Explicit representations of the Drazin inverse of block matrix and modified matrix

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Abstract

A new result on the Drazin inverse of $2 \times 2$ block matrix $M = \begin{bmatrix} A & B \\ D & C \end{bmatrix}$, where $A$ and $C$ are square matrices are presented, extended in the case when $D = 0$, the well known representation for the Drazin inverse of $M$, given by Hartwig, Meyer and Rose in 1977, respectively. Using that new result, an explicit representation for the Drazin inverse of a modified matrix $P + RS$ and its generalized matrix $PQ + RS$ are also presented.

Key words: Drazin inverse, Modified matrix, Index, Block matrix

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1 Introduction

Let $A \in \mathbb{C}^{n \times n}$. The smallest nonnegative integer $k$ such that $\text{rank}(A^{k+1}) = \text{rank}(A^k)$, denoted by $\text{Ind}(A)$, is called the index of $A$. If $\text{Ind}(A) = k$, there exists a unique matrix $A^D \in \mathbb{C}^{n \times n}$ satisfying the following equations [2]

\[ A^{k+1}A^D = A^k, \quad A^DAA^D = A^D, \quad AA^D = A^DA. \]

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A\^D is called the Drazin inverse of \( A \). In particular, when \( \text{Ind}(A) \leq 1 \), the matrix \( A\^D \) is called the group inverse of \( A \) and denoted by \( A\# \). Clearly, \( \text{Ind}(A)=0 \) if and only if \( A \) is nonsingular, and in this case \( A\^D = A^{-1} \).

The theory of Drazin inverses has seen a substantial growth over the past few decades. It is an area which is of great theoretical interest and finds applications in many diverse areas, including Statistics, Numerical analysis, Differential equations, Markov chains, Population models, Cryptography, and Control theory (cf. [1,2,8,10,13,16,17,19]). One topic of Drazin inverse of considerable interest concerns the explicit representations for the Drazin inverse of a \( 2 \times 2 \) block matrix (cf. [3,5,11,15,18,22]) and explicit representations for the Drazin inverse of the sum of two matrices (cf. [4,12,14,20,21,23]).

A sizeable literature devoted to these topics has sprouted. The representation of the Drazin inverse of \( M = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \) (abbreviated hereafter as Hartwig-Meyer-Rose formula, cf. [2, Theorem 7.7.1], [9], or [18]) is employed frequently and plays an essential role, where \( A \) and \( C \) are square matrices. Therefore, it would be desirable to extend the Hartwig-Meyer-Rose formula to the case \( M = \begin{bmatrix} A & B \\ D & C \end{bmatrix} \), when \( D \neq 0 \). Many papers have considered this open problem and each of them offered a formula for the Drazin inverse and specific conditions for the \( 2 \times 2 \) block matrix \( M = \begin{bmatrix} A & B \\ D & C \end{bmatrix} \) to satisfy so that the formula is valid. Some of those conditions are listed below:

(i) \( \text{rank}(M) = \text{rank}(A) \) and \( \text{Ind}(A) = 0 \), (see [2]);

(ii) \( B = 0 \) (or \( D = 0 \)), (see [9],[18]);

(iii) \( BD = 0, CD = 0 \) and \( BC = 0 \) (see [7]);

(iv) \( BD = 0, CD = 0 \) (or \( BC = 0 \)) and \( C \) is nilpotent, (see [11]);

(v) \( A\pi B = 0, DA\pi = 0 \) and \( C = DA\^D B \) is either nonsingular or zero, (see [22]);

(vi) \( AA\pi B = 0, DA\pi B = 0 \) and \( C = DA\^D B \) is either nonsingular or zero, (see [11]);

(vii) \( AA\pi B = 0, DA\pi B = 0, BDA\^D = 0 \) and \( CDAA\^D = 0 \) (see [15]);

(viii) \( A = I \) and \( CD = 0 \) (or \( BC = 0 \)), (see [5]).

In this paper we will give some new conditions for a \( 2 \times 2 \) matrix \( M \) under which we obtain an explicit representation of the Drazin inverse of \( M \). Based on these results, in Section 3 we present some interesting explicit representations of the Drazin inverse of a modified matrix. Finally, in Section 4 we present several numerical examples showing that there exist matrices for which none
of the conditions (i)-(viii) are satisfied whereas the conditions from Theorem 2.1 and Theorem 2.2 are satisfied, so in such cases our results are very useful in finding Drazin inverses. Also, a numerical example is given showing an explicit representation of the Drazin inverse of a modified matrix using Theorem 3.1.

Throughout this paper we denote the identity matrix and the transpose matrix of a matrix $A$ as $I$, and $A^T$ respectively. For notational convenience, if the lower limit of a sum is bigger than its upper limit, we always define the sum to be 0. For example, we define a sum $\sum_{i=0}^{k-2} *$ to be 0 for $k < 2$. We also use that $G^0 = I$, for any square matrix $G$. By $l_A$ we denote the $\text{Ind}(A)$.

2 Some new results for the Drazin inverse of $2 \times 2$ block matrix

In this section, we present two results concerning the Drazin inverse of a $2 \times 2$ block matrix, which extend the well-known Hartwig-Meyer-Rose formula and play an important role in finding explicit representations of the Drazin inverse of a modified matrix in the Section 3.

**Theorem 2.1.** Let $M = \begin{bmatrix} A & B \\ D & C \end{bmatrix} \in \mathbb{C}^{n \times n}$, where $A$ and $C$ are square matrices. If $DA = 0$ and $DB = 0$, then

$$M^D = \begin{bmatrix} A^D + X_2D & X_1 \\ (C^D)^2D & C^D \end{bmatrix},$$

where

$$X_i = \left(\sum_{j=0}^{l-A^D -1} (A^D)^{i+j+1}BC^j\right)(I - CC^D)$$

$$+ (I - AA^D)\left(\sum_{j=0}^{l-A^D -1} A^jB(C^D)^{i+j+1}\right) - \sum_{j=0}^{i-1}(A^D)^{j+1}B(C^D)^{i-j},$$

for $i = 1, 2$. Furthermore, $\text{Ind}(M) \leq \text{Ind}(A) + \text{Ind}(C) + 2$.

**Proof.** Denote the right-hand side of (2.1) by $X$ and show that $X = M^D$ by the direct verification of (1.1). Firstly, a simple computation shows that

$$A^D A + X_1D = AA^D + AX_2D + B(C^D)^2D$$

$$A^DB + X_1C = AX_1 + BC^D$$

$$DX_1 = 0, \quad DX_2 = 0.$$
which implies that
\[
MX = XM = \begin{bmatrix}
A^D A + X_1 D & A^D B + X_1 C \\
C^D D & C^D C
\end{bmatrix}.
\] (2.3)

Noting that
\[
DX_1 = 0, \quad DX_2 = 0
\]
\[
A^D A X_2 D + A^D B (C^D)^2 D + X_1 C^D D = X_2 D
\]
\[
A A^D X_1 + A^D B C^D + X_1 C C^D = X_1,
\]
from (2.3) it follows that
\[
X M X = \begin{bmatrix}
A^D A + X_1 D & A^D B + X_1 C \\
C^D D & C^D C
\end{bmatrix} \begin{bmatrix}
A^D + X_2 D & X_1 \\
(C^D)^2 D & C^D
\end{bmatrix}
\]
\[
= \begin{bmatrix}
A^D + X_2 D & X_1 \\
(C^D)^2 D & C^D
\end{bmatrix},
\]
i.e., \(X M X = X\).

Finally, we assert that \(M^{k+1} X = M^k\), for every integer \(k \geq \text{Ind}(A) + \text{Ind}(C) + 2\). In fact, by induction on integer \(k \geq 2\), we have
\[
M^k = \begin{bmatrix}
A^k + \sum_{i=0}^{k-2} A^i B C^{k-i-2} D & \sum_{i=0}^{k-1} A^i B C^{k-i-1} \\
C^{k-1} D & C^k
\end{bmatrix},
\] (2.4)
noting that \(DA = DB = 0\). Combining (2.3) and (2.4), we obtain
\[
M^{k+1} X = M^k M X
\]
\[
= \begin{bmatrix}
A^k + \sum_{i=0}^{k-2} A^i B C^{k-i-2} D & \sum_{i=0}^{k-1} A^i B C^{k-i-1} \\
C^{k-1} D & C^k
\end{bmatrix} \begin{bmatrix}
A^D A + X_1 D & A^D B + X_1 C \\
C^D D & C^D C
\end{bmatrix}
\]
\[
= \begin{bmatrix}
A^k A^D A + A^k X_1 D + \sum_{i=0}^{k-1} A^i B C^{k-i-1} C^D D & A^k A^D B + A^k X_1 C + \sum_{i=0}^{k-1} A^i B C^{k-i-1} C^D C \\
C^k C^D D & C^k C^D C
\end{bmatrix}
\]
\[
= \begin{bmatrix}
A^k + \sum_{i=0}^{k-2} A^i B C^{k-i-2} D & \sum_{i=0}^{k-1} A^i B C^{k-i-1} \\
C^{k-1} D & C^k
\end{bmatrix},
\]
that is to say, $M^{k+1}X = M^k$, for every integer $k \geq \text{Ind}(A) + \text{Ind}(C) + 2$. Hence $X = MD$ and $\text{Ind}(M) \leq \text{Ind}(A) + \text{Ind}(C) + 2$. This completes the proof. □

**Theorem 2.2.** Let $M = \begin{bmatrix} A & B \\ D & C \end{bmatrix} \in \mathbb{C}^{n \times n}$, where $A$ and $C$ are square matrices. If $BD = 0$ and $CD = 0$ then

$$M^D = \begin{bmatrix} A^D & X_1 \\ D(A^D)^2 & C^D + DX_2 \end{bmatrix},$$

where $X_1$ and $X_2$ are defined in (2.2). Furthermore, $\text{Ind}(M) \leq \text{Ind}(A) + \text{Ind}(C) + 2$.

**Proof.** The proof is similar to the proof of Theorem 2.1. □

3 Explicit representations of the Drazin inverse of a modified matrix and its generalized matrix

We will give, under some conditions several explicit representations of the Drazin inverse of a modified matrix $P + RS$ and its generalized matrix $PQ + RS$. First, we present two lemmas and then we derive several new results concerning the modified matrix and its generalized matrix.

**Lemma 3.1.** ([6]) Let $A \in \mathbb{C}^{n \times n}$ and $B \in \mathbb{C}^{n \times m}$. Then $(AB)^D = A[(BA)^D]^2B$.

**Lemma 3.2.** Let $M = \begin{bmatrix} A & B \\ D & C \end{bmatrix} \in \mathbb{C}^{n \times n}$, where $A$ and $C$ are square matrices. Then the following assertions hold

If $DA = 0$ and $DB = 0$, then $(M^D)^2 = \begin{bmatrix} (A^D)^2 + X_2D & X_1 \\ (C^D)^3D & (C^D)^2 \end{bmatrix};$ (3.1)

If $BD = 0$ and $CD = 0$, then $(M^D)^2 = \begin{bmatrix} (A^D)^2 & X_1 \\ D(A^D)^3 & (C^D)^2 + DX_2 \end{bmatrix}.$ (3.2)

where

$$X_i = X_i(A, B, C) = \left( \sum_{j=0}^{l_i-1} (A^D)^{i+j+2}BC^j \right)(I - CC^D) \quad (3.3)$$

$$+ (I - AA^D) \left( \sum_{j=0}^{l_{i-1}} A^jB(C^D)^{i+j+2} \right) - \sum_{j=0}^{l_i} (A^D)^{i+1}B(C^D)^{i-j+1},$$

for $i = 1, 2.$
Proof. After straightforward computation, the above assertions (3.1) and (3.2) follow readily from Theorem 2.1 and Theorem 2.2, respectively. □

**Theorem 3.1.** Let $P \in \mathbb{C}^{m \times n}$, $Q \in \mathbb{C}^{n \times m}$, $R \in \mathbb{C}^{n \times l}$ and $S \in \mathbb{C}^{l \times m}$. If $SPQP = 0$ and $SPQR = 0$, then

$$(PQ + RS)^D = P((QP)^D)^2 Q + PX_2 SPQ + PX_1 S + R((SR)^D)^3 SPQ + R((SR)^D)^2 S$$

$$= [P, R] \begin{bmatrix} ((QP)^D)^2 + X_2 SP & X_1 \\ ((SR)^D)^3 SP & ((SR)^D)^2 \end{bmatrix} \begin{bmatrix} Q \\ S \end{bmatrix}$$

where $X_i = X_i(QP, QR, SR), i = 1, 2$ are defined as in (3.3).

Proof. By Lemma 3.1, we have

$$(PQ + RS)^D = \left( \begin{bmatrix} P, R \\ Q/S \end{bmatrix} \right)^D = \begin{bmatrix} [QP, QR] \end{bmatrix}^D \begin{bmatrix} Q \\ S \end{bmatrix}$$ (3.4)

Using (3.4) and (3.1), we obtain

$$(PQ + RS)^D = P((QP)^D)^2 Q + PX_2 SPQ + PX_1 S + R((SR)^D)^3 SPQ + R((SR)^D)^2 S,$$

where $X_i = X_i(QP, QR, SR), i = 1, 2$ are defined as in (3.3). □

**Theorem 3.2.** Let $P \in \mathbb{C}^{m \times n}$, $Q \in \mathbb{C}^{n \times m}$, $R \in \mathbb{C}^{m \times l}$ and $S \in \mathbb{C}^{l \times m}$. If $QRSP = 0$ and $SRSP = 0$, then

$$(PQ + RS)^D = P((QP)^D)^2 Q + RSPX_2 S + PX_1 S + RSP((QP)^D)^3 Q + R((SR)^D)^2 S$$

$$= \begin{bmatrix} P, R \\ SP((QP)^D) \end{bmatrix} \begin{bmatrix} X_1 \\ SP((QP)^D)^3 SP + ((SR)^D)^2 \end{bmatrix} \begin{bmatrix} Q \\ S \end{bmatrix}$$

where $X_i = X_i(QP, QR, SR), i = 1, 2$ are defined as in (3.3).

Proof. The proof is equivalent to that of Theorem 3.1. □

Applying Theorem 3.1 to the case where $Q = I$ and $R = I$, we obtain the following explicit representation of the Drazin inverse of matrix $S + P$, which is the main result of [12].
Corollary 3.1. ([12]) Let $S, P \in \mathbb{C}^{m \times m}$. If $SP = 0$, then

$$
(S + P)^D = \sum_{j=0}^{m-1} (P^D)^{j+1}S^j(I - SS^D) + \sum_{j=0}^{m-1} (I - PP^D)P^j(S^D)^{j+1}
$$

If we consider Theorem 3.1 in the case where $R = P$ and $Q = P^{k-1}$ ($k \geq 1$) and in the case where $Q = I$ and Theorem 3.2 in the case where $R = Q^{k-1}$ and $S = Q$ and in the case where $R = I$, we deduce the following:

Corollary 3.2. (1) Let $P, Q, S \in \mathbb{C}^{m \times m}$. For any integer $k \geq 1$, the following assertions hold:

(i) If $SP^{k+1} = 0$, then

$$
(P^k + PS)^D = (P^D)^k + (PS)^D + P((SP)^D)^2P^{k-1} + PX_1S + PX_2SP^k,
$$

(ii) If $Q^{k+1}P = 0$, then

$$
(PQ + Q^k)^D = (Q^D)^k + (PQ)^D + Q^{k-1}(QP)^DQ + PY_1Q + Q^kPY_2Q,
$$

where $X_i = X_i(P^k, P^k, SP)$ and $Y_i = X_i(QP, Q^k, Q^k)$, for $i = 1, 2$ are defined as in (3.3);

(2) Let $P \in \mathbb{C}^{m \times n}$, $R \in \mathbb{C}^{m \times l}$, and $S \in \mathbb{C}^{l \times m}$. Then the following assertions hold:

(i) If $SP^2 = 0$ and $SPR = 0$, then

$$
(P + RS)^D = P^D + R((SR)^D)^2S + R((SR)^D)^3SP + PX_1S + PX_2SP.
$$

(ii) If $SPR = 0$, and $P^2R = 0$, then

$$
(P + RS)^D = P^D + R((SR)^D)^2S + PR((SR)^D)^3S + RY_1P + PRY_2P
$$

where $X_i = X_i(P, R, SR)$ and $Y_i = X_i(SR, S, P)$, $i = 1, 2$ are defined as in (3.3).

4 Examples

The following example describes a $2 \times 2$ matrix $M$ which does not satisfy any of the conditions (i)-(viii) that can be found in the papers [2,5,7,9,11,15,18,22], whereas the conditions of Theorem 2.1 are met, which allows us to compute $M^D$. 

7
Example 1. Let \( M = \begin{bmatrix} A & B \\ D & C \end{bmatrix} \), where \( A = \begin{bmatrix} 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \), \( B = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & -1 \\ 0 & 0 \end{bmatrix} \),

\( C = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \) and \( D = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix} \).

It is calculated that

\[ A^D = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad A^\pi = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

Now, it is evident that:

(i) \( \text{rank}(A) = 2 \neq \text{rank}(M) \),

(ii) \( B \neq 0 \) and \( D \neq 0 \),

(iii) \( CD \neq 0 \),

(iv) \( BD \neq 0 \),

(v) \( DA^\pi \neq 0 \),

(vi) \( C - DA^DB = C \) is neither nonsingular nor zero,

(vii) \( AA^\pi B \neq 0 \),

(viii) \( CD \neq 0, \ BC \neq 0 \),

so the representations of \( M^D \) in the literature ([2,7,9,11,18,15,22]) fail to apply, while Theorem 2.1 on the other hand, gives the exact value of \( M^D \). Since, \( C \) is idempotent, it follows that \( C^D = C \) and by \( A^DB = 0 \), we get,
\[ X_1 = X_2 = BC = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} \]  

Using a representation (2.1) in Theorem 2.1,

\[ M^D = \begin{bmatrix} 2 & 0 & -2 & 1 & 1 & 1 \\ 2 & 0 & -2 & 1 & 1 & 1 \\ 2 & 0 & -2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & -2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

The following example shows that in the case when none of the conditions (i)-(viii) are satisfied, Theorem 2.2 can also come in handy:

**Example 2.** Let \( M = \begin{bmatrix} A & B \\ D & C \end{bmatrix} \), where \( A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \), \( B = C = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \) and \( D = \begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix} \).

It can be checked that none of the conditions (i)-(viii) are satisfied and the conditions from Theorem 2.2 are. So, using the representation for \( M^D \) given in Theorem 2.2, we get that

\[ M^D = \begin{bmatrix} 1 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & -2 & 12 & 0 \end{bmatrix} \]

The next example illustrates how useful Theorem 3.1 can be for computing the Drazin inverse of generalized modified matrices.
Example 3. Let

\[ M = PQ + RS = \begin{bmatrix} 26 & 45 & -28 & 15 & 30 \\ -7 & -13 & 11 & -23 & 6 \\ -43 & -14 & 29 & -34 & 72 \\ -7 & -13 & 11 & -23 & 6 \\ 17 & -1 & -1 & -11 & 42 \end{bmatrix}, \]

where

\[
P = \begin{bmatrix} 3 & 1 \\ 1 & 2 \\ 2 & 3 \\ 1 & 2 \\ 1 & 2 \end{bmatrix},
Q = \begin{bmatrix} -5 & 5 & -5 & 0 \\ 5 & 0 & 5 & 0 \end{bmatrix},
R = \begin{bmatrix} 0 & -6 \\ -3 & 3 \\ 6 & 6 \\ -3 & 3 \\ -6 & 0 \end{bmatrix},
S = \begin{bmatrix} -2 & 1 & 1 & -7 \\ -6 & -5 & 3 & -5 \end{bmatrix}.
\]

Then it can be verified that \(SPQP = 0\) and \(SPQR = 0\). Hence, by Theorem 3.1 one can find the Drazin inverse of the \(5 \times 5\) matrix \(M\) by computing the Drazin inverse of \(2 \times 2\) matrices \(QP\) and \(SR\) and performing a few other straightforward computations. Thus:

\[
(QP)^D = \begin{bmatrix} -25 & -20 \\ 25 & 20 \end{bmatrix}^D = \frac{1}{5} \begin{bmatrix} -5 & -4 \\ 5 & 4 \end{bmatrix},
(SR)^D = \begin{bmatrix} 42 & 24 \\ 78 & 24 \end{bmatrix}^D = \frac{1}{144} \begin{bmatrix} -4 & 4 \\ 13 & -7 \end{bmatrix}
\]

which implies

\[
X_1 = (I - QP(QP)^D)QR((SR)^D)^3 - (QP)^DQR((SR)^D)^2
\]
\[
- ((QP)^D)^2QR(SR)^D
\]
\[
= \frac{1}{4320} \begin{bmatrix} -229 & 199 \\ 229 & -199 \end{bmatrix},
\]

\[
X_2 = (I - QP(QP)^D)QR((SR)^D)^4 - (QP)^DQR((SR)^D)^3
\]
\[
- ((QP)^D)^2QR((SR)^D)^2 - ((QP)^D)^3QR(SR)^D
\]
\[
= \frac{1}{3110400} \begin{bmatrix} 38251 & -32281 \\ -38251 & 32281 \end{bmatrix},
\]

10
noting that \((I - QP(QP)^D)QP = 0\) and \(I - SR(SR)^D = 0\). Now we have

\[
M^D = (PQ + RS)^D = \left[ P, R \right] \begin{bmatrix}
((QP)^D)^2 + X_2SP & X_1 \\
((SR)^D)^3 SP & (SR)^D^2 
\end{bmatrix} \begin{bmatrix}
Q \\
S
\end{bmatrix}
\]

\[
= \begin{bmatrix}
3 & 1 & 0 & -6 \\
1 & 2 & -3 & 3 \\
2 & 3 & 6 & 6 \\
1 & 2 & -3 & 3 \\
1 & 2 & -6 & 0
\end{bmatrix}
\begin{bmatrix}
3991 & 3559 & 229 & 199 \\
10800 & 10800 & 4320 & 4320 \\
-3991 & -3559 & 229 & 199 \\
10800 & 10800 & 4320 & 4320 \\
7 & 2592 & 17 & 11 \\
2592 & 2592 & 5184 & 5184 \\
61 & 61 & 143 & 101 \\
10368 & 10368 & 20736 & 20736 
\end{bmatrix}
\begin{bmatrix}
-5 & 5 & -5 & -5 \\
5 & 0 & 5 & 0 \\
-2 & 1 & 1 & -7 \\
-6 & -5 & 3 & -5 & -5
\end{bmatrix}
\]

\[
= \begin{bmatrix}
-35 & 27127 & 149 & 33679 & 149 \\
54 & 8640 & 540 & 8640 & 1080 \\
11 & 9049 & 53 & 10753 & -23 \\
36 & 5760 & 360 & 5760 & -729 \\
17 & 13591 & 77 & 16687 & -77 \\
54 & 8640 & 540 & 8640 & 1080 \\
11 & 9049 & 53 & 10753 & -23 \\
36 & 5760 & 360 & 5760 & -729 \\
1 & 47 & 2 & 59 & 1 \\
3 & 30 & 15 & 30 & 15
\end{bmatrix}
\]

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