and [21–23] in multi-agent systems.

3. Distributed adaptive consensus tracking protocols design

In this section, based on the output information of neighboring agents, a distributed dynamic output-feedback controller with the independent coupling gain is proposed as follows

\[
u_i = c K \hat{\theta}_i - \hat{W}_i \phi_i (\hat{x}_i), \tag{5}
\]

where $c \in \mathbb{R}$ is the coupling gain independent of the Laplacian matrix, $\hat{\theta}_i$ is the state of the following adaptive estimator with adaptive couplings as

\[
\begin{align*}
\dot{\hat{\theta}}_i &= A \hat{\theta}_i + c B K \hat{\theta}_i + L (C \hat{x}_i - y_i) + \rho (\alpha_i + \beta_i) B K (e_i - \xi_i), \\
\dot{\xi}_i &= - \sigma_i (e_i - 1) + (e_i - \xi_i) \Gamma (e_i - \xi_i), \\
\beta_i &= (e_i - \xi_i)^T P^{-1} (e_i - \xi_i),
\end{align*} \tag{6}
\]

and $\hat{x}_i$ is the state of the following local state observer as

\[
\begin{align*}
\dot{\hat{x}}_i &= A \hat{x}_i + C B K \hat{\theta}_i + L (\hat{y}_i - y_i) \\
\dot{\hat{y}}_i &= C \hat{x}_i,
\end{align*} \tag{7}
\]

where $\sigma_i > 0$ is a small design parameter, $\rho > 0$ is a constant scalar which is included to tune the bounded control input of the leader, $e_i = \sum_{j=1}^{N} a_{ij} (\theta_i - \hat{\theta}_i) + a_{0i} \theta_i$, $\xi_i = \sum_{j=1}^{N} a_{ij} (\hat{x}_i - \hat{x}_j) + a_{0i} (\hat{x}_i - x_0)$. $\Gamma = P^{-1} B B^T P^{-1}$, $L = - \Omega^{-1} C^T$, $K = - B^T P^{-1}$, and $P > 0$, $Q > 0$ are the positive solutions of the following LMIs:

\[
AP + PA^T - 2 c B B^T < 0 \tag{8}
\]

\[
\begin{bmatrix}
A^T Q + QA & - C^T Q B & - Q B \\
\end{bmatrix}
\begin{bmatrix}
C^T - Q B \\
B^T Q \\
0 - d I
\end{bmatrix} < 0 \tag{9}
\]

where $\mu > 0$, $d > 0$, $c > 0$, $l > 0$ are independent positive scalars.

Moreover, the update laws are designed based on the output information as

\[
\dot{\hat{W}}_i = \tau_i [- \hat{y}_i - y_i] \psi_i^T (\hat{x}_i) - \frac{1}{2 \mu} \hat{W}_i \phi_i (\hat{x}_i) \psi_i^T (\hat{x}_i) - k_i \hat{W}_i, \tag{10}
\]

where $\tau_i$ is the adaptive gain, and $k_i > 0$ is a small design parameter.

Before moving forward, let $\hat{x}_i = \hat{x}_i - x_i$, $\delta_i = \hat{x}_i - x_0$, as the estimation error and estimated tracking error, respectively, then with (2), we can get

\[
\begin{align*}
\dot{\hat{x}}_i &= (A + LC) \hat{x}_i (t) - B [\hat{W}_i \phi_i (\hat{x}_i) - \hat{W}_i \phi_i (\hat{x}_i) + \epsilon_i], \\
\hat{\delta}_i &= A \hat{\delta}_i (t) + c B K \hat{\theta}_i L C \hat{x}_i - B \hat{f}_0
\end{align*} \tag{11}
\]

where $\hat{\phi}_i = \phi_i (\hat{x}_i) - \phi_i (x_0)$, $\hat{W}_i = \hat{W}_i - W_i$. Let $\bar{x} = [\bar{x}_1^T \ldots \bar{x}_N^T]^T$, $\bar{\xi}_N^T (t)$, $\bar{\delta}_N^T (t)$, $\bar{\xi} (t) = [\bar{\xi}_1^T \ldots \bar{\xi}_N^T]^T$, $e (t) = [e_1^T \ldots e_N^T]^T$. Then we have the fact that

\[
\begin{align*}
\bar{\xi} (t) &= (L_1 \otimes I_N) \bar{\delta} (t), \\
e (t) &= (L_1 \otimes I_N) \bar{\theta} (t)
\end{align*} \tag{12}
\]

where $L_1$ is defined in (1). As $L_1$ is nonsingular according to Lemma 1, $e (t)$ are similar to $\delta (t), \bar{\theta} (t)$, respectively.

Then we obtain the dynamics of $\bar{x}_i, \bar{\xi}_i, e_i$ in the compact form as

\[
\begin{align*}
\dot{\bar{x}} &= \{I_N \otimes (A + LC) \bar{x} - (I_N \otimes B) [\hat{W}_i \phi_i (\hat{x}_i) - \hat{W}_i \phi_i (\hat{x}_i) + \epsilon_i], \\
\dot{\bar{\xi}} &= \{I_N \otimes \bar{\xi} - (I_N \otimes B) \bar{\theta} + \epsilon_i, \\
\dot{e} &= \{I_N \otimes (A + CBK) \} e + (L_1 \otimes C) \bar{x} - (L_1 \otimes B) \bar{f}_0
\end{align*} \tag{13}
\]

where $\bar{\theta} = \text{diag}(\hat{W}_1, \ldots, \hat{W}_N), \bar{\xi} = \text{diag}(\hat{W}_1, \ldots, \hat{W}_N), \psi (\bar{x}) = \text{col}(\psi (\hat{x}_1), \ldots, \psi (\hat{x}_N)) \in \mathbb{R}^{m \times 1}$, $\hat{\psi} = \text{col}(\hat{\psi}_1, \ldots, \hat{\psi}_N) \in \mathbb{R}^{m \times 1}$, and $\bar{\alpha} = \text{diag}(\alpha_1, \ldots, \alpha_N), \bar{\beta} = \text{diag}(\beta_1, \ldots, \beta_N)$.

**Remark 2.** The consensus tracking protocols consist of (5)–(7) and (10), in which Eq. (7) is a local observer to estimate the state of the system. Eq. (10) is the update law to estimate the unknown constant matrix $W_i$. The main idea lies in the adaptive estimator (6) with dynamic coupling gain, which has improved the order of the system and provides extra freedom for the controller design.

The following theorem states the main result of this section.

**Theorem 6.** Suppose Assumption 1 holds, then with the dynamic output-feedback controller (5)–(7) and the adaptive update law (10), all the closed-loop signals including $\xi_i, \theta_i, \hat{x}_i, \alpha_i, \hat{W}_i$ are uniformly bounded and the consensus tracking error converges to a small neighborhood around the origin as

\[
\lim_{t \to \infty} \|x - I \otimes x_0\| \leq \kappa_1 + \frac{1}{\sigma_{\min} (L_1)} (\kappa_2 + \kappa_3)
\]

where the constants $\kappa_1, \kappa_2, \kappa_3$ are determined in the following proof.
Proof. Consider the following candidate Lyapunov function

\[
V_1 = e^T(I_N \otimes P^{-1})e + \frac{\gamma_1}{2} \sum_{i=1}^N g_i(2\alpha_i + \beta_i)\hat{\alpha}_i + \frac{\gamma_1}{2} \sum_{i=1}^N (\alpha_i - \alpha)^2 + 2\gamma_2 e^T(I_N \otimes Q)\hat{x} \\
+ \gamma_2 \sum_{i=1}^N \text{tr} \left( \frac{1}{v_i} W_i^T W_i \right),
\]

(14)

where \(\gamma_1, \gamma_2, \alpha\) are constants to be determined later, \(P, Q\) are positive definite matrices defined in (8) and (9). In addition, with \(\alpha_i(0) \geq 1\), it can be seen from Eq. (6) that \(\alpha_i(t) \geq 1\). Thus, the candidate Lyapunov function \(V_1\) is positive definite.

The time derivative of \(V_1\) along the trajectory of (6), (10) and (13) is obtained as

\[
\dot{V}_1 = 2e^T(I_N \otimes P^{-1})\dot{e} + \gamma_1 \left[ \sum_{i=1}^N g_i(2\alpha_i + \beta_i)\dot{\hat{\alpha}}_i + \sum_{i=1}^N g_i\dot{\alpha}_i\hat{\alpha}_i \right] \\
+ \gamma_1 \sum_{i=1}^N (\alpha_i - \alpha)\dot{\alpha}_i + 2\gamma_2 e^T(I_N \otimes Q)\dot{\hat{x}} \\
+ 2\gamma_2 \sum_{i=1}^N \text{tr} \left( \frac{1}{v_i} W_i^T W_i \right).
\]

(15)

Define \(P^{-1}A + A^TP^{-1} - 2CP^{-1}BB^TP^{-1} = -H_1 < 0\), by using Eq. (13) and Young’s inequality in Lemma 5, we can obtain

\[
2e^T(I_N \otimes P^{-1})\dot{e} = -e^T(I_N \otimes H_1)e + 2e^T(I_L \otimes P^{-1}L)e \\
- 2e^T[\mu\hat{L}_1(\hat{\alpha} + \hat{\beta}) \otimes \Gamma](e - \xi) \\
\leq -e^T(I_N \otimes \frac{1}{2} H_1)e \\
+ 2\gamma_1 \left[ \sum_{i=1}^N \frac{4\sigma_2^2 \sigma_{\max}^2(L_L) \lambda_{\max}(C^TC)}{\lambda_{\min}(H_1)} \lambda_{\min}(P) \lambda_{\min}(Q) \lambda_{\min}(Q) \lambda_{\min}(Q) \right] \hat{x} \\
+ (e - \xi)^T \left[ \frac{4\sigma_2^2 \sigma_{\max}^2(L_L) \lambda_{\max}(\Gamma)}{\lambda_{\min}(H_1)} \right] (\hat{\alpha} + \hat{\beta})^2 \otimes \Gamma \]

(16)

where the positive definite matrix \(Q_2 > 0\) will be determined later.

By substituting the second and third equation of the adaptive estimator (6) into part of Eq. (15), we have

\[
\gamma_1 \left[ \sum_{i=1}^N g_i(2\alpha_i + \beta_i)\dot{\hat{\alpha}}_i + \sum_{i=1}^N g_i\dot{\alpha}_i\hat{\alpha}_i \right] + \gamma_1 \sum_{i=1}^N (\alpha_i - \alpha)\dot{\alpha}_i \\
= 2\gamma_1 (e - \xi)^T \left[ \frac{G\beta \otimes \Gamma + (\hat{\alpha} - \alpha L_h) \otimes \Gamma}{\lambda_{\min}(H_1)} \right] \\
\times \left[ (\hat{\alpha} + \hat{\beta})^2 \otimes \Gamma \right] \\
+ \gamma_1 \sum_{i=1}^N \sigma_i\dot{\alpha}_i(\alpha - \alpha - 1) \\
\leq \gamma_1 (e - \xi)^T \left[ \frac{G\beta \otimes \Gamma + (\hat{\alpha} - \alpha L_h) \otimes \Gamma}{\lambda_{\min}(H_1)} \right] \\
- \frac{3\rho \lambda_0}{4} (\hat{\alpha} + \hat{\beta})^2 \otimes \Gamma \\
+ (G\beta \otimes \Gamma) + (\hat{\alpha} - \alpha L_h) \otimes \Gamma \right](e - \xi) - \gamma_1 \sum_{i=1}^N \sigma_i\dot{\alpha}_i^2 \\
+ \gamma_1 \sum_{i=1}^N \sigma_i(\alpha - 1)^2 + \frac{4\gamma_1}{\rho \lambda_0} \left( \sum_{i=1}^N g_i^2 \sigma_2^2 \right) r_i^T r_i \]

(17)

where we have used the fact that \(G\lambda_1 + \lambda_1^2G \geq \lambda_0 L_h - \alpha_i(\hat{\alpha}_i + \alpha - 1) \leq -\frac{1}{2}\hat{\alpha}_i^2 + \frac{1}{2}(\alpha - 1)^2\), and Lemma 5 to get the inequality.

Observe that

\[
\gamma_1 (e - \xi)^T \left[ \frac{G\beta \otimes \Gamma + (\hat{\alpha} - \alpha L_h) \otimes \Gamma}{\lambda_{\min}(H_1)} \right] \\
- \frac{3\rho \lambda_0}{4} (\hat{\alpha} + \hat{\beta})^2 \otimes \Gamma \\
+ (G\beta \otimes \Gamma) + (\hat{\alpha} - \alpha L_h) \otimes \Gamma \right](e - \xi) - \gamma_1 \sum_{i=1}^N \sigma_i\dot{\alpha}_i^2 \\
+ \gamma_1 \sum_{i=1}^N \sigma_i(\alpha - 1)^2 + \frac{4\gamma_1}{\rho \lambda_0} \left( \sum_{i=1}^N g_i^2 \sigma_2^2 \right) r_i^T r_i \]

where \(A^TQ + QA = 2C^TC = -H_2 - 2C^TC = -Q_2 < 0\), the constant scalar \(\hat{\alpha}\) is determined by the fact that \(\frac{d\hat{\alpha}}{dt} + (\hat{\alpha} + \hat{\beta})^2 + \hat{\alpha}^2_n \geq \sqrt{\rho \lambda_0} \hat{\alpha}(\hat{\alpha} + \hat{\beta}) \geq 2cG(\hat{\alpha} + \hat{\beta})\) if \(\hat{\alpha}(\hat{\alpha} + \hat{\beta})\) is large enough.

In addition, by using the adaptive update law (10), one can get

\[
2\gamma_2 (I_N \otimes Q)\dot{\hat{x}} + 2\gamma_2 \sum_{i=1}^N \text{tr} \left( \frac{1}{v_i} W_i^T W_i \right) \\
= -2\gamma_2 (I_N \otimes H_2)\dot{x} - 2\gamma_2 (I_N \otimes (C^T - QB))\dot{W}(\dot{x}) \\
+ 2\gamma_2 (I_N \otimes QB)(W(\dot{x}) - \epsilon) - 2\gamma_2 \sum_{i=1}^N \text{tr}[k_i W_i^T W_i] \\
- \gamma_2 \sum_{i=1}^N \text{tr}[\tilde{L}_i(\dot{x})(\dot{x})] \\
\leq 2\gamma_2 (I_N \otimes P^{-1}A + A^TP^{-1} - 2CP^{-1}BB^TP^{-1}) \\
+ \frac{1}{\Delta B} B^TB^T Q \dot{x} - \gamma_2 \sum_{i=1}^N \text{tr}[k_i W_i^T W_i] + \Delta_1
\]

(19)

Based on (16)–(19), we obtain that

\[
\dot{V}_1 \leq -e^T(I_N \otimes \frac{1}{2} H_1)e + \gamma_1 (e - \xi)^T [(G\beta + \hat{\beta})^2 \otimes \Gamma] \\
+ (P^{-1}A + A^TP^{-1} - 2cG(\hat{\alpha} + \hat{\beta}))\dot{x} \\
+ (1 - e^{-\xi})^T [(G\beta + \hat{\beta})^2 \otimes \Gamma] \\
+ (\hat{\alpha} - \alpha L_h) \otimes \Gamma \right](e - \xi) - \gamma_1 \sum_{i=1}^N \sigma_i\dot{\alpha}_i^2 \\
+ \gamma_1 \sum_{i=1}^N \sigma_i(\alpha - 1)^2 + \frac{4\gamma_1}{\rho \lambda_0} \left( \sum_{i=1}^N g_i^2 \sigma_2^2 \right) r_i^T r_i \]

(20)
where $\Delta = \Delta_1 + \frac{2\gamma_1}{\mu_1}(\sum_{i=1}^{N} \sigma_i^2 \gamma_1 r_0^2 + \gamma_2 \sum_{i=1}^{N} \sigma_i(\alpha - 1)^2)$. The constants $\gamma_1$, $\gamma_2$ are chosen as $\gamma_1 > \frac{16\gamma_1^2(3) \Delta_1}{4\rho_{\min}(H_1)}$, $\gamma_2 > \frac{4\rho_{\min}(H_1) \Delta_1 \lambda_{\max}(C(C))}{\rho_{\min}(H_1)} + 1$ to get the last inequality. In light of Lemma 3, it can be obtained that all the signals including $x_i$, $\theta_i$, $\tilde{x}_i$, $\alpha_i$, $\tilde{W}_i$ in the closed-loop network systems are uniformly ultimately bounded.

Moreover, in order to get the residual set of the tracking error, (20) can be rewritten as
\[
\dot{V}_1 \leq -\eta V_1(t) + W + \varsigma Z - \varsigma Z
\]
\[
= -\eta V_1(t) - e^T \left[ \sum_{i=1}^{N} \left( \frac{\varsigma}{2} x_i^T H_i - \eta b^{-1} \right) \right] - \gamma_1(e - \xi)^T G(\dot{\alpha} + \dot{\beta}) \otimes (\varsigma H_1 - \eta \rho^{-1})(e - \xi) - \tilde{x}^T \left[ \sum_{i=1}^{N} (\varsigma Q_i - \gamma_2 Q_i) \tilde{x} \right] + e^T \left[ \sum_{i=1}^{N} \left( \frac{1}{2} (\varsigma - 1) H_1 \right) \right] e + \tilde{x}^T \left[ \sum_{i=1}^{N} (\varsigma - 1) Q_i \tilde{x} \right] + \gamma_1(e - \xi)^T G(\dot{\alpha} + \dot{\beta}) \otimes (\varsigma - 1) H_1 (e - \xi) \leq -\eta V_1(t) - \left( \frac{\varsigma}{2} - \frac{\eta}{\lambda_{\min}(H_1) \lambda_{\min}(P)} \right) \lambda_{\min}(H_1) \| e \|^2
\]
\[
- \left( \frac{\varsigma - \gamma_2}{2} \lambda_{\min}(Q_i) \right) \lambda_{\min}(Q_i) \| \tilde{x} \|^2 + \Delta
\]
(21)
where $Z = e^T \left[ \sum_{i=1}^{N} (\varsigma - 1) H_1 \right] e + \gamma_1(e - \xi)^T G(\dot{\alpha} + \dot{\beta}) \otimes (\varsigma - 1) H_1 (e - \xi) + \tilde{x}^T \left[ \sum_{i=1}^{N} (\varsigma - 1) Q_i \tilde{x} \right]$, $\varsigma$ is a constant that satisfies $0 < \varsigma < 1$, and
\[
\eta \leq \min_{i=1,\ldots,N} \left( \kappa_i, \sigma_i, \frac{\varsigma \lambda_{\min}(H_1) \lambda_{\min}(P)}{2}, \frac{\varsigma \lambda_{\min}(Q_i)}{\gamma_2 \lambda_{\min}(Q_i)} \right).
\]
(22)
It is not difficult to get
\[
\dot{V}_1 \leq -\eta V_1(t),
\]
if $\| \tilde{x} \|^2 \geq \kappa_1$ or $\| e \|^2 \geq \kappa_2$ or $\| (e - \xi) \|^2 \geq \kappa_3$, where
\[
\kappa_1 = \Delta / \left[ \left( \frac{\varsigma}{2} - \frac{\eta}{\lambda_{\min}(H_1) \lambda_{\min}(P)} \right) \lambda_{\min}(H_1) \right],
\]
\[
\kappa_2 = \Delta / \left[ \left( \frac{\varsigma - \gamma_2}{2} \lambda_{\min}(Q_i) \right) \lambda_{\min}(Q_i) \right],
\]
\[
\kappa_3 = \Delta / \left( \gamma_1 \min_{i=1,\ldots,N} \left[ \kappa_i, \sigma_i, \frac{\varsigma \lambda_{\min}(H_1) \lambda_{\min}(P)}{2}, \frac{\varsigma \lambda_{\min}(Q_i)}{\gamma_2 \lambda_{\min}(Q_i)} \right] \right).
\]
(23)

Remark 3. In Theorem 6, the inequality $A^T Q + QA = 2TC^T C + 2\mu(C^T - QB)(C^T - QB)^T + \frac{1}{d} QBB^T Q + 2\mu NN^T < 0$ can be rewritten in the following parameter-dependent Riccati inequality form [39]
\[
A^T Q + QA = 2TC^T C + \frac{1}{d} QBB^T Q + 2\mu NN^T < 0
\]
(24)
\[
N = C^T - QB.
\]

Once the independent constants $l$, $d$ have been chosen, $Q > 0$ depends continuously on the parameter $\mu$. Note that $\mu = 0$ corresponds to the Riccati inequality which can be easily computed to obtain the positive definite solution $Q$. Thus, there exists solution $Q(\mu) > 0$ on the interval $\mu \in (0, \mu_{\max})$, which defines the largest set within which there exists a positive definite solution for $Q$. By virtue of the Schur Complement Lemma, the parameter-dependent Riccati inequality (24) can be transformed into the LMI (9). Moreover, the adaptive estimator (6) in Theorem 6 has been designed partially motivated by [32].

Remark 4. In Theorem 6, the consensus protocols with adaptive update laws guarantee that all the signals in the closed-loop systems including the parameter estimate $\tilde{W}_i$ are uniformly ultimately bounded. The results do not imply that the adaptive parameter estimates will converge to the real unknown parameters. If the persistent excitation(PE) condition and some other assumptions are satisfied, the parameter convergence may be obtained [25]. Without the PE condition, the consensus tracking object still can be achieved with the designed protocols.

Remark 5. Compared with the related references [17,22,23], in which the unknown nonlinear functions in the followers are also linearly parameterized, the consensus tracking protocols in this paper are designed for the general linear multi-agent systems including the integrator-type system as a special case. Furthermore, the protocols are developed based on output information which is more practical than the state-feedback ones [19–21,25].

Remark 6. By virtue of the output information of the neighboring agents, the consensus protocols are designed in a fully distributed fashion in the sense that only the local information of each agent and its neighbors is utilized. In the related references [23,24], the protocols are also developed based on the output information, but the common limitation is that the coupling gain in the controller requires the eigenvalues of the Laplacian matrix associated with the whole communication topology. In comparison with these two related references, the coupling gain $c$ in the protocol (5) is a constant independent without the knowledge of the Laplacian matrix a priori. Besides, the protocols of this paper are applicable for the general directed graph, while the adaptive protocols in [26,27,29] just work for the undirected graphs.

4. Simulation

In this section, a simulation example is provided to illustrate the validity of the theoretical results. Consider a network of five followers and one leader whose dynamics are third-order integrators as
\[
A = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
\end{bmatrix}, 
B = \begin{bmatrix}
0 \\
0 \\
1 \\
\end{bmatrix}, 
C = \begin{bmatrix}
1 & 0 & 0 \\
\end{bmatrix}.
\]

The communication topology is given by Fig. 1 in which the node indexed by 0 is the leader and only the follower indexed by 1 is available to the leader’s information. It is easy to see that the communication graph contains a directed spanning tree. The basis functions in the linearly parameterized model (4) are defined by
\[
\varphi_i(x_i(t)) = \left[ \sin((i + 1)D_1 x_i(t)) \cos((i + 1)D_2 x_i(t)) \right]^T, 
i = 1, \ldots, 5
\]

where $D_1 = [1, 0]$, $D_2 = [0, 1]$. Based on Section 3, we can get a full picture of the consensus tracking protocol with local observer and estimator for the $i$th
Fig. 1. Communication graph.

Fig. 2. The evolutions of tracking error $x_{i1} - x_{01}$.

Fig. 3. The evolutions of tracking error $x_{i2} - x_{02}$.

Fig. 4. The evolutions of tracking error $x_{i3} - x_{03}$.

Fig. 5. The evolutions of coupling gains $\alpha_i(t)$.

agent as

$$
\begin{aligned}
  u_i &= cK\theta_i - \hat{W}_i\hat{\psi}(\hat{x}_i) \\
  \dot{x}_i &= Ax_i + cBK\theta_i + L(C\hat{x}_i - y_i) \\
  \dot{\theta}_i &= A\theta_i + cBK\theta_i + L(C\hat{x}_i - y_i) + \rho(\alpha_i + \beta_i)BK(e_i - \xi_i), \\
  \dot{\alpha}_i &= -\sigma_\alpha(\alpha_i - 1) + (e_i - \xi_i)^T\Gamma(e_i - \xi_i), \\
  \dot{\beta}_i &= (e_i - \xi_i)\Gamma^{-1}(e_i - \xi_i), \\
  \dot{\hat{W}}_i &= t_i \left[ -(\hat{y}_i - y_i)\psi_i^T(\hat{x}_i) - \frac{1}{2\mu}\hat{W}_i\psi_i(\hat{x}_i)\psi_i^T(\hat{x}_i) - k_i\hat{W}_i \right]
\end{aligned}
$$

where the parameters $K$, $L$, $\Gamma$, $c$, $l$, $\rho$, $\mu$, $d$, $\sigma_\alpha$, $t_i$, $k_i$ in the consensus protocol are determined in the following part.

By solving the matrix inequality (8) and (9) with the LMI toolbox of Matlab, the matrices $K$, $L$ and $\Gamma$ are given as

$$
K = \begin{bmatrix} -0.9194 & -1.9056 & -2.5039 \end{bmatrix},
$$

(25)

$$
L = \begin{bmatrix} -45.2909 \\ -191.0680 \\ -112.3833 \end{bmatrix},
$$

$$
\Gamma = \begin{bmatrix} 0.8453 & 1.7520 & 2.3021 \\ 1.7520 & 3.6314 & 4.7715 \\ 2.3021 & 4.7715 & 6.2696 \end{bmatrix}.
$$

where the constants $c = 1$, $l = 25$, $\rho = 1$, $\mu = 20$, $d = 0.05$.

The initial states of agents are chosen randomly within $[-20, 20]$, and the initial states of $\alpha_i$ are chosen to satisfy $\alpha_i(0) \geq 1$. The parameters in the adaptive protocol are taken as $t_i = 100$, $\sigma_\alpha = 0.005$, $k_i = 0.0008$, $i = 1, \ldots, 5$. The tracking consensus errors $x_i - x_0$, $i = 1, \ldots, 5$ are depicted in Figs. 2-4, denoting that the errors converge to a small neighborhood around the origin. Fig. 5 shows clearly that the coupling gains are bounded with $\alpha_i \geq 1$, $i = 1, \ldots, 5$.

5. Conclusion

This paper has addressed the distributed consensus tracking problem for the linear multi-agent systems with unknown dynamics under general directed graphs. Distributed adaptive output-feedback consensus tracking protocols have been designed to guarantee that the tracking errors converge to a small neighborhood around the origin, and all the signals in the closed-loop system are uniformly ultimately bounded. In addition, the consensus protocols are in fully distributed fashion for the control gains which are independent of the eigenvalues of the Laplacian matrix associated with the whole communication graph which is actually the global information. Finally, the validity of the theoretical results was illustrated through an example.

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References