Finite-time Optimal Formation Control for Linear Multi-agent Systems

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Abstract: This paper investigates the problem of finite-time formation control for multi-agent systems with general linear dynamics. First, the formation problem considered is converted into the motion planning problem. Then, by using Pontryagins maximum principle (PMP), an optimal formation control law is developed for multi-agent systems which satisfy some invertible conditions. With this control law, the multi-agent systems achieve the desired formation in finite time, where the formation configurations and the settling time are specified in advance according to task requirements. Meanwhile, a performance index is guaranteed to be optimal. Further, it is proved that the conditions concerned are invertible if and only if the linear systems we considered are controllable.

Key Words: Finite Time, Formation Control, Linear System, Motion Planning

1 Introduction

In the past decades, the coordination problem of multi-agent systems has been considerably studied due to its broad applications in such areas as spacecraft formation flying, unmanned aircraft formation flying and sampling, distributed sensor networks, automated highway systems (AHS), etc [1, 11, 32]. One critical issue arising from multi-agent systems is to develop control laws that enable all agents to reach a desired formation, which is known as the formation problem. In contrast to the traditional monolithic systems, the system with multiple agents working in a form of formation reduces the system cost, breaches the size constraints, and prolongs the life span of the system [11]. Furthermore, the robustness and flexibility are enhanced.

Consensus, as an interesting topic of the coordination control, is aimed at guaranteeing that the states of all agents to reach an agreement. Through appropriately choosing information states on which consensus is reached, consensus algorithms can be applied to tackle formation control problems [21]. Consensus of multi-agent systems with various dynamics have been well studied in recent decades. Early seminal works for systems with first-order dynamics have been launched by Jadbabaie, Lin and Morse [8], Olfati-Saber and Murray [18], Ren and Beard [24], to name a few, where various consensus algorithms are developed and many relevant engineering issues are considered, such as time-delays, switching topology, communication constraints. In [14, 22, 31, 33], consensus for second-order dynamics systems has also been extensively investigated. Specially, Yu et al. in [33] proposed some necessary and sufficient conditions for second-order consensus in directed networks containing a directed spanning tree. Further, consensus results are extended to the systems with high-order integrator dynamics in [9, 25]. Nevertheless, most actual multi-agent systems have very complex physical dynamics. Motivated by this observation, consensus for systems with general linear dynamics also receives dramatic attention. Corresponding results can be found in [13, 16, 26–28]. In these works, some asymptotic protocols have been developed for systems to converge to a consistent view of their information states. Besides, consensus for multiple Euler-Lagrange systems and nonlinear systems are studied in [15] and [20], respectively. In summery, a common feature in all above works is that the asymptotic control laws are designed for systems such that the desired consensus or formation are achieved when the time variable tends to infinity. Nevertheless, in many practical applications, it is more desirable for systems to achieve consensus or formation with a fast convergence rate.

As an important performance indicator of the consensus protocol, convergence rate is another hot research topic in the area of consensus problems. For the above mentioned asymptotic control laws, the convergence rate is at best exponential with infinite settling time [12]. Finite-time consensus algorithms, by contrast, are more desirable. Besides a faster convergence rate, the closed-loop systems under finite time control usually demonstrate better disturbance rejection properties [2]. Finite-time consensus for multiple dynamic agents were first introduced in [4]. Then, several kinds of finite-time consensus protocols have been developed for the first-order multi-agent systems in [6, 7, 30], to name a few. In particular, the finite-time consensus protocols are developed to deal with the time-invariant and time-variant formation problem in [30]. Extensions to the second-order multi-agent systems are also studied in [10, 12, 23, 29, 34, 35], to name a few, where the consensus protocol problem for systems with multiple second-order integrators and coupled harmonic oscillators are addressed. In [12], for systems with multiple second-order integrators, the authors give the estimation of the settling time, which depends on the designed Lyapunov function, controller parameters, and the information of the commutation topology. Further, the finite-time consensus for systems with multiple rigid bodies are also investigated by many researchers. Corresponding results can be found in [5, 17, 19].

This paper is concerned with the finite-time formation control...
The problem of multi-agent systems with general linear dynamics, which may also be considered as the linearized model of a nonlinear network. A new framework is introduced, which converts the finite-time formation problem of multi-agent systems into the motion planning problem. By using Pontryagin’s maximum principle (PMP), an optimal formation control law is proposed for multi-agent systems to achieve the desired formation in finite time. The contributions of this paper are three-fold. First, it is the first time that the finite-time control law is proposed for systems with general linear systems dynamics, as far as we know. Compared with the first-order and second-order dynamics, general linear systems are more common in practical applications. Second, unlike the settling time conservatively estimated according to the designed control laws and the constructed Lyapunov function, in this paper, the settling time can be precisely specified in advance, which is more meaningful and is consistent with practical needs. Last but not the lest, with such control laws, a performance index is minimized, which is crucial for systems with limited fuel.

The remainder of this paper is organized as follows. The problem formulation are given in Section 2. Main theoretical results are provided in Section 3. In Section 4, a numerical example is reported to illustrate the theoretical results. Concluding remarks are finally given in Section 5.

Notations: Let \( \mathbb{R}^{n \times n} \) be the sets of \( n \times n \) real matrices. The superscript \( T \) means transpose for real matrices. \( I_N \) represents the identity matrix of dimension \( N \), and \( I \) denotes the identity matrix of an appropriate dimension. Denote by \( 1 \) the column vector with all entries equal to one. Matrices, if not explicitly stated, are assumed to have compatible dimensions. The Kronecker product of matrices \( A \in \mathbb{R}^{m \times n} \) and \( B \in \mathbb{R}^{p \times q} \) denoted by \( A \otimes B \), which satisfies the following properties:

\[
(A \otimes B)^T = A^T \otimes B^T,
\]

\[
(A \otimes B)^{-1} = A^{-1} \otimes B^{-1},
\]

\[
A \otimes B + A \otimes C = A \otimes (B + C),
\]

\[
(A \otimes B)(C \otimes D) = (AC) \otimes (BD),
\]

where \( C \) and \( D \) are matrices with compatible dimensions.

2 Problem Formulation

Consider a group of \( N \) identical agents with general linear dynamics, which may be regarded as the linearized model of some nonlinear systems. The dynamics of the \( i \)th agent are described by

\[
\dot{x}_i = Ax_i + Bu_i,
\]

\[
x_i(t_0) = x_i^0, \quad i = 1, \ldots, N,
\]

where \( x_i \in \mathbb{R}^n \) is the state, \( u_i \in \mathbb{R}^m \) is the control input, and \( t_0 \) is the initial time. Matrices \( A \) and \( B \) are constant real matrices with compatible dimensions.

Now the problem is for given relative states \( D_{ij} \in \mathbb{R}^n \), \( i, j = 1, \ldots, N \), determined by formation tasks, and initial states \( x_i^0, \quad i = 1, \ldots, N \), to find control input \( u_i \), \( i = 1, \ldots, N \), such that

\[
x_i(t_f) - x_i(t_0) = D_{i-1}, \quad i = 2, \ldots, N,
\]

where \( t_f > t_0 \) is the final time which is given by formation tasks in advance, meanwhile, to minimize the following cost function,

\[
J = \frac{1}{2} \int_{t_0}^{t_f} \sum_{i=1}^{N} (u_i^T(t)u_i(t)) \, dt.
\]

The above problem is referred to as finite time optimal formation control. When \( D_{ij} = 0, \quad i = 2, \ldots, N \), the corresponding problem is referred to as finite time optimal consensus control. Note that the latter is the special case of the former. Thus, we shall focus on the formation control problem.

Denote a graph \( G = (\mathcal{V}, \mathcal{E}) \) to be a communication topology among \( N \) agents, where \( \mathcal{V} = 1, \ldots, N \) and \( \mathcal{E} \subseteq \mathcal{V} \times \mathcal{V} \) represent the set of nodes and the set of edges, respectively. A graph \( G \) is said to be complete graph if there exists an edge among each pair of distinct nodes.

To achieve the desired formation, it needs to coordinate the behaviors of different agents. The control for the individual agent must rely on the information of all other agents’ states that can be obtained. From the communication point of view, this multi-agent systems can be viewed as a graph. For this graph, the following assumption is proposed.

Assumption 1 In this paper, it is assumed that the communication topology between all the agents is a complete graph.

This assumption implies that every agent can derive the information of all other agents’ states.

3 Main results

The problem of finite-time optimal formation control is an optimal control problem. Therefore, the next task is to solve the corresponding optimal control problem.

In order to use Pontryagin’s minimum principle [3], the Hamiltonian for this problem can be constructed as follows

\[
H = -\frac{1}{2} \sum_{i=1}^{N} u_i(t)^T u_i(t) + \sum_{i=1}^{N} p_i^T (Ax_i + Bu_i),
\]

where \( p_i \in \mathbb{R}^n \) is the co-state (Lagrangian multiplier). Then, the corresponding Hamiltonian system can be written as

\[
\dot{x}_i = \frac{\partial H}{\partial p_i} = Ax_i + Bu_i,
\]

\[
\dot{p}_i = -\frac{\partial H}{\partial x_i} = -A^T p_i, \quad i = 1, \ldots, N.
\]

According to PMP, the optimal control \( u_i \) must satisfies the necessary condition that

\[
\frac{\partial H}{\partial u_i} = -u_i + B^T p_i = 0, \quad i = 1, \ldots, N.
\]

Since these equations have the unique solutions, the above condition is also sufficient. Then, it is followed from (7) that

\[
u_i = B^T p_i, \quad i = 1, \ldots, N.
\]

Let \( x = (x_1^T, \ldots, x_N^T)^T, \quad p = (p_1^T, \ldots, p_N^T)^T \) and \( u = (u_1^T, \ldots, u_N^T)^T \). Then, the Hamiltonian systems (5,6) and the control (8) can be written as

\[
\dot{x} = (I_N \otimes A)x + (I_N \otimes B)u,
\]

\[
\dot{p} = -(I_N \otimes A^T)u,
\]

\[
u = (I_N \otimes B^T)p.
\]

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Similarly, it is followed from the given formation conditions (2) that
\[
\begin{bmatrix}
I_{N-1} & -I_{N-1}
\end{bmatrix} \otimes I_n \cdot x(t_f) = D,
\]
(12)
where \(D = (D^T_1, \ldots, D^T_{N-1})^T\). Substituting (11) into Hamiltonian systems (9,10) gives
\[
\begin{bmatrix}
\dot{x} \\
p
\end{bmatrix} = \begin{bmatrix}
I_N \otimes A & I_N \otimes BB^T \\
0 & -(I_N \otimes A^T)
\end{bmatrix} \begin{bmatrix}
x \\
p
\end{bmatrix},
\]
(13)
By integrating the above equations from \(t_0\) to \(t\), one gets
\[
\begin{bmatrix}
x(t) \\
p(t)
\end{bmatrix} = \exp\left((t-t_0) \begin{bmatrix}
I_N \otimes A & I_N \otimes BB^T \\
0 & -(I_N \otimes A^T)
\end{bmatrix}\right) \begin{bmatrix}
x_0 \\
p_0
\end{bmatrix},
\]
(14)
where \(x_0 = (x_0^T, \ldots, x_0^T)^T\) and \(p_0 \in \mathbb{R}^{Nn}\) is the initial value for \(p\). Thus, the determination of the optimal control (8) is boiled down to finding the initial value \(p_0\). In order to solve \(p_0\), we give a lemma related to the transversality condition corresponding to the formation conditions (2).

**Lemma 1** If \(x_i(t_f) - x_i(t_f) = D_i, i = 2, \ldots, N\), then
\[
1_N^T \otimes I_n \cdot p(t_f) = 0.
\]
(15)

**Proof.** For the given formation conditions (2), let
\[
f_{ij}(x(t_f)) = x_i^j(t_f) - x_{i+1}^j(t_f) = D_i,
\]
where \(x_i^j(t_f)\) and \(x_{i+1}^j(t_f)\) are the \(j\)th entries of \(x_i(t_f)\) and \(x_{i+1}(t_f)\), respectively. Then according to PMP, the transversality condition corresponding to \(f_{ij} = D_i\) for the given \(D_i\) can be written as
\[
p_i^j(t_f) = \sum_{i=1}^{N-1} \lambda_{ij} \frac{\partial f_{ij}}{\partial x_i^j}(t_f) = \sum_{i=1}^{N-1} \lambda_{ij},
\]
(16)
which can be rewritten as
\[
p_k^{j+1}(t_f) = \sum_{i=1}^{N-1} \lambda_{ij} \frac{\partial f_{ij}}{\partial x_i^{j+1}}(t_f) = -\lambda_{kj},
\]
(17)
where \(p_i^j\) and \(p_k^{j+1}\) represent the \(j\)th entries of \(p_1^j\) and \(p_{k+1}\), respectively, and \(\lambda_{ij}\) is the parameter to be determined. Combining (16) with (17), it follows that
\[
p_i^j(t_f) = -\sum_{k=1}^{N} p_k^{j+1}(t_f).
\]
Thus
\[
\sum_{k=1}^{N} p_k^j(t_f) = 0, j = 1, \ldots, n.
\]
In the matrix form, we get
\[
1_N^T \otimes I_n \cdot p(t_f) = 0.
\]
The proof is completed.

Let
\[
\Phi(t-t_0) = \exp\left((t-t_0) \begin{bmatrix}
I_N \otimes A & I_N \otimes BB^T \\
0 & -(I_N \otimes A^T)
\end{bmatrix}\right),
\]
\[
\Delta = \begin{bmatrix}
1_N & -I_{N-1} & I_n \\
0 & 1T_N \otimes I_n
\end{bmatrix}.
\]
It is followed from (12) and (15) that
\[
\begin{bmatrix}
1_N & -I_{N-1} & I_n \\
0 & 1T_N \otimes I_n
\end{bmatrix} \begin{bmatrix}
x(t_f) \\
p(t_f)
\end{bmatrix} = \begin{bmatrix}
D \\
0
\end{bmatrix},
\]
(18)
i.e.
\[
\Delta \Phi(t_f-t_0) [x(t_0) \ p(t_0)] = \begin{bmatrix}
D \\
0
\end{bmatrix}.
\]
Thus
\[
\Delta \Phi(t_f-t_0) E_1 x_0 + \Delta \Phi(t_f-t_0) E_2 p_0 = \begin{bmatrix}
D \\
0
\end{bmatrix}.
\]
Assuming that \(\Delta \Phi(t_f-t_0) E_2\) is invertible, it follows that
\[
p_0 = -\left(\Delta \Phi(t_f-t_0) E_2\right)^{-1}\Delta \Phi(t_f-t_0) E_1 x_0 - \begin{bmatrix}
D \\
0
\end{bmatrix}.
\]
Substituting the above equation into (14), we get
\[
p(t) = E_2^T \Phi(t-t_0) (E_1 x_0 + E_2 p_0).
\]
Then, recalling the control law (11), it is obtained that
\[
u = (I_N \otimes B^T) E_2^T \Phi(t-t_0) (E_1 x_0 + E_2 p_0).
\]
(19)
Now we have the following theorem.

**Theorem 1** Suppose that Assumption 1 holds. Then under the control law
\[
u = (I_N \otimes B^T) E_2^T \Phi(t-t_0) (E_1 x_0 + E_2 p_0),
\]
\[
p_0 = -\left(\Delta \Phi(t_f-t_0) E_2\right)^{-1}\Delta \Phi(t_f-t_0) E_1 x_0 - \begin{bmatrix}
D \\
0
\end{bmatrix},
\]
the specified formation (2) is achieved at the given formation time \(t_f\), if and only if \(\Delta \Phi(t_f-t_0) E_2\) is invertible.

In practical applications, the invertibility of \(\Delta \Phi(t_f-t_0) E_2\) can be determined using numeric methods. However, for a group of linear systems with high dimensional system matrix, it is unavoidable to perform exact complex calculations. Therefore, we propose the following theorem, where a sufficient condition is given to guarantee the feasibility of the proposed control law (19).

Before we continue, the following lemma is given.

**Lemma 2** For matrices \(A \in \mathbb{R}^{n \times n}\) and \(B \in \mathbb{R}^{n \times m}\), \((A, BB^T)\) is controllable, if and only if \((A, B)\) is controllable.
Proof. First, we prove the sufficiency. It is known that \((A, B)\) is controllable. We desire to prove that \((A, BB^T)\) is controllable.

By reductio, assume that \((A, BB^T)\) is uncontrollable. Then, it is followed that there is at least one \(\lambda \in \mathbb{C}\), such that \([A - \lambda I, BB^T]\) is not row full rank; that is to say, there exists a nonzero vector \(\alpha \in \mathbb{R}^n\), which makes that
\[
\alpha^T [A - \lambda I, BB^T] = 0.
\]
Thus
\[
\alpha^T (A - \lambda I) = 0, \tag{20}
\]
\[
\alpha^T BB^T = 0. \tag{21}
\]
Multiplying (21) at the right by \(\alpha\) gives \(\alpha^T BB^T \alpha = 0\). So,
\[
\alpha^T B = 0. \tag{22}
\]
Considering (20) and (22), it obtains
\[
\alpha^T [A - \lambda I, B] = 0.
\]
Obviously, this contradicts with the condition that \((A, B)\) is controllable. Thus, \((A, BB^T)\) is controllable. Similarly, we can prove that \((A, B)\) is controllable if \((A, BB^T)\) is controllable. The proof is completed.

\[
\begin{bmatrix}
1_{N-1} & -I_{N-1} & 0 \\
0 & 1_N \otimes I_n & 0 \\
1_N^T \otimes I_n & 0 & 1_N \\
\end{bmatrix}
\begin{bmatrix}
\Phi_{11} \\
\Phi_{21} \\
\Phi_{12} \\
\end{bmatrix} x_0 +
\begin{bmatrix}
1_{N-1} & -I_{N-1} & 0 \\
0 & 1_N \otimes I_n & 0 \\
1_N^T \otimes I_n & 0 & 1_N \\
\end{bmatrix}
\begin{bmatrix}
\Phi_{11} \\
\Phi_{21} \\
\Phi_{12} \\
\end{bmatrix} p_0 =
\begin{bmatrix}
D \\
0 \\
\end{bmatrix}.
\]
Thus,
\[
\begin{bmatrix}
1_{N-1} & -I_{N-1} & 0 \\
0 & 1_N \otimes I_n & 0 \\
1_N^T \otimes I_n & 0 & 1_N \\
\end{bmatrix}
\begin{bmatrix}
\Phi_{11} \\
\Phi_{21} \\
\Phi_{12} \\
\end{bmatrix} p_0 =
\begin{bmatrix}
D - \begin{bmatrix}
1_{N-1} & -I_{N-1} & 0 \\
0 & 1_N \otimes I_n & 0 \\
1_N^T \otimes I_n & 0 & 1_N \\
\end{bmatrix}\Phi_{12} x_0 \\
\end{bmatrix}. \tag{23}
\]

It is followed from the theory of matrix and the system of linear equations that, if \(\Phi_{12}\) is invertible, then the above equation of \(p_0\) is solvable. Thus, if we show that \(\Phi_{12}\) is invertible, the result follows.

Let
\[
\Pi = tBB^T + \frac{t^2}{2!} (BB^T - BB^T A^T) + \frac{t^3}{3!} (A^2 BB^T - A^2 BB^T A^T - ABT^2 + ABT^2 A^T) + \frac{t^4}{4!} (A^3 BB^T - A^3 BB^T A^T + ABB^T A^T - ABB^T A^T - ABT^2 + ABT^2 A^T) + \ldots.
\]

By reductio, assume that \(\Pi\) is singular. Thus, there is at least one nonzero vector \(\alpha \in \mathbb{R}^n\), which makes that \(\alpha^T \Pi = 0\). Taking the derivatives of the above equation to \((n - 1)\) order with respect to the time \(t\) and then setting \(t = 0\), we get
\[
\begin{align*}
\alpha^T BB^T &= 0, \\
\alpha^T (BB^T - BB^T A^T) &= 0, \\
\alpha^T (A^2 BB^T - ABB^T A^T + BB^T A^T) &= 0, \\
&\ldots, \\
\alpha^T (A^{n-1} BB^T - A^{n-2} BB^T A^T + A^{n-3} BB^T A^T - \ldots + (-1)^{n-1} BB^T A^{n-1}) &= 0.
\end{align*}
\]

Simplifying the above equations, we have
\[
\begin{align*}
\alpha^T BB^T &= 0, \\
\alpha^T ABT &= 0, \\
\alpha^T A^2 BB^T &= 0, \\
&\ldots, \\
\alpha^T A^{n-1} BB^T &= 0.
\end{align*}
\]
Let
\[
Q = \begin{bmatrix}
BB^T & ABB^T & A^2 BB^T & \ldots & A^{n-1} BB^T
\end{bmatrix}.
\]
Then,
\[
\alpha^T Q = 0.
\]

It follows from \(\alpha \neq 0\) that the matrix \(Q\) is linearly dependent. Considering Lemma 2, this contradicts with the condition that \((A, B)\) is controllable. Thus, it is proved that \(\Pi\) is nonsingular. Further, \(\Phi_{12}\) is invertible. Then, the result follows. The proof is completed.

Remark 1 Note that the proposed control law (19) is an open-loop control law, which depends on the initial states \(x_0\). Theoretically, agents only need the information of the others at the initial time. Therefore, under the control law concerned, packet losses are allowed, and further, agents do not need communication after the initial time. During the derivation of control laws (19), integrating the equation (13)
from $t$ to $t_f$ gives the following real-time feedback control law

$$u = -(I_N \otimes B^T) \cdot (\Delta \Phi(t_f-t)E_2)^{-1} \cdot \left(\Delta \Phi(t_f-t)E_1x(t) - \begin{bmatrix} D \\ 0 \end{bmatrix}\right),$$

which depends on the current states, not the initial states. Thus, it can achieve the desired formation even if disturbance exists in the initial conditions and/or control inputs. When there is no disturbance, the above new control law is equivalent to (19).

**Remark 2** Considering that disturbances exist after the terminal time $t_f$, the following switching control law is introduced

$$U = \left\{ \begin{array}{ll} u, & t \in [t_0, t_f) \\ u^*, & t \geq t_f \end{array} \right.,$$

where $u^*$, given in [13], is the asymptotic formation protocol for multi-agent systems with general linear dynamics. The above control law achieves the desired formation at the given terminal time $t_f$, and then switches to the general asymptotic protocol to keep the formation.

4 Numerical example

In this section, a numerical example is given to illustrate the effectiveness of the control laws proposed in the above sections. For simplicity, the initial time is given by $t_0 = 0$. Consider the systems with three agents described by (1), where

$$A = \begin{pmatrix} 0 & -4 \\ 5 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$  

The desired formation is that the configurations of these three agents achieve consensus. Therefore, the formation conditions is given by $D_1 = D_2 = (0,0)^T$. And the desired formation time is 4. Select the initial states $x_1(t_0) = (-1,1)^T$, $x_2(t_0) = (2,0)^T$, and $x_3(t_0) = (-10,-0.5)^T$. The simulation time is 5.

Fig. 1: The x,y states of these three agents and the formation time is 4.

Fig.1 shows the simulation results. It can be seen that the agents achieve consensus at the given terminal time $t_f$.

5 Conclusions

In this paper, we have studied the problem of finite-time formation control for multi-agent systems with general linear dynamics. By using Pontryagin’s maximum principle (PMP), we first develop an optimal formation control law for multi-agent systems which satisfy some invertible conditions. With this control law, the multi-agent systems achieve the desired formation in finite time, where the formation mode and the settling time are specified in advance according to task requirements. Meanwhile, the given integral performance index is minimized. Further, it has been proven that the proposed invertible condition concerned is equivalent to the controllability of the linear dynamics we considered.

Nevertheless, there are still some problems remained to be solved, such as finite-time formation control for multi-agent systems with general linear dynamics under variable network topologies. The solutions of these problems could be important both for theoretical research and for practical applications.

References


