Critical points and positive solutions of singular elliptic boundary value problems

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Abstract

Usually we do not think there is variational structure for singular elliptic boundary value problems, so it cannot be considered by using critical points theory. In this paper, we use critical theory on certain convex closed sets to solve positive solutions for singular elliptic boundary value problems, especially use the ordinary differential equation theory of Banach spaces to obtain new results on the existence of multiple positive solutions. The method is useful for other singular problems.

Keywords: Critical points; Singular problems; Positive solutions

1. Introduction

Let us consider the following singular elliptic boundary value problem

\[
\begin{cases}
-\Delta u = p(x)u^{-\gamma} + \lambda f(u) & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u|_{\partial\Omega} = 0,
\end{cases}
\]  

(1)

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\]

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has been extensively studied (see [1,2] and the references therein). In [12], the authors consider the following singular problem involving superlinear nonlinearity

\[ -\Delta u + \lambda u^{\gamma} + p(x)u^{\gamma-1} = 0, \quad u > 0 \text{ in } \Omega, \\
\lambda u|_{\partial \Omega} = 0 
\]

and \( \varphi_1 \) is the first eigenfunction.

We assume \( f \in C(R^+, R^+) \) has superlinear or asymptotically linear nonlinearities \( (R^+ = [0, +\infty)) \),

\[
(H_1) \quad \lim_{u \to 0^+} \frac{f(u)}{u} = a_1. \\
(H_2) \quad \lim_{u \to +\infty} \frac{f(u)}{u} = a_2. \\
(H_3) \quad \text{When } a_1 = 0, a_2 = +\infty, \text{ we assume } \exists \theta > 2 \text{ such that } \theta F(t) \leq t^\theta, \forall t \in R^+; \text{ for all } t \in R^+, f(t) \leq C(1 + |t|^\alpha), \alpha \in (1, \min\{\frac{\gamma+2}{\gamma-2}, \frac{2n}{\gamma(n-2)}\}), n \geq 3 \text{ (if } n \leq 2, \alpha \text{ has no restriction), where } F(t) = \int_0^t f(s) \, ds. 
\]

Furthermore, we always assume that \( f(0) = 0, f(u) \geq 0, \forall u > 0, \text{ and } f(u) \) is locally Lipschitz continuous in \( u \).

Let \( \lambda_j \) be the \( j \)th eigenvalue of \(-\Delta\) with zero Dirichlet boundary data, \( 0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots \), denote by \( n_k \) the dimension of \( \ker(-\Delta - \lambda_k I) \) and let \( \{\varphi_{k, 1}, \varphi_{k, 2}, \ldots, \varphi_{k, n_k}\} \) be an orthogonal basis of \( \ker(-\Delta - \lambda_k I) \) with the property that \( \int_{\Omega} |\nabla \varphi_{k,j}|^2 = 1 \) for \( j = 1, 2, \ldots, n_k \) and \( k = 1, 2, \ldots \). It is well known that \( \lambda_1 > 0, \lambda_1 \) is simple, i.e., \( n_1 = 1 \), and \( \varphi_{1,1} > 0 \). We denote \( \varphi_{1,1} \) by \( \varphi_1 \).

**Definition 1.** We call a function \( u \in H := H_0^1(\Omega) \) is a weak solution of (1) if \( u \) satisfies the following equality

\[
\int_{\Omega} \left[ \nabla u \cdot \nabla \varphi - p(x)u^{-\gamma} \varphi - \lambda f(u) \varphi \right] \, dx = 0, \quad \forall \varphi \in H. 
\]

Since (1) has singularity, usually we cannot say nontrivial solutions of (1) correspond to nontrivial critical points of the following functional on \( H := H_0^1(\Omega) \)

\[
J_\lambda(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx - \frac{1}{1-\gamma} \int_{\Omega} p(x)u^{1-\gamma} - \lambda \int_{\Omega} F(u) \, dx, 
\]

where \( F(u) = \int_0^u f(s) \, ds \). In \( H \) we take the inner product

\[
(u, v) = \int_{\Omega} \nabla u \cdot \nabla v, \quad \forall u, v \in H; \quad \text{and} \quad \|u\| = \sqrt{(u, u)}, \quad \forall u \in H. 
\]

The existence of solutions to the elliptic equation

\[
\Delta u + p(x)u^{\gamma-1} = 0 \quad \text{in } \Omega, \\
u|_{\partial \Omega} = 0
\]

has been extensively studied (see [1,2] and the references therein). In [12], the authors consider the following singular problem involving superlinear nonlinearity

\[
\Delta u + \lambda u^{\theta} + p(x)u^{\gamma-1} = 0, \quad u > 0 \text{ in } \Omega, \\
u|_{\partial \Omega} = 0,
\]

\[
(5)
\]
where $p: \Omega \to \mathbb{R}$ is a given nonnegative nontrivial function in $L^2(\Omega)$, $1 < \beta < 2^* - 1$, $0 < \gamma < 1$ are two constants, $2^*$ is the limiting exponent in the Sobolev embedding, $n \geq 3$, $\lambda > 0$ is a real parameter. By the Ekeland’s variational principle and careful estimates inspired by Lair–Shaker, the authors show (5) has at least two weak positive solutions for suitable $p$’s provided $\lambda > 0$ is small. They pointed out that $J_\lambda$ fails to be Frechet differential, critical point theory could not be applied to obtain the existence of solutions.

In fact, $J_\lambda$ fails to be Gâteaux differential on the whole $H$, but $J_\lambda$ is really Gâteaux differential on certain convex closed set of $C^1_0(\bar{\Omega})$, and positive solutions of singular problems are in this set under suitable conditions. We can still use critical point theory on this convex closed set to obtain the existence of solutions. For more information about critical point theory and cones, see [3–5,7,9,11,13].

2. Some lemmas

Let $g_\lambda(x,u) := p(x)u^{-\gamma} + \lambda f(u)$, $K = (-\Delta)^{-1/2}$; $\bar{G}_\lambda(u) = g_\lambda(x,u(x))$ for $x \in \bar{\Omega}$, $u \in H^1_0(\Omega)$, $u > 0$.

Let $X = C^1_0(\bar{\Omega})$ with the usual norm, $\|u\|_X = \max_{0 \leq |\alpha| \leq 1} \sup_{x \in \bar{\Omega}} |D^\alpha u(x)|$, it is well known that $X \subset H$ is densely embedded into $H$.

Let $P_H := \{u \in H : u \geq 0 \text{ almost everywhere}, P := P_H \cap X = \{u \in C^1_0(\bar{\Omega}) : u \geq 0\}$, $P$ has nonempty interior $\hat{P}$ in $X$. Let $\Gamma_\varepsilon = \{u \in P \mid u \geq \varepsilon \varphi_1\}$ for any fixed $\varepsilon > 0$, then $\Gamma_\varepsilon$ is closed convex set in $X$. Then we have

Lemma 1. $J_\lambda$ is Gâteaux differential on the convex closed set $\Gamma_\varepsilon$, and $J'_\lambda(u) = u - K\bar{G}_\lambda(u)$, $\forall u \in \Gamma_\varepsilon$.

Proof. In fact we only need check the functional $\Phi(u) = \frac{1}{1-\gamma} \int_\Omega p(x)u^{1-\gamma}$ is Gâteaux differential on the convex closed set $\Gamma_\varepsilon$. For $u \in \Gamma_\varepsilon$, $\forall h \in X$, by the integral mean value theorem, we get (as $t$ sufficiently small)

$$
\frac{1}{t}[\Phi(u + th) - \Phi(u)] = \int_{\Omega} dx \int_{u(x)}^{u(x)+th(x)} p(x)s^{-\gamma} ds \\
= \int_{\Omega} dx \left[ p(x)(u(x) + t\theta(x)h(x))^{-\gamma}th(x) \right] \\
= \int_{\Omega} dx \left[ (u(x) + t\theta(x)h(x))^{-\gamma} - u^{-\gamma}(x) \right]h(x) dx \\
+ \int_{\Omega} dx p(x)u^{-\gamma}(x)h(x) dx,
$$

(6)

where $0 \leq \theta(x) \leq 1$ is a measurable function. Note that since $u \geq \varepsilon \varphi_1$ for some $\varepsilon > 0$, $[u(x) + t\theta(x)h(x)]^{-\gamma} > 0$ is defined for $x \in \Omega$ as $t$ sufficiently small. And by $p(x)\varphi_1^{-\gamma} \in$
\[
\epsilon(\lambda) > L
\]
Under Lemma 3.

For any fixed \(J\), if any sequence \(\epsilon(\lambda) > 0\) sufficiently large, and for all weak positive solutions \(u\) of (1), there exists \(\epsilon(\lambda) > 0\) such that \(u\) is a weak positive solution of (1), by Corollary 1.1 of [6] (p. 143), then \(u \in L^2(\Omega)\). Since \(g_2(x, u) > \lambda(a_2 - \delta_2)u (0 < \delta_2 < a_2)\), as \(u > 0\) sufficiently large, and \(g_2(x, u) > p(x)\) as \(u > 0\) sufficiently small, and \(g_2(x, u) > 0, \forall u > 0\). So \(g_2(x, u) > a_2(x) > \delta_2 > a_2\) for some nontrivial \(a_2(x)\). Thus if \(u\) is a positive solution of (1), then there exists \(\epsilon(\lambda) > 0\) such that \(u \geq K \delta_2 \geq \epsilon \phi_1\) \((\epsilon(\lambda) > 0\) independent of the solution \(u\)). Thus \(u \in \Gamma_{\gamma(\lambda)}\). This finishes the proof. \(\Box\)

**Lemma 2.** For any fixed \(\lambda > 0\), and for all weak positive solutions \(u\) of (1), there exists \(\epsilon(\lambda) > 0\) such that \(u \in \Gamma_{\gamma(\lambda)}\).

**Proof.** Since \(H \hookrightarrow L^\infty(\Omega), g_2(x, u) : L^\infty(\Omega) \rightarrow L^q(\Omega) (0 < \lambda < +\infty, u \geq \phi_1)\) under our assumptions, if \(u\) is a weak positive solution of (1), by Corollary 1.1 of [6] (p. 143), then \(u \in C_0^1(\Omega)\). Since \(g_2(x, u) > \lambda(a_2 - \delta_2)u (0 < \delta_2 < a_2)\), as \(u > 0\) sufficiently large, and \(g_2(x, u) > p(x)\) as \(u > 0\) sufficiently small, and \(g_2(x, u) > 0, \forall u > 0\). So \(g_2(x, u) > a_2(x) > 0, \forall u > 0\) for some nontrivial \(a_2(x)\). Thus if \(u\) is a positive solution of (1), then there exists \(\epsilon(\lambda) > 0\) such that \(u \geq K \delta_2 \geq \epsilon \phi_1\) \((\epsilon(\lambda) > 0\) independent of the solution \(u\)). Thus \(u \in \Gamma_{\gamma(\lambda)}\). This finishes the proof. \(\Box\)

Let \(M = \{u \in \Gamma \mid J^*_c(u) = \emptyset\}\), and let \(u(t, u_0), 0 < t < \eta(u_0)\) be the unique right-direction saturation solution of the initial value problem in \(H\):

\[
\begin{cases}
u' = -J^*_c(u), \\
u(0) = u_0, \quad u_0 \in \hat{P}.
\end{cases}
\]

We know by [8] that

\[
u(t, u_0) = e^{-t} \left[u_0 + \int_0^t e^s K \bar{G}(u(s, u_0)) \, ds\right].
\]

**Definition 1.** We call \(N \subset H\) is an invariant set of descent flow of \(J_2\) if \(\{u(t, u_0) \mid 0 < t < \eta(u_0), u_0 \in N\} \subset N\).

**Definition 2.** Suppose that \(J \in C^1(X, R^1), c \in R^1, N\) is an invariant set of descent flow of \(J\), we call \(J\) has retracting property for \(c\) on \(N\), if \(\forall b > c\), \(J^{-1}(c, b) \cap N \cap M = \emptyset\), then \(J^c \cap N\) is a retract of \(J^b \cap N\), i.e., there exists \(\eta : J^b \cap N \rightarrow J^c \cap N\) continuous in \(X\), such that \(\eta(J^b \cap N) \subset J^c \cap N\), \(\eta|_{J^b \cap N} = \text{id}|_{J^b \cap N}\).

By [8] we can get that \(X\) is an invariant set of descent flow of \(J_\lambda\), i.e., \(\forall u_0 \in X\), then \(u(t, u_0) \subset X\).

**Lemma 3.** Under \((H_1)\), \((H_2)\) and \(0 < \lambda < \frac{1}{\delta_2}\) \((\text{or} (H_3))\), the functional \(J_\lambda\) satisfies PS condition on \(\Gamma_{\gamma(\lambda)}\) (we say \(J_\lambda\) satisfies PS condition on \(\Gamma_{\gamma(\lambda)}\) if any sequence \(\{x_n\} \subset L^2(\Omega)\), we know \(p(x)u^{-\gamma} \in L^2(\Omega)\), thus the linear functional \(\int_\Omega p(x)u^{-\gamma} h(x)\) is defined for \(h \in \Omega\). Since \(p(x)u^{-\gamma} : \Gamma_{\gamma^0} \rightarrow L^2(\Omega)\) is continuous, we have that

\[
\left| \frac{1}{t} \left[ \Phi(u + th) - \Phi(u) \right] - \int_\Omega p(x)u^{-\gamma} h(x) \, dx \right| 
\leq \left\| p(u + t\theta h) - pu^{-\gamma} \right\|_{L^2(\Omega)} \left\| h \right\|_{L^2(\Omega)} \rightarrow 0 \quad (as \ t \rightarrow 0).
\]
\( \Gamma_{\varepsilon(\lambda)} \) along which \( J_{\lambda}(x_n) \) are bounded and \( J'_{\lambda}(x_n) \to 0 \) (strongly) possesses a convergent subsequence in \( \Gamma_{\varepsilon(\lambda)} \).

**Proof.** We only give the asymptotically linear case here. Let \( \{u_n\} \subset \Gamma_{\varepsilon} \) be a sequence such that

\[
|J_{\lambda}(u_n)| \leq c, \quad J'_{\lambda}(u_n) \to 0. \tag{10}
\]

It is easy to see that

\[
g_{\lambda}(x,u) \leq C + (\lambda_1 - \varepsilon)u, \quad (\lambda a_2 + \varepsilon < \lambda_1), \quad \forall u \in \Gamma_{\varepsilon(\lambda)}. \tag{11}
\]

Now (10) implies that \( \forall \phi \in H^1_0 \)

\[
\int_{\Omega} (\nabla u_n \nabla \phi - g_{\lambda}(x,u_n) \phi) \, dx \to 0.
\]

Set \( \phi = u_n \) we have

\[
\|u_n\|^2 = \int_{\Omega} g_{\lambda}(x,u_n) u_n \, dx + \left\{ J'_{\lambda}(u_n), u_n \right\} \leq \int_{\Omega} g_{\lambda}(x,u_n) u_n \, dx + o(1)\|u_n\|
\]

\[
\leq C\|u_n\|_{L^2(\Omega)}^2 + (\lambda_1 - \varepsilon)\|u_n\|_{L^2(\Omega)}^2 + o(1)\|u_n\|
\]

\[
\leq C\lambda_{\delta}^{-\frac{1}{2}}\|u_n\| + \frac{\lambda_1 - \varepsilon}{\lambda_{\delta}}\|u_n\|^2 + o(1)\|u_n\|. \tag{12}
\]

So \( \|u_n\| \) is bounded. A standard argument shows that \( \{u_n\} \) has a convergent subsequence such that \( u_{n_k} \to u^* \) in \( H \), and \( u^* \geq \varepsilon \varphi_1 \), and \( J'_{\lambda}(u^*) = 0 \), by the regular argument at the beginning of Lemma 2, we get that \( u^* \in \Gamma_{\varepsilon(\lambda)} \). Therefore, \( J_{\lambda} \) satisfies the (PS) condition on \( \Gamma_{\varepsilon(\lambda)} \).

For the superlinear case, we can prove as that of Lemma 2.2 (p. 147) of [6]. \( \square \)

### 3. Existence results

Now we give the main results

**Theorem 1.** Under the assumptions (H\(_1\)), (H\(_2\)), when \( 0 < \lambda < \frac{\lambda_1}{a_2} \), (1) has at least one positive solution.

**Proof.** For fixed \( 0 < \lambda < \frac{\lambda_1}{a_2} \), by (9) for any \( u_0 \in \Gamma_{\varepsilon(\lambda)} \), \( u(t, u_0) \subset \Gamma_{\varepsilon(\lambda)}, t \geq 0 \), so \( \Gamma_{\varepsilon(\lambda)} \) is an invariant set of descent flow of \( J_{\lambda} \).

By (H\(_2\)), we get that \( \exists \delta > 2\delta < \lambda_1 - \lambda a_2 \) such that

\[
g_{\lambda}(x,u) \leq \lambda a_2 u + \delta u + C, \quad \forall u \in \Gamma_{\varepsilon(\lambda)},
\]

then let \( G_{\lambda}(x,u) = \int_0^t g_{\lambda}(x,t) \, dt \), then

\[
\lambda G_{\lambda}(x,u) \leq \frac{\lambda a_2 + \delta}{2} u^2 + C_1 u, \quad u \in \Gamma_{\varepsilon(\lambda)}.
\]
Therefore, by Poincare inequality,
\[
J_\lambda(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \frac{1}{1-\gamma} \int_\Omega p(x)u^{1-\gamma} - \lambda \int_\Omega F(u) \, dx
\]
\[
= \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \int_\Omega G_\lambda(x,u) \, dx
\]
\[
\geq \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \frac{\lambda \alpha_2 + \delta}{2} \|u\|_{L^2_1(\Omega)}^2 - C_2\|u\|
\]
\[
\geq \frac{1}{2} \|u\|^2 - \frac{1}{2}(\lambda_1 - \delta)\|u\|_{L^2_1(\Omega)}^2 - C_2\|u\|^2 \geq \frac{1}{2}\|u\|^2 - C_2\|u\|.
\]

So we know that $J_\lambda$ is bounded from below on $\Gamma_{c(\lambda)}$ when $\lambda < \frac{\lambda_1}{\delta_2}$. $J_\lambda$ has retracting property for any $c \in R^1$ on $\Gamma_{c(\lambda)}$ by [8], so for any $u_0 \in \Gamma_{c(\lambda)}$, $u(t, u_0) \to u^*$ in $X$ as $t \to +\infty$, $u^*$ is a critical point of $J_\lambda$. By Lemma 2, we know $u^* \in \Gamma_{c(\lambda)}$. \hfill \square

**Remark 1.** Since when $\tilde{\lambda}_2 < \tilde{\lambda}_1$, the positive solution $u_{\tilde{\lambda}_1}$ of (13) is a super-solution for $\lambda = \tilde{\lambda}_2$ in (13), so the solutions $u_{\lambda} \to u_0$ in $H$ as $\lambda \to 0$ (up to a subsequence), by (2) we know $u_0$ is the solution of

\[
\begin{cases}
-\Delta u = p(x)u^{1-\gamma} & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u|_{\partial \Omega} = 0.
\end{cases}
\]

(13)

**Theorem 2.** Under the assumptions $(H_1)$, $(H_3)$, $\exists \lambda_0 > 0$ such that as $\lambda \in (0, \lambda_0)$, (13) has at least 2 positive solutions.

**Proof.** First we find a positive super-solution of (13) for all $\lambda \in (0, \lambda_0]$ for some $\lambda_0 > 0$.

Since $p(x)\varphi_1^{1-\gamma} \in L^q(\Omega)$, by Corollary 1.1 of [6] (p. 143), we know that $K(p(x)\varphi_1^{1-\gamma}) \in C^1_0(\bar{\Omega}) \cap \bar{P}$. And since $\varphi_1 \in \bar{P}$, for some $t_0 > 1$ we can obtain

\[
t_0\lambda_1\varphi_1 - t_0^{-\gamma}K(p(x)\varphi_1^{-\gamma}) \in \bar{P}.
\]

For such $t_0$, we know that $\exists M > 0$ such that $|f(t\varphi_1(x))| \leq M$, $\forall t \in (0, t_0]$. Thus $\exists \lambda_0 > 0$ such that $\forall \lambda \in (0, \lambda_0]$, we have

\[
\lambda Kf(t_0\varphi_1) \leq t_0\lambda_1\varphi_1 - t_0^{-\gamma}K(p(x)\varphi_1^{-\gamma}).
\]

Therefore, $t_0\varphi_1$ is a positive super-solution for all $\lambda \in (0, \lambda_0]$. Clearly, for each $\lambda \in (0, \lambda_0]$, $\varepsilon(\lambda)\varphi_1$ is its sub-solution. By (9), for $u_0 \in [\varepsilon(\lambda)\varphi_1, t_0\varphi_1]$, we first have $u(t, u_0) > \varepsilon(\lambda)\varphi_1$, $\forall t > 0$, where $\varepsilon(\lambda)\varphi_1, t_0\varphi_1 = \{u \in X : \varepsilon(\lambda)\varphi_1(x) \leq u(x) \leq t_0\varphi_1(x)\}$. And for $u \in [\varepsilon(\lambda)\varphi_1, \sup_{x \in \Omega}[t_0\varphi_1(x)]]$, there is $m > 0$ such that $p(x)u^{1-\gamma} + \lambda f(u) + mu$ is increasing for $u$ for fixed $x \in \Omega$. Now in $H$ we take the inner product $\|u\|_1 = \|u\| + \|u\|_{L^2_1(\Omega)}$ which


is equivalent to $\|u\|$. Under this norm, $J'_\lambda(u) = u - (-\Delta + m)^{-1}(p(x)u^{-\gamma} + \lambda f(u) + mu)$, and (9) becomes

$$u(t, u_0) = e^{-t}\left[u_0 + \int_0^t e^s(-\Delta + m)^{-1}\left(p(x)u^{-\gamma} + \lambda f(u) + mu\right)ds\right].$$  \hspace{1cm} (14)

$(-\Delta + m)^{-1}$ is a linear compact operator, and has same property as that of $K$, it is strongly order-preserving too, i.e., for any $v \in P$, $(-\Delta + m)^{-1}v \in \hat{P}$. Thus, if $u_0 \in [\varepsilon(\lambda)\varphi_1, t_0\varphi_1]$, then $u(t, u_0) \subset [\varepsilon(\lambda)\varphi_1, t_0\varphi_1]$ by (14). And the solution of (8) is the same under the two norms by the uniqueness. So $[\varepsilon(\lambda)\varphi_1, t_0\varphi_1]$ is an invariant set of descent flow of $J_\lambda$. It is clear that $J_\lambda$ is bounded from below and has retracting property on $\varepsilon(\lambda)\varphi_1$, so by Lemma 2 of [10], there is a critical point $u_1$ in $[\varepsilon(\lambda)\varphi_1, t_0\varphi_1]$, i.e.,

$$\varepsilon(\lambda)\varphi_1(x) \leq u_1(x) \leq t_0\varphi_1(x), \quad \forall x \in \bar{\Omega}.$$  

Remark 2. It is possible that the conditions of this paper can be weaken, the main purpose of this paper is to give a new method to solve some singular problems.

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References

