Effects of transverse magnetic field and viscosity on the Richtmyer–Meshkov instability

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The effects of transverse magnetic field and viscosity on the Richtmyer–Meshkov instability are considered under the framework of two semi-infinite fluids with densities ρ_+ and ρ_-, magnetic fields $\mathbf{B}_+$ and $\mathbf{B}_-$, and viscosities μ_+ and μ_-, respectively. The amplitude of the perturbations is analytically obtained. It is found that the magnetic field provides oscillation and damping, and viscosity provides damping. When both are present, the perturbations of the interface undergo damped oscillation provided that the magnetic field is strong enough; otherwise the perturbations will damp from the beginning. © *2008 American Institute of Physics*. [DOI: 10.1063/1.2888512]

I. INTRODUCTION

The Richtmyer–Meshkov (RM) (Refs. 1–3) instability arises when an incident shock wave strikes a corrugated interface of a stratified heterogeneous fluid. The RM instability was theoretically predicted by Richtmyer[2] and confirmed in experiment by Meshkov.[3] Richtmyer's work, which dealt with the interaction of a shock wave with a perturbed contact discontinuity separating fluids of different densities, concluded that the perturbations grow linearly with time $\eta \sim kA_T\Delta u \eta_0 t$, where η is the perturbation amplitude, k is the mode wave number, A_T is the Atwood number, and Δu is the difference between the shocked and unshocked mean interface velocity.

The RM instability is important in a wide variety of applications including supernova blast wave interaction with surrounding matter in astrophysics[4] and inertial confinement fusion (ICF),[5] in which we are primarily interested. In these applications, the fluids may be ionized, thus the feature of the instability may be affected by magnetic field. In practice, it is expected that magnetic field can exist in ionized fluids. The temperature and density gradient of the laser-induced plasma in ICF can generate large-scale circulating magnetic field.[6] Other mechanisms, such as Weible instability, resonant absorption, and motion of superthermal electrons, can also generate such a magnetic field.[7] So it is important to investigate the effect of magnetic field on RM instability, which has partly been discussed by previous authors. Samtany's numerical simulation[8] of a shock interacting with an oblique planar contact discontinuity demonstrated that the growth of the RM instability is suppressed in the existence of a magnetic field. Wheatly *et al.*[9] studied the case with a magnetic field perpendicular to the interface by solving the linearized initial value problem and found that the initial growth rate of the RM instability was unaffected by the magnetic field, but for a finite magnetic field the interface amplitude was asymptotic to a constant value. Thus the RM instability is suppressed. However, the effect of the magnetic field which is parallel to the interface (namely, the transverse magnetic field) has not been analytically investigated. In fact, when transverse magnetic field exists, in a superconductive plasma, it will act somewhat as a surface tension. The transverse magnetic field will hinder the surface perturbations from growing and as a result will stabilize the instability.

The effects of viscosity on the RM instability are often ignored by previous authors. Viscosity is often thought of as trivial importance on plasma instabilities because of the calculated viscosity from kinetic equations are often very small.[10] However, recent measurements of viscosity are up to 10^8 times larger than predicted by classical collisional theory.[11] Mikaelian examined the effects of viscosity on RM instability, and concluded that the viscosity contributes to the suppression of the instability due to the dissipation of energy.[12] A nonlinear viscous theory of RM instability is built by Carlès and Popinet (CP) and agreed very well with the result of direct numerical simulation.[13] The effect of viscosity on RM instability was also investigated experimentally and the theories of Mikaelian and CP were compared in Ref. 14. Recently, it was found that viscosity can suppress the "kink-singularities" of a shocked interface.[15]

In this work, we present a linear analysis of the RM instability under the effects of magnetic field and viscosity with a single-mode sinusoidal perturbation in amplitude. The equilibrium magnetic field is parallel to the material interface. By analytically deriving the expression of the interface perturbations, we find out that both the transverse magnetic field and the viscosity act as stabilizing mechanisms; besides, the magnetic field also contributes to oscillation. The work is organized as follows: In Sec. II, a physical model is built and some basic assumptions are set. In Sec. III, a specific calculation is presented and an analytical expression of the interface perturbation is obtained. The effects of the quantities

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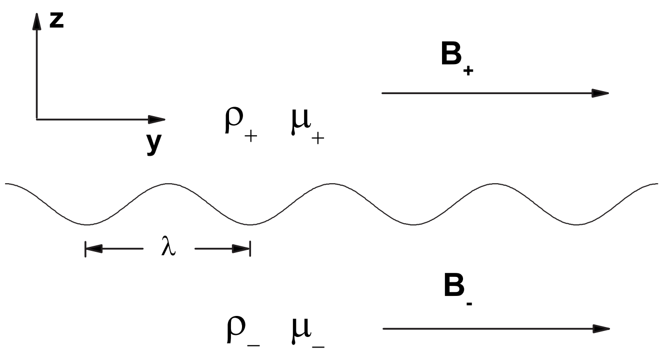

FIG. 1. The initial condition geometry. Two semi-infinite fluids of mass density ρ_- and ρ_+, magnetic field $\mathbf{B}_-$ and $\mathbf{B}_+$ in the y direction and coefficients of dynamic viscosity μ_- and μ_+ are separated by an interface at $z=0$. The interface perturbations are in the y direction and with the wavelength $\lambda(\lambda=2\pi/k)$.

such as viscosity, magnetic field, etc. on the RM instability are discussed in Sec. IV. The results are summarized in Sec. V.

II. EQUATIONS AND ASSUMPTION

In this paper, an incompressible viscid magnetized fluid is considered. Dissipation effects, except viscosity, are ignored on RM instability. The equilibrium profiles, such as mass density ρ_0, magnetic field $\mathbf{B}_0$, and coefficient of dynamic viscosity μ, are illustrated in Fig. 1. We take a sharp boundary model, i.e., two semi-infinite uniform fluids with density ρ_+ and ρ_- separated by a horizontal interface at $z=0$. The equilibrium quantities are considered to be constant at both sides of the interface, while at the interface, there is a jump. The complete equilibrium profiles are given by

$$\mathbf{u}_0=0, \quad \mathbf{B}_0=B_0(z)\mathbf{e}_y,$$

$$\rho_0=\rho_-, \quad B_0(z)=B_-, \quad \mu=\mu_- \quad \text{for} \quad z<0,$$

$$\rho_0=\rho_+, \quad B_0(z)=B_+, \quad \mu=\mu_+ \quad \text{for} \quad z>0,$$

where $\rho_+(B_+,\mu_+)$ and $\rho_-(B_-,\mu_-)$ are, respectively, the mass density (magnetic field, coefficient of dynamics viscosity) at both sides of the interface. We will examine the evolution of a rippled interface separating two incompressible fluids in the noninertial reference frame that is impulsively accelerated at $t=0$. We begin the analysis with the magnetohydrodynamics equations,

$$\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\mathbf{u})=0, \tag{1}$$

$$\rho\frac{\partial \mathbf{u}}{\partial t}+\rho\mathbf{u}\cdot\nabla\mathbf{u}=-\nabla p+\mathbf{J}\times\mathbf{B}+\nabla\cdot\chi+\rho\mathbf{g}, \tag{2}$$

$$\mathbf{J}=\frac{1}{\mu_0}\nabla\times\mathbf{B}, \tag{3}$$

$$\frac{\partial \mathbf{B}}{\partial t}=-\nabla\times\mathbf{E}, \tag{4}$$

$$\mathbf{E}+\mathbf{u}\times\mathbf{B}=0, \tag{5}$$

where μ_0 is the magnetic permeability, and $\mathbf{J}$, $\mathbf{E}$ are the electric current density and electric field, respectively. $\mathbf{g}=-\Delta u\delta(t)\mathbf{e}_z$ approximates the effect of the incident shock wave, where Δu is the imparted jump velocity and $\delta(t)$ is the Dirac delta function. The viscous tensor χ can be expressed in an incompressible fluid as

$$\chi_{ij}=\mu\left(\frac{\partial u_j}{\partial x_i}+\frac{\partial u_i}{\partial x_j}\right). \tag{6}$$

As we have mentioned, the fluid is incompressible, thus we have

$$\nabla\cdot\mathbf{u}=0. \tag{7}$$

Density, pressure, and viscosity gradients are all parallel to the shock acceleration which is in the direction of the negative z axis, so it is logical to assume that the inhomogeneity exists only in this very direction. Assume that all quantities are of the following form $\varphi=\varphi_0+\varphi_1$, where φ_0 denote the equilibrium quantities, and φ_1, which are much smaller than φ_0, denote the perturbed quantities. It is also assumed that all perturbations have the sinusoidal form $\varphi_1(y,z,t)=\tilde{\varphi}_1(z,t)e^{iky}$. Employing the condition that the magnetic field is solenoidal $(\nabla\cdot\mathbf{B}=0)$, we get the linearized equations from Eqs. (1)–(7),

$$\frac{\partial \rho_1}{\partial t}+u_{1z}\frac{d\rho_0}{dz}=0, \tag{8}$$

$$\rho_0\frac{\partial u_{1y}}{\partial t}=-ik\left(p_1+\frac{\mathbf{B}_0\cdot\mathbf{B}_1}{\mu_0}\right)+\frac{1}{\mu_0}\left(ikB_0B_{1y}+B_{1z}\frac{dB_0}{dz}\right)+\mu\left(\frac{\partial^2}{\partial z^2}-k^2\right)u_{1y}+\left(\frac{\partial u_{1y}}{\partial z}+iku_{1z}\right)\frac{d\mu}{dz}, \tag{9}$$

$$\rho_0\frac{\partial u_{1z}}{\partial t}=-\frac{\partial}{\partial z}\left(p_1+\frac{\mathbf{B}_0\cdot\mathbf{B}_1}{\mu_0}\right)+\frac{1}{\mu_0}(ikB_0B_{1z})+\mu\left(\frac{\partial^2}{\partial z^2}-k^2\right)u_{1z}+2\frac{\partial u_{1z}}{\partial z}\frac{d\mu}{dz}-\rho_1\Delta u\delta(t), \tag{10}$$

$$\frac{\partial \mathbf{B}_1}{\partial t}=ikB_0\mathbf{u}_1-u_{1z}\frac{d\mathbf{B}_0}{dz}, \tag{11}$$

$$iku_{1y}+\frac{\partial u_{1z}}{\partial z}=0. \tag{12}$$

III. ANALYTICAL DEVELOPMENT

Combining Eqs. (10) and (11), we have

$$ik\rho_0\frac{\partial u_{1z}}{\partial t}=\frac{\partial}{\partial z}\left[\rho_0\frac{\partial u_{1y}}{\partial t}-\frac{1}{\mu_0}\left(ikB_0B_{1y}+B_{1z}\frac{dB_0}{dz}\right)-\mu\left(\frac{\partial^2}{\partial z^2}-k^2\right)u_{1y}-\left(\frac{\partial u_{1y}}{\partial z}+iku_{1z}\right)\frac{d\mu}{dz}\right]-\frac{1}{\mu_0}k^2B_0B_{1z}+ik\mu\left(\frac{\partial^2}{\partial z^2}-k^2\right)u_{1z}+2ik\frac{\partial u_{1z}}{\partial z}\frac{d\mu}{dz}-ik\rho_1\Delta u\,\delta(t). \tag{13}$$

Introducing the perturbation of the interface $\xi=\bar{\xi}(z,t)e^{iky}$, the normal component of the perturbed velocity can be written as

$$u_{1z}=\frac{d\xi}{dt}. \tag{14}$$

Thus we may get the perturbations of the mass density, magnetic field and transverse velocity,

$$\rho_1=-\xi\frac{d\rho_0}{dz}, \tag{15}$$

$$B_{1y}=-\frac{\partial(\xi B_0)}{\partial z},\quad B_{1z}=ikB_0\xi, \tag{16}$$

$$u_{1y}=\frac{i}{k}\frac{\partial}{\partial t}\frac{\partial\xi}{\partial z}. \tag{17}$$

Substituting Eqs. (14)–(17) into Eq. (13), we get the equation describing the perturbations of the interface,

$$k^2\rho_0\frac{\partial^2\xi}{\partial t^2}=\frac{\partial}{\partial z}\left[\rho_0\frac{\partial^2}{\partial t^2}\frac{\partial\xi}{\partial z}+\frac{k^2B_0^2}{\mu_0}\frac{\partial\xi}{\partial z}-\mu\frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial z^2}-k^2\right)\frac{\partial\xi}{\partial z}-\frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial z^2}+k^2\right)\xi\frac{d\mu}{dz}\right]-\frac{k^4B_0^2}{\mu_0}\xi+k^2\mu\frac{\partial}{\partial t}\times\left(\frac{\partial^2}{\partial z^2}-k^2\right)\xi+2k^2\frac{\partial}{\partial t}\frac{\partial\xi}{\partial z}\frac{d\mu}{dz}+k^2\Delta u\frac{d\rho_0}{dz}\xi\delta(t). \tag{18}$$

Since at both sides of the interface, the equilibrium profiles such as ρ_0, μ, and $\mathbf{B}_0$ are constant, we have $d\varphi_0/dz=0$. Equation (18) reduces to

$$\left[\rho_0\frac{\partial^2}{\partial t^2}+\frac{k^2B_0^2}{\mu_0}-\mu\frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial z^2}-k^2\right)\right]\left(\frac{\partial^2}{\partial z^2}-k^2\right)\xi=0. \tag{19}$$

The exact form of the interface perturbations can be obtained by solving Eq. (19), but to do so one must know the time dependence of the perturbations. This apparently circular argument suggests the fact that Eq. (19) is a fourth-order eigenvalue equation.[12] Following Mikaelian's approach, we may get an explicit, analytical albeit approximate expression of the interface perturbations by taking the following assumption:

$$\frac{\partial^2\xi}{\partial z^2}-k^2\xi=0. \tag{20}$$

The general solution of Eq. (20) is a linear combination of e^{+kz} and e^{-kz}. Consider the physical boundary condition, i.e., $\xi|_{z=\pm\infty}=0$ and the continuous boundary condition, i.e., $\xi|_{z=0^+}=\xi|_{z=0^-}$, we have

$$\xi=\begin{cases}\eta(t)e^{-kz}e^{iky} & z>0\\ \eta(t)e^{+kz}e^{iky} & z<0\end{cases}. \tag{21}$$

Spatially integrating Eq. (18) over the interface from 0^- to 0^+, and taking into account Eq. (21), we have

$$\frac{d^2\eta(t)}{dt^2}+k^2v_a^2\eta(t)+2k^2\nu\frac{d\eta(t)}{dt}-kA_T\Delta u\,\eta(t)=0, \tag{22}$$

where $v_a\equiv\sqrt{(B_+^2+B_-^2)/[\mu_0(\rho_++\rho_-)]}$ is the modified *Alfvén* velocity, $\nu\equiv(\mu_++\mu_-)/(\rho_++\rho_-)$ is the mean kinematic viscosity, and $A_T\equiv(\rho_+-\rho_-)/(\rho_++\rho_-)$ is the Atwood number. Solving this equation, we may get the amplitude of the perturbations under the effects of transverse magnetic field and viscosity,

$$\frac{\eta(t)}{\eta(0)}=\left[\cosh(\omega t)+\frac{k^2\nu+kA_T\Delta u}{\omega}\sinh(\omega t)\right]e^{-k^2\nu t}, \tag{23}$$

where $\omega=k\sqrt{k^2\nu^2-v_a^2}$.

IV. DISCUSSION

In this section, effects of the magnetic field, viscosity, wavelength, and density gradient on the RM instability are discussed.

For pure magnetic field we have $\nu=0$, $\omega^2=-k^2v_a^2$, and Eq. (23) can be simplified to

$$\eta(t)/\eta(0)=\cos(kv_at)+\frac{A_T\Delta u}{v_a}\sin(kv_at). \tag{24}$$

The linear growth of the instability is suppressed to oscillation with constant amplitude.

For pure viscosity ($v_a=0$), we get

$$\frac{\eta(t)}{\eta(0)}=1+\frac{A_T\Delta u}{2k\nu}(1-e^{-2k^2\nu t}), \tag{25}$$

which is consistent with Mikaelian's work.[12]

If both $v_a=0$ and $\nu=0$, we recover the classical linear-growth result of Richtmyer[2]

$$\eta(t)/\eta(0)=1+\Delta ukA_Tt. \tag{26}$$

The combined effect of magnetic field and viscosity, shown by Eq. (23), may be qualitatively described as follows: viscosity provides damping; magnetic field provides oscillation and damping. When both are present, the motion of the perturbations will be oscillated damping provided $\omega^2<0$, otherwise there will be no oscillation. As time goes by, the perturbations will be damped out due to the dissipation of the viscosity.

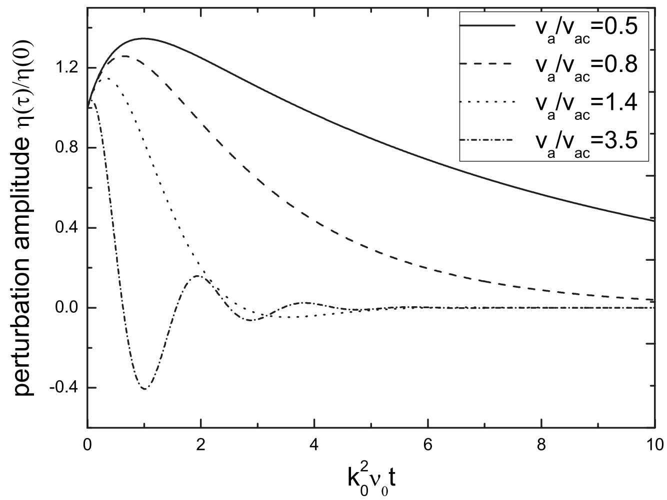

FIG. 2. The time evolution of perturbations amplitude $\eta(t)/\eta(0)$ at v_a/v_{ac} $=0.5$, 0.8, 1.4, and 3.5, where v_a is the modified Alfvén velocity and v_{ac} serves as a scale. The interface perturbations oscillate at stronger magnetic field ($v_a/v_{ac}>1$). When the viscosity is ignorable, the interface oscillates at constant amplitude.

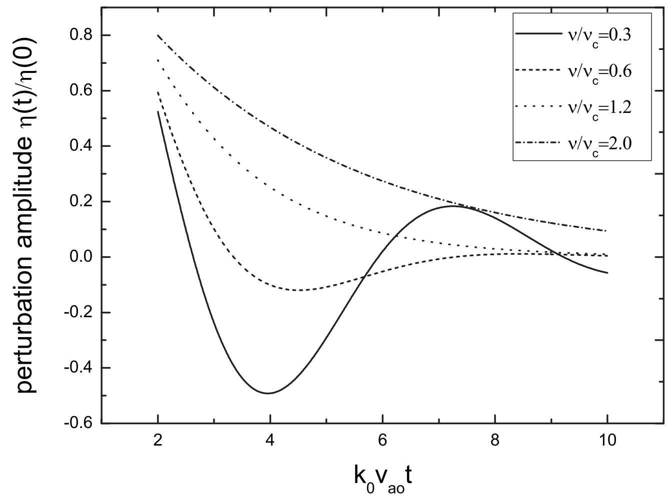

FIG. 3. $\eta(t)/\eta(0)$ vs t at $\nu/\nu_c=0.3$, 0.6, 1.2, and 2.0, where ν is the mean kinematic viscosity, ν_c serves as a scale. The perturbations damp faster for stronger viscosities. When the viscosity is not strong enough ($\nu/\nu_c<1$), the perturbed interface oscillates under the effect of the magnetic field.

To discuss the effect of the magnetic field, we have the wave number k_0 and viscosity ν_0 fixed. The natural parameter to denote the effect of magnetic field, we believe, is defined by setting $\omega=0$, i.e., $v_{ac}=k_0\nu_0$. The effect of the magnetic field on RM instability is illustrated in Fig. 2. We may see that when the magnetic field is not big enough ($v_a/v_{ac}<1$), the perturbations will be damped from the beginning. While $v_a/v_{ac}>1$, ω becomes a pure imaginary number, thus the amplitude of the perturbations will be oscillated damping with frequency $(k\sqrt{v_a^2-v_{ac}^2})$ increasing with magnetic field. The magnetic field also contributes to the suppression of the perturbations. For the simplicity of discussion, we will take the $v_a/v_{ac}>1$ case as an instance. In practice, the effect of magnetic field is often much stronger than that of the viscosity. Then the modulus of the perturbations at time t can be expressed as $\|\eta(t)/\eta(0)\| = \sqrt{1+(k\nu+A_T\Delta u)^2/(v_a^2-k^2\nu^2)}e^{-k^2\nu t}$; we may see that the amplitude of the perturbations decreases with the magnetic field. When $\omega^2>0$, the stabilization effect of the magnetic field is also present. In a nonresistive plasma, the magnetic lines are frozen together with the fluid. When perturbations occur, the magnetic tension hinders the perturbations from growing, and as a result, the instability is suppressed. At the same time, this tension acts as a restoring force, therefore the interface oscillates.

For viscosity, if we fix the magnetic field and wave number, $\omega=0$ gives the critical viscosity $\nu_c=v_{a0}/k_0$. The dependence of the perturbations on viscosity is illustrated in Fig. 3. We may observe that for larger viscosity, the perturbations are more suppressed. In this figure, we also see the criterion of oscillation at $\omega=0$. With the combined effect of magnetic field and viscosity, if the magnetic field is strong enough, the perturbations will be oscillated damping, and if the effect of viscosity dominates, it is just damping.

The dependence of the interface perturbations on the reduced wave number ($k_c=v_{a0}/\nu_0$) is shown in Fig. 4. We may see that larger k, i.e., shorter-wavelength perturbations, oscillate faster and are more damped, while longer-wavelength perturbations oscillate slower and are less damped.

Finally, we would like to discuss the effect of the density gradient on the instability. Differing from the similar Rayleigh–Taylor instability, RM instability can be initialized by shock wave propagating from the heavy fluid to the light (heavy/light, $A_T<0$), as well as that from the light to the heavy (light/heavy, $A_T>0$). The dependence of the perturbation amplitude on Atwood number is illustrated in Fig. 5. We can see that the amplitude of perturbations grows from the start for the light/heavy case and initially decreases before reversing its phase and growing for the heavy/light case. When $\omega^2<0$, the amplitude of the perturbations can be ex-

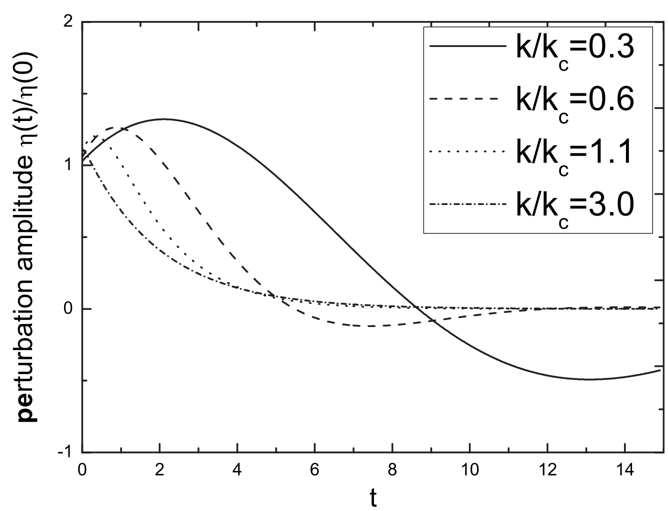

FIG. 4. Time evolution of the perturbations amplitude $\eta(t)/\eta(0)$ at k/k_c $=0.3$, 0.6, 1.1, and 3.0, where k is the wave number, k_c is the scale defined by $\omega=0$. We may see that smaller (bigger) k, i.e., longer (shorter) wavelength perturbations, oscillate slower (faster), achieve larger (smaller) maxima, and are less (more) damped. When $k/k_c>1$, ω is imaginary, and there are no oscillations.

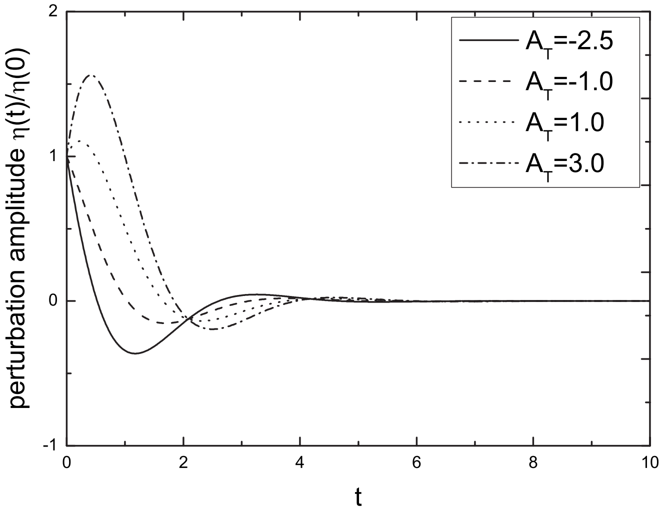

FIG. 5. Time evolution of the perturbation amplitude $\eta(t)/\eta(0)$ at $A_T=-2.5$, -1, 1, 3.0. Consider the linear stage of the instability, when $A_T<0$, corresponding to the shock wave propagate from the heavy fluid to the light, the perturbations initially decrease before reverse phase and growing; while $A_T>0$, the perturbations grow in amplitude from the beginning.

pressed as $\|\eta(t)/\eta(0)\| = \sqrt{1+(\nu+A_T\Delta u/k)^2/|\omega^2|}e^{-k^2\nu t}$, from which we can see that Atwood number influences the RM instability via the maximum amplitude of the perturbations. Especially, when $A_T=-k\nu/\Delta u$, the perturbations will not grow at all, i.e., the system is stable. Otherwise, the perturbations will increase their amplitude. However, in a long run, the perturbations will damp out due to the dissipation effect of the viscosity.

V. CONCLUSION

The effects of transverse magnetic field and viscosity on the RM instability are examined by considering the behavior of an impulsively accelerated corrugated interface separating two incompressible fluids. The expression of the interface perturbations is analytically obtained. Then effects of transverse magnetic field, viscosity, wave number, and density gradient on the RM instability are discussed, respectively. We may see that the perturbations are amplified as long as the shock interacts with the interface, then damped due to the stabilizing effect of the transverse magnetic field and the viscosity. Pure magnetic field will suppress the linear growth of the RM instability to oscillation with constant amplitude. For pure viscosity, the RM instability will reach an asymptotic limit of $\eta(0)(1+\Delta u A_T/2k\nu)$. When $\Delta u A_T=-2k\nu$, the asymptote is zero, i.e., the perturbations are completely suppressed. A criterion of oscillation at $\omega^2=k^2(k^2\nu^2-v_a^2)=0$ is obtained. When $\omega^2>0$, the amplitude of the interface perturbations is damped from the beginning; otherwise the perturbation amplitude will undergo oscillated damping with frequency $k\sqrt{v_a^2-k^2\nu^2}$. Thus we may see that systems with stronger magnetic field, longer wavelength, and weaker viscosity tend to oscillate more. We also discussed the effects of density gradient on the RM instability. The perturbations grow from the start for the light/heavy case; whereas for the heavy/light configuration, the perturbations initially decrease in amplitude before reversing its phase and growing.

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