Isomorphism of Cyclic Cubes and Bidi Shufflenet

Zhen Chen  
State Key Lab of ISN, Xidian University.  
E-mail: zhenchen@xidian.edu.cn.

Zengji Liu  
State Key Lab of ISN, Xidian University.  
E-mail: zjlj@xidian.edu.cn.

Zhiliang Qiu  
State Key Lab of ISN, Xidian University.  
E-mail: zdqiu@mail.xidian.edu.cn.

Abstract

Cyclic Cubes, proposed as a new family of Interconnection Networks, is proved not new and is equivalent to the wrapped butterfly networks. There is a similar fact that the Cyclic Cubes is also equivalent to the Bidi Shufflenet, which is investigated recently. But this fact is almost neglected. In this paper, we discuss the equivalence between the Cyclic Cubes and the Bidi Shufflenet.

1. Introduction

Let $G$ be a finite group and let the generator set $\Omega \subseteq G$ satisfy $1 \notin \Omega$, where 1 denotes the identity element of $G$, and $h \in \Omega$ implies $h^{-1} \in \Omega$. The Cayley Graph $(G, \Omega)$ is the graph whose vertices are the elements of $G$ and for each $g \in G$, an edge joining $g$ to $g \cdot h$ for all $h \in \Omega$ (· represents group operation).

In the Cayley graph, I is excluded from the $\Omega$ to prevent self-loops. And, when there is an edge from $g$ to $g \cdot h$, there is also an edge from $g \cdot h$ to $(g \cdot h) \cdot h^{-1} = g$, thus the Cayley graph is undirected.

For its good characteristics, Cayley graphs have been proposed by several authors as candidates for Interconnection Networks [8]. Akers and Krishnamurthy [1] present two new families of Cayley graph, which are called star networks and pancake networks. P. Vadapalli and K. Srimani [2] present a family of degree four Cayley called star networks and pancake networks. P. Vadapalli [1] presents two new families of Cayley graph, which are equivalent to the wrapped butterfly networks [10]. The purpose of this paper is to show that the Cyclic Cubes [5] is also equivalent to the Bidi Shufflenet [7], which is investigated recently.

2. Isomorphism of Cyclic Cubes and Bidi Shufflenet

2.1 Cyclic Cubes

We first review the Cyclic Cubes $CC(p, k)$ . In the $CC(p, k)$, each vertex is represented as a circular permutation of $k$ different symbols in lexicographic order, where every symbol is superscripted with $1, 2, \ldots, p$. Let $t_i$, $0 \leq i \leq k-1$, denote the $i$-th symbol in the set of $n$ symbols. We can use the alphabet as symbols: thus, for $n = 4$, $t_0 = a$, $t_1 = b$, $t_2 = c$, and $t_3 = d$. We use $t_i^j$ to denote a symbol with superscript $j$, $0 \leq j \leq p-1$. Therefore, for $k$ distinct symbols, there are exactly $k$ different cyclic permutations of the symbols in lexicographic order. Since each symbol can be presented in the form $t_i^j$, $0 \leq j \leq p-1$, the vertex set of $CC(p, k)$ has a cardinality of $k \times p^k$.

Since each vertex is some cyclic permutation of the $k$ symbols in lexicographic order, then, if $a_0a_1\cdots a_{k-1}$ denotes the label of an arbitrary vertex and $a_0 = t_i^j$ for some integer $i, j$, for all $l$, $1 \leq l \leq k - 1$, we have $a_l = t_{(i+l)modn}^j = t_{(i+l)modn}^j$. Thus, the definition of $CC(p, k)$ is given as follows.

Definition 1. The graph $CC(p, k)$ is a Cayley graph whose vertex are the $k \times p^k$ cyclic permutations of $k$ distinct symbols in lexicographic order. Each symbol has the form $t_i^j$, $0 \leq j \leq p-1$. Given a vertex represented as a string $a_0a_1\cdots a_{k-1}$, its edges are defined by the following generators functions:

$$g(a_0^{i_0}a_1^{i_1}\cdots a_{k-1}^{i_{k-1}}) = a_1^{i_1}a_2^{i_2}\cdots a_{k-1}^{i_{k-1}}a_0^{i_0}$$

$$g^{-1}(a_0^{i_0}a_1^{i_1}\cdots a_{k-1}^{i_{k-1}}) = a_{k-1}^{i_{k-1}}a_0^{i_0}\cdots a_1^{i_1}$$

$$f^i(a_0^{i_0}a_1^{i_1}\cdots a_{k-1}^{i_{k-1}}) = a_0^{i_0}a_1^{2i_1}\cdots a_{k-1}^{i_{k-1}}a_0^{i_0}$$

$$f^{-i}(a_0^{i_0}a_1^{i_1}\cdots a_{k-1}^{i_{k-1}}) = a_{k-1}^{i_{k-1}}a_0^{i_0}\cdots a_1^{i_1}$$

where $i = 1, 2, \ldots, p-1$ and $\oplus$ represents the addition modulo $p$ (we also assume the superscript of a symbol is taken modulo $p$).
If the identity permutation is \( t_0 t_1 \cdots t_{k-1} \), then the generator set: \( \Omega = \{ g, g^{-1}, f, f^{-1} | 1 \leq i \leq p - 1 \} \) is given as:

\[
\begin{align*}
g &= t_0 t_1 \cdots t_{k-1} t_0 \\
g^{-1} &= t_{k-1} t_0 t_1 \cdots t_{k-2} \\
\vdots \\
f^i &= t_{k-1}^i t_0^i t_1^i \cdots t_{k-2}^i \\
f^{-i} &= t_{k-1}^{-i} t_0^{-i} t_1^{-i} \cdots t_{k-2}^{-i} \\
\vdots \\
f^{p-1} &= t_0 t_1 \cdots t_{k-2} t_0 \\
f^{-(p-1)} &= t_{k-1} t_0 t_1 \cdots t_{k-2}
\end{align*}
\]

As examples, Fig. 1(a) and Fig. 2(a) give the Cyclic Cubes CC(3, 2) and CC(2, 3) proposed in [5].

2.2 Bidi Shufflenet

The Bi-directional Shufflenet or Bidi Shufflenet [7] is the enhanced version of Shufflenet [6] with Bi-directional link. It is a wrapped k-level graph. Each level has \( p^k \) vertices. Every vertex has 2p edges, are connected to the vertices in the next column, and p are connected to the vertices in the previous column. Every edge in each side is labeled with sequence number \( l \) from the top down, \( 0 \leq l \leq p-1 \).

Now we give the formal definition of \( BSEN(p, k) \).

**Definition 2.** The \( BSEN(p, k) \) has vertex-set \( Z_k \times Z_{p^k} \).

Each vertex is represented as a pair \( \langle c, r \rangle \), where \( c \in Z_k \) is the column of the vertex and \( r \in Z_{p^k} \) is the row of the vertex. In the \( BSEN(p, k) \), the connecting pattern between consecutive columns of vertices is **perfect p-shuffle**. That means the edge of vertex \( \langle c, r \rangle \), assumes \( X \), is connected to the edge \( Y \) of vertex \( \langle (c+1) \mod k, r' \rangle \), and \( X, Y \) satisfies the perfect p-shuffle \( \sigma^p \):

\[
Y = \sigma^p(X) = (pX + \left\lfloor \frac{pX}{N} \right\rfloor) \mod N, \tag{1}
\]

where \( N = p^{k+1}, X = p \cdot r + m, Y = p \cdot r' + n, 0 \leq m, n \leq p-1, 0 \leq c \leq k-1, 0 \leq r, r' \leq p^k-1 \).

From above, the row position \( r_i \) for each vertex can be represented with a base \( p \) string \( W(i) \) of length \( k \), called string (or \( p \)-ary string) representation of vertex \( i \).

If \( X, Y \) are presented in string representation, then \( X, Y \)
satisfies:  
\[ x_{k-1}x_{k-2}\cdots x_2x_1x_0 = \sigma^p (x_{k-1}x_{k-2}\cdots x_1x_0) \]
where \( 0 \leq x_i \leq p - 1 \). (2)

By the knowledge of congruence property of Number theory [9], (1) and (2) are proved to be are equivalent.

The inverse perfect shuffle does the opposite as one expect:
\[ \sigma^p \sigma^{-1} (x_{k-1}x_{k-2}\cdots x_1x_0) = x_0x_kx_{k-1}\cdots x_2x_1 \]
It is obvious that the perfect p-shuffle connection performs a cyclic shifting of the digits in X to the left for one position while the inverse perfect p-shuffle connection performs a cyclic shifting of the digits in X to the right for one position.

These cyclic shifting property of perfect shuffle can also be applied to the vertex connection, as the following propositions show.

**Proposition 1** In a \( BSEN(p, k) \), the vertex \( \langle c+1 \rangle \mod k, y \rangle \) connecting to vertex \( \langle c, x \rangle \) can be determined by the formulae
\[ y = (px + m) \mod (p^k), 0 \leq m \leq p - 1, \]
where m is the sequence number of the edge of vertex \( \langle c, x \rangle \).

Let vertex x and y be in string representation, \( x = (x_0x_1\cdots x_{k-1})_p \) and \( y = (y_0y_1\cdots y_{k-1})_p \), from (4), it shows that
\[ y = (y_0y_1\cdots y_{k-1})_p = (x_1\cdots x_{k-1}m)_p. \] (5)

**Proposition 2** In a \( BSEN(p, k) \), the vertex \( \langle c, x \rangle \) connecting to vertex \( \langle c+1 \rangle \mod k, y \rangle \) can be determined by the formulae:

Fig. 2. Cyclic Cubes CC(2, 3) and Shufflenet BSEN(2,3).
\[ x = (y - m) / d + n \cdot r', \] where \( d = (p, p') = p \) and \[ r' = p^k / p = p^{k-1}. \] (6)

where \( n \) is the sequence number of the edge of vertex \( \langle (c + 1) \mod k, y \rangle \).

Let vertex \( x \) and \( y \) be in string representation, \( x = (x_0 x_1 \cdots x_{k-1})_p \) and \( y = (y_0 y_1 \cdots y_{k-1})_p \), from (6), it shows that
\[ x = (0 y_0 y_1 \cdots y_{k-2}) + n \cdot p^{k-2} = (n y_0 y_1 \cdots y_{k-2}), \] where \( 0 \leq n \leq p-1 \). (7)

By using congruence property and the solution of the congruence of Number theory [9], the correctness of proposition 1 and 2 is easily verified. From Proposition 1 and 2, the connection between two adjacent vertices also satisfies the cyclic shifting property of perfect shuffle if the vertices are represented in string representation.

Examples of an eighteen-vertex and a twenty-four-vertex Bidi Shufflenet are shown in Fig. 1(b) and Fig. 2(b).

### 2.3 Isomorphism of two networks

In this section, we show that the CC\((p, k)\) is isomorphic to the BSEN\((p, k)\).

Let \( T = (V', E) \) and denote two graphs, an isomorphism from the graph \( T \) to graph \( T' \) is a one-to-one mapping \( \Phi : V \to V' \), such that \((\Phi(v), \Phi(u)) \in E' \) if and only if \((v, u) \in E \). By \( T \cong T' \) mean that \( T \) and \( T' \) is isomorphism, i.e., two graphs are equivalent.

An isomorphic mapping \( \Phi \) between CC\((p, k)\) and BSEN\((p, k)\) is as follows: Given an arbitrary vertex \( a_0 a_1 \cdots a_{k-1} \) in CC\((p, k)\) and \( a_i = t_0^{j} \) for some \( i \), the vertex becomes \( a_i a_{i+1} \cdots a_{k-1} a_0 \cdots a_{i-1} \) after \(( (k - i) \mod k ) g \) operations. If we substitute \( j \) for every symbol \( t_0^j \), and let the resulting p-ary string be \( r \), then vertex \( a_0 a_1 \cdots a_{k-1} \) in CC\((p, k)\) corresponds to vertex \( \langle k - i (\mod k), r \rangle \) in BSEN\((p, k)\).

Furthermore, the generator functions \( g \) and \( f^j \) act as same as the cyclic-left-shifting while \( g^{-1} \) and \( f^{-j} \) as the cyclic-right-shifting. So the generator functions are equivalent to the perfect p-shuffle operation. Every g or f-edge in CC\((p, k)\) corresponds to one and only one edge of BSEN\((p, k)\) and preserves the adjacent vertices. This suggests that \( \Phi \) is an isomorphic mapping. That is, the CC\((p, k)\) and BSEN\((p, k)\) are equivalent.

Fig.1 and Fig.2 show clearly that the CC\((3, 2)\) and CC\((2, 3)\) proposed in [5] can turn into the BSEN\((3, 2)\) and BSEN\((2, 3)\) respectively, by reordering the vertices in the graphs.

### 3. Conclusions

We have shown that the graph CC\((p, k)\) is equivalent to the Bidi Shufflenet. Indeed, CC\((p, k) \cong BSEN(p, k)\).

And the minimal (optimal) routing algorithms proposed in [5] and [7] are identical in essential. Even so, the contributions of [5] and [7] are independent in most respects, i.e. [5] has discussed the Embedding Graph while [7] has done the deadlock-avoidance wormhole routing. And the group-theoretical representation in [5] may shed some lights on the topological properties of CC\((p, k)\) (or BSEN\((p, k)\)).

### References


