Note on a paper by N. Ujević

Zheng Liu

Institute of Applied Mathematics, Faculty of Science, Anshan University of Science and Technology, Anshan 114044, Liaoning, China

Received 10 July 2006; accepted 14 September 2006

Abstract

A generalization of two sharp inequalities in a recent paper by N. Ujević is established. Applications in numerical integration are also given and the results of N. Ujević are revised and improved.

© 2006 Elsevier Ltd. All rights reserved.

Keywords: Sharp inequality; Generalization; Composite quadrature formula

1. Introduction

In a recent paper [1], Ujević has proved the following two interesting sharp inequalities of Simpson type and Ostrowski type:

**Theorem 1.** Let $f : [a, b] \rightarrow \mathbb{R}$ be an absolutely continuous function whose derivative $f' \in L_2(a, b)$. Then

$$\left| \frac{b-a}{6} \left[ f(a) + 4f \left( \frac{a+b}{2} \right) + f(b) \right] - \int_a^b f(t) \, dt \right| \leq \frac{(b-a)^{3/2}}{6} \sqrt{\sigma(f')},$$

(1)

where $\sigma(\cdot)$ is defined by

$$\sigma(f) = \|f\|_2^2 - \frac{1}{b-a} \left( \int_a^b f(t) \, dt \right)^2,$$

(2)

and $\|f\|_2 := \left( \int_a^b f^2(t) \, dt \right)^{1/2}$. Inequality (1) is sharp in the sense that the constant $\frac{1}{6}$ cannot be replaced by a smaller one.

**Theorem 2.** Under the assumptions of Theorem 1, for any $x \in [a, b]$, we have

$$\left| (b-a) f(x) - \left( x - \frac{a+b}{2} \right) [f(b) - f(a)] - \int_a^b f(t) \, dt \right| \leq \frac{(b-a)^{3/2}}{2\sqrt{3}} \sqrt{\sigma(f')}.$$

(3)

Inequality (3) is sharp in the sense that the constant $\frac{1}{2\sqrt{3}}$ cannot be replaced by a smaller one.

E-mail address: lewzheng@163.net.

0893-9659/$ - see front matter © 2006 Elsevier Ltd. All rights reserved.
doi:10.1016/j.aml.2006.09.001
Remark 1. It should be noticed that the inequality (3) first appeared in [2] without a proof of its sharpness.

In this work, we will derive a new sharp inequality with a parameter for absolutely continuous functions with derivatives belonging to $L_2(a, b)$, which will not only provide a generalization of inequalities (1) and (3), but also give some other interesting sharp inequalities as special cases and showing that the averaged mid-point–trapezoid quadrature rule is optimal. Applications in numerical integration are also given. The results on estimating error bounds for the corresponding composite quadrature formulas in [1] are revised and improved.

2. The results

Theorem 3. Let the assumptions of Theorem 1 hold. Then for any $\theta \in [0, 1]$ and $x \in [a, b]$ we have
\[
|b - a| \left[(1 - \theta)f(x) + \theta \frac{f(a) + f(b)}{2}\right] - (1 - \theta)\left(x - \frac{a + b}{2}\right) \frac{f(b) - f(a)}{2} - \int_a^b f(t) \, dt | \\
\leq \left[\theta (1 - \theta) (b - a) \left(x - \frac{a + b}{2}\right)^2 + \left(\frac{1}{12} - \frac{\theta}{4} + \frac{\theta^2}{4}\right) (b - a)^3\right]^{\frac{1}{2}} \sqrt{\sigma(f')}.
\]

The inequality (4) is sharp in the sense that the coefficient constant 1 of the right-hand side cannot be replaced by a smaller one.

Proof. Let us define the function
\[
K(x, t) := \begin{cases} 
 t - \left( a + \theta \frac{b - a}{2} \right), & t \in [a, x], \\
 t - \left( b - \theta \frac{b - a}{2} \right), & t \in (x, b].
\end{cases}
\]

Integrating by parts, we obtain
\[
\int_a^b K(x, t) f'(t) \, dt = (b - a) \left[(1 - \theta)f(x) + \theta \frac{f(a) + f(b)}{2}\right] - \int_a^b f(t) \, dt. \tag{5}
\]

We also have
\[
\int_a^b K(x, t) \, dt = (1 - \theta)(b - a) \left(x - \frac{a + b}{2}\right) \tag{6}
\]
and
\[
\int_a^b f'(t) \, dt = f(b) - f(a). \tag{7}
\]

From (5) to (7), it follows that
\[
\int_a^b \left[K(x, t) - \frac{1}{b - a} \int_a^b K(x, s) \, ds\right] f'(t) - \frac{1}{b - a} \int_a^b f'(s) \, ds \, dt \\
= (b - a) \left[(1 - \theta)f(x) + \theta \frac{f(a) + f(b)}{2}\right] - (1 - \theta)\left(x - \frac{a + b}{2}\right) \frac{f(b) - f(a)}{2} - \int_a^b f(t) \, dt. \tag{8}
\]

On the other hand, we have
\[
\left| \int_a^b \left[K(x, t) - \frac{1}{b - a} \int_a^b K(x, s) \, ds\right] f'(t) - \frac{1}{b - a} \int_a^b f'(s) \, ds \, dt \right| \\
\leq \left\| K(x, \cdot) - \frac{1}{b - a} \int_a^b K(x, s) \, ds \right\|_2 \left\| f' - \frac{1}{b - a} \int_a^b f'(s) \, ds \right\|_2. \tag{9}
\]
We also have
\[
\left\| K(x, \cdot) - \frac{1}{b-a} \int_a^b K(x, s) \, ds \right\|_2^2 = \theta(1 - \theta)(b - a) \left( x - \frac{a + b}{2} \right)^2 + \left( \frac{1}{12} - \frac{\theta}{4} + \frac{\theta^2}{4} \right) (b - a)^3
\]
(10)
and
\[
\left\| f' - \frac{1}{b-a} \int_a^b f'(s) \, ds \right\|_2^2 = \|f'\|_2^2 - \frac{(f(b) - f(a))^2}{b-a}.
\]
(11)
From (8) to (11), we can easily get (4), since by (2) we have
\[
\sqrt{\sigma(f')} = \left[ \|f'\|_2^2 - \frac{(f(b) - f(a))^2}{b-a} \right]^{\frac{1}{2}}.
\]
In order to prove that the inequality (4) is sharp for any \( \theta \in [0, 1] \) and \( x \in [a, b] \), we define the function
\[
f(t) = \begin{cases} 
\frac{1}{2}t^2 - \frac{\theta}{2}t, & \text{if } t \in [0, x], \\
\frac{1}{2}t^2 - \left(1 - \frac{\theta}{2}\right)(1 - \theta)x, & \text{if } t \in (x, 1].
\end{cases}
\]
(12)
Clearly, the function given in (12) is absolutely continuous on \([a, b]\), since it is a continuous piecewise polynomial function.

We now suppose that (4) holds with a constant \( C > 0 \) as
\[
(b - a) \left[ (1 - \theta)f(x) + \frac{f(a) + f(b)}{2} \right] - (1 - \theta) \left( x - \frac{a + b}{2} \right) [f(b) - f(a)] - \int_a^b f(t) \, dt \\
\leq C \left[ \theta(1 - \theta)(b - a) \left( x - \frac{a + b}{2} \right)^2 + \left( \frac{1}{12} - \frac{\theta}{4} + \frac{\theta^2}{4} \right) (b - a)^3 \right]^{\frac{1}{2}} \sqrt{\sigma(f')}.
\]
(13)
Choosing \( a = 0, b = 1 \), and \( f \) defined in (12), we get
\[
\int_0^1 f(t) \, dt = -\frac{1}{2}(1 - \theta)x^2 + (1 - \theta)x + \frac{\theta}{4} - \frac{1}{3},
\]
\[
f(0) = 0, \quad f(1) = (1 - \theta) \left( x - \frac{1}{2} \right), \quad f(x) = \frac{x}{2}(x - \theta)
\]
and
\[
\int_0^1 (f'(t))^2 \, dt = (1 - \theta) \left( x - \frac{1}{2} \right)^2 + \frac{1}{12} - \frac{\theta}{4} + \frac{\theta^2}{4},
\]
such that the left-hand side becomes
\[
\text{L.H.S.}(13) = \theta(1 - \theta) \left( x - \frac{1}{2} \right)^2 + \frac{1}{12} - \frac{\theta}{4} + \frac{\theta^2}{4}.
\]
(14)
We also find that the right-hand side is
\[
\text{R.H.S.}(13) = C \left[ \theta(1 - \theta) \left( x - \frac{1}{2} \right)^2 + \frac{1}{12} - \frac{\theta}{4} + \frac{\theta^2}{4} \right].
\]
(15)
From (13) to (15), we find that \( C \geq 1 \), proving that the coefficient constant 1 is the best possible in (4). □
Corollary 1. Let the assumptions of Theorem 3 hold. Then for any \( \theta \in [0, 1] \) we have
\[
\left| (b-a) \left[ (1-\theta) f \left( \frac{a+b}{2} \right) + \theta f(a) + f(b) \right] - \int_a^b f(t) \, dt \right| \leq \frac{(b-a)^3}{2\sqrt{3}} (1-3\theta^2)^{\frac{1}{2}} \sqrt{\sigma(f')} . \tag{16}
\]

Proof. We set \( x = \frac{a+b}{2} \) in (4) to get (16). \( \square \)

Remark 2. If we take \( \theta = 0 \) and \( \theta = 1 \) in (16), we get a sharp mid-point type inequality
\[
\left| (b-a) f \left( \frac{a+b}{2} \right) - \int_a^b f(t) \, dt \right| \leq \frac{(b-a)^3}{2\sqrt{3}} \sqrt{\sigma(f')},
\]
and a sharp trapezoid type inequality
\[
\left| \frac{b-a}{2} [ f(a) + f(b) ] - \int_a^b f(t) \, dt \right| \leq \frac{(b-a)^3}{2\sqrt{3}} \sqrt{\sigma(f')}, \tag{17}
\]
respectively.

If we take \( \theta = \frac{1}{4} \), we recapture the sharp Simpson type inequality (1), and if we take \( \theta = \frac{1}{2} \), we get a sharp averaged mid-point–trapezoid type inequality as
\[
\left| \frac{b-a}{4} \left[ f(a) + 2 f \left( \frac{a+b}{2} \right) + f(b) \right] - \int_a^b f(t) \, dt \right| \leq \frac{(b-a)^3}{4\sqrt{3}} \sqrt{\sigma(f')} . \tag{18}
\]
It is interesting to notice that the smallest bound for (4) is obtained at \( x = \frac{a+b}{2} \) and \( \theta = \frac{1}{2} \). Thus the averaged mid-point–trapezoid rule is optimal in the current situation.

Remark 3. If we take \( x = a \) or \( x = b \) or \( \theta = 1 \) in (4), then the inequality (17) is recaptured.

Remark 4. If we take \( \theta = 0 \) in (4), then the sharp Ostrowski type inequality (3) is recaptured. Thus Theorem 3 may be regarded as a generalization of Theorems 1 and 2.

3. Applications in numerical integration

We restrict further considerations to the averaged mid-point–trapezoid quadrature rule. We also emphasize that similar considerations can be given for all quadrature rules considered in the previous section.

Theorem 4. Let \( \pi = \{ x_0 = a < x_1 < \cdots < x_n = b \} \) be a given subdivision of the interval \( [a, b] \) such that \( h_i = x_{i+1} - x_i = h = \frac{b-a}{n} \) and let the assumptions of Theorem 3 hold. Then we have
\[
\left| \int_a^b f(t) \, dt - \frac{h}{4} \sum_{i=0}^{n-1} \left[ f(x_i) + 2 f \left( \frac{x_i + x_{i+1}}{2} \right) + f(x_{i+1}) \right] \right| \leq \frac{(b-a)^3}{4\sqrt{3} n} \sqrt{\sigma(f')} . \tag{19}
\]

Proof. From (18) in Corollary 1 we obtain
\[
\left| \frac{h}{4} \left[ f(x_i) + 2 f \left( \frac{x_i + x_{i+1}}{2} \right) + f(x_{i+1}) \right] - \int_{x_i}^{x_{i+1}} f(t) \, dt \right| \leq \frac{h^3}{4\sqrt{3}} \int_{x_i}^{x_{i+1}} (f'(t))^2 \, dt - \frac{1}{h} \left( f(x_{i+1}) - f(x_i) \right)^2 \frac{1}{2} . \tag{20}
\]
By summing (20) over \( i \) from 0 to \( n-1 \) and using the generalized triangle inequality, we get
\[
\left| \int_a^b f(t) \, dt - \frac{h}{4} \sum_{i=0}^{n-1} \left[ f(x_i) + 2f \left( \frac{x_i + x_{i+1}}{2} \right) + f(x_{i+1}) \right] \right|
\leq \frac{h \frac{7}{2}}{4\sqrt{3}} \sum_{i=0}^{n-1} \left[ \int_{x_i}^{x_{i+1}} (f'(t))^2 \, dt - \frac{1}{h} (f(x_{i+1}) - f(x_i))^2 \right]^{\frac{1}{2}}.
\]

By using the Cauchy inequality twice, it is not difficult to obtain
\[
\sum_{i=0}^{n-1} \left[ \int_{x_i}^{x_{i+1}} (f'(t))^2 \, dt - \frac{1}{h} (f(x_{i+1}) - f(x_i))^2 \right]^{\frac{1}{2}} \leq \sqrt{n} \left[ \|f'\|_2^2 - \frac{n}{b-a} \sum_{i=0}^{n-1} (f(x_{i+1}) - f(x_i))^2 \right]^{\frac{1}{2}}
\leq \sqrt{n} \left[ \|f'\|_2^2 - \frac{\|f(b) - f(a)\|^2}{b-a} \right]^{\frac{1}{2}} = \sqrt{n} \sigma(f').
\]

Consequently, the inequality (19) follows from (21) and (22). □

**Remark 5.** Finally, we would like to point out that under the same assumptions as for Theorem 4, the inequalities (30), (34) in [1] may be revised and improved as
\[
\left| \int_a^b f(t) \, dt - \frac{h}{6} \sum_{i=0}^{n-1} \left[ f(x_i) + 4f \left( \frac{x_i + x_{i+1}}{2} \right) + f(x_{i+1}) \right] \right| \leq \frac{(b-a)^{\frac{3}{2}}}{6n} \sqrt{\sigma(f')}
\]
and
\[
\left| \int_a^b f(t) \, dt - \sum_{i=0}^{n-1} \left[ hf(\xi_i) - \left( \frac{\xi_i - x_i + x_{i+1}}{2} \right) (f(x_{i+1}) - f(x_i)) \right] \right| \leq \frac{(b-a)^{\frac{3}{2}}}{2\sqrt{3n}} \sqrt{\sigma(f')},
\]
where \(\xi_i \in [x_i, x_{i+1}], i = 0, 1, \ldots, n - 1\).

**References**