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Continuous-state reliability measures based on fuzzy sets

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This article proposes to use the theory and methods of fuzzy sets to model the reliability of a component or system experiencing continuous stochastic performance degradation. The performance characteristic variable, which indicates the continuous performance levels of degradable systems, is used to fuzzify the states of a component or system. The concept of an engineering or technological performance variable is understood by both customers and system designers and can be used to represent different degrees of success. Thus, the imprecision in the meaning of success/failure is quantified through the fuzzy success/failure membership function, which is defined over the performance characteristic variable. The proposed fuzzy reliability measures provide an alternative to model the continuous state behavior for a component or system as it evolves from a binary state to a multi-state and finally to a fuzzy state. The dynamic behavior of fuzzy reliability is investigated using the concept of a fuzzy random variable under appropriate stochastic performance degradation processes. This article also develops some reliability performance metrics that are able to capture the cumulative experiences of customers with the system. In addition, the perception and utility from the customers are utilized to develop customer-centric reliability performance measures.

Keywords: Fuzzy reliability measures, performance characteristics, fuzzy sets, multi-state reliability, customer-centric measures

1. Introduction

It is commonly known that binary state reliability modeling for components and systems is too simplistic and does not fully capture the real behavior of most systems, which can have many, or even continuous, levels of performance. This has been the motivation to use reliability models that consider multi-state systems with multi-state components (Barlow and Wu, 1978; Hudson and Kapur, 1983; Aven, 1993; Brunelle and Kapur, 1999; Lisnianski and Levitin 2003). It has been extensively documented in the literature that models with multi-state behavior are superior in terms of capturing the performance of systems and also making sure that the performance measures capture the experience of a customer with a system over time (Boedighheimer and Kapur, 1994; Liu and Kapur, 2006, 2008). In this article, a new perspective and methodology is proposed to model the reliability behavior of a component or system with continuous stochastic performance degradation using the theory and methods of fuzzy sets. We demonstrate how fuzzy modeling of the states for the component/system can capture a more realistic picture of the performance of the system that can be used in evaluation and decision-making processes by the system designer as well as the customer. This article also shows the evolving relationship from binary state to multi-state and to the proposed fuzzy reliability performance measures.

Since the seminal work on fuzzy sets performed by Zadeh (1965), fuzzy set theory has been widely applied in engineering areas, such as design of a fuzzy controller, fuzzy image processing, fuzzy optimization techniques, fuzzy probabilistic risk assessment and fuzzy reliability analysis. In reliability-related research, fuzzy set theory was first applied in fault tree analysis where the probabilities of basic events in the fault tree are treated as fuzzy numbers and the top event's fuzzy probability is evaluated using fuzzy number arithmetic operations (Tanaka et al., 1983; Singer, 1990). Misra and Weber (1990) proposed applying fuzzy set theory to handle imprecision and uncertainties in probabilistic risk analysis by treating the exact probabilities as fuzzy ones. The concept of fuzzy reliability was introduced by Cai and Wen (1990) and further developed by Cai et al. (1993) and Cai (1996). Pandey and Tyagi (2007) presented an application of this method for a two-unit parallel-series system by assuming that each unit experiences multiple fuzzy states in terms of the number of operating components in the unit.

To model the imprecision and uncertainty in estimating component reliability due to insufficient data, fuzzy set theory has been applied in fuzzy system reliability analysis by fuzzifying exact reliability values as fuzzy numbers.
Cheng and Mon (1993) proposed evaluating a system’s fuzzy reliability by using interval arithmetic operations on the level sets of the component’s fuzzy reliability numbers. Chen (1994) and Hong and Do (1997) applied various fuzzy number arithmetic operations for a system’s fuzzy reliability evaluation with the subsystem/component’s reliability values considered as fuzzy numbers. Verma et al. (2004) evaluated the fuzzy dynamic reliability for a deteriorating system using the concept of probist reliability as a fuzzy number. Fuzzy set theory has also been applied in Bayesian structural reliability analysis (Wu, 2004; Huang et al., 2006) and structural reliability analysis (Liu et al., 1997; Li et al., 2000). More recently, fuzzy set theory was applied to evaluate the multi-state reliability performance where both probabilities and performance rates are regarded as fuzzy values (Ding and Lislanski, 2008; Ding et al., 2008; Liu et al., 2008).

The existing fuzzy reliability models and methods can be used to deal with the imprecision involved in estimating a component’s reliability/probability, performance rate, transition rate, and failure time by fuzzifying certain parameters in the traditional binary state and multi-state reliability methodologies. However, the universal set over which the membership function of the fuzzy number or fuzzy variable is defined is not well motivated and is difficult to interpret in a meaningful way by customers and system designers. For a system with continuous stochastic performance degradation, the meaning that a system functions or succeeds needs to be easily understood and defined from the viewpoint of the customer as well as to be measurable from the engineering perspective. The acceptable level of functionality for a system or component is related to a performance or indicator variable, also called the substitute characteristic (Kapur, 1998; Li and Kapur, 2011), since it is a substitute to quantify the level of performance for the system. The technological or engineering substitute/performance characteristic can be used as a means to translate the voice and needs of the customer based on their language and thought processes, and we propose to use this performance characteristic variable to represent the states for a component or system. Some examples of the performance characteristics are the strength of a structure, the resistance of a resistor, the current or voltage in an electrical system, shrinkage, wear or some other deterioration of a part, the level of flow of a fluid to meet some demand, and the amount of power generated relative to the demand/maximum capacity. The degrees of success or failure are directly related to the levels of this technical/engineering performance characteristic, and customers can easily relate the degree of success to the values of the performance characteristic. Because there is continuous performance degradation for degradable systems, the success state can be considered as a fuzzy state based on using the performance characteristic as a variable for the degree of success membership. For a larger-the-better performance characteristic variable $y$, higher values of $y$ result in better system performance and should have higher values of success membership using the concept of fuzzy sets. Thus, the fuzzy success membership function based on the performance characteristic variable $y$ is another way to model the multi-state and continuum state behavior for a system or component.

Typically, we have a considerable amount of empirical data on performance characteristics and we can thus develop probability models for their behavior. Both static fuzzy reliability and dynamic fuzzy reliability models use probability models to describe the stochastic behavior of the performance variable, and the imprecision of success states is quantified using the fuzzy success membership function. In this article numerical examples are given to compute the proposed dynamic fuzzy reliability measures that also illustrate their advantages over the traditional reliability measures.

The main contributions of this article are in the following two aspects. First, this article develops fuzzy reliability measures by redefining the traditional crisp success/failure events as fuzzy success/failure events over some performance variables, and the evolving relationship from binary reliability to multi-state reliability and to fuzzy reliability is demonstrated. Second, the concept of a fuzzy random variable is introduced to model the dynamic behavior of the time to fuzzy failure when the failure/success state is fuzzy and no single failure threshold value exists.

2. Static fuzzy reliability modeling

2.1. Reliability measures based on performance variables

In classic reliability models, the states of a component or system are assumed to be binary. The binary state assumption implies that the success and failure of a component can be precisely determined in terms of a single threshold value $y_0$ of some performance characteristic random variable $Y$ (Fig. 1(a)). For the larger-the-better performance characteristic, the precise definition of success and failure can be expressed as

$$
\mu_S(Y) = \begin{cases} 
0 & \text{failure } \iff Y < y_0, \\
1 & \text{success } \iff Y \geq y_0.
\end{cases} \tag{1}
$$

where $\mu_S(y)$ is the characteristic function of the crisp success set, which is essentially a state indicator variable. Then the reliability of a component can be evaluated as

$$
R = \Pr \left[ \mu_s(Y) = 1 \right] = E[\mu_S(Y)] = \Pr [Y \geq y_0]
= \int_{y \geq y_0} 1 \times f(y)dy, \tag{2}
$$

where $f(y)$ is the probability density function of the performance variable.

For degradable components/systems with continuous performance levels, it is difficult or unrealistic to use a single threshold value to divide success from failure. In
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Fig. 1. Reliability models under binary and fuzzy state assumption: (a) binary state reliability modeling and (b) fuzzy state reliability modeling.

Fig. 2. Reliability models under fuzzy state and multi-state assumption: (a) fuzzy state representation and (b) multi-state representation.

In this article, the component’s success and failure are defined as fuzzy sets over the performance characteristic variable \( Y \). For any given value of the performance characteristic \( y \), the component exhibits a certain degree of success \( \mu_S(y) \) as shown in Fig. 1(b), where \( y_0 \) corresponds to a degree of success. Mathematically, a fuzzy set \( A \) can be defined as

\[
A = \{(y, \mu_A(y)); y \in U\},
\]

where \( \mu_A(y) \) is called the membership function, which indicates the degree of membership of an element \( y \) belonging to the set of \( A \); \( U \) is any universal set that is a traditional crisp set; and \( \mu_A: U \rightarrow [0, 1] \). It is worth noting that the membership function of a fuzzy set is the generalization of the characteristic function of a crisp set.

### 2.2. Fuzzy state representation by membership functions

The success membership function \( \mu_S(y) \) in Fig. 2(a) shows another example of how success is defined in fuzzy reliability modeling using the concept of fuzzy sets. This membership function for the fuzzy success event is defined over the universal set \( \mathbb{R} = [0, y_M] \), where \( y_M \) is the maximum value for the performance variable \( Y \), and \( \mu_s(y) = 0 \) if \( 0 \leq y < y_1 \); \( \mu_s(y) = (1/(y_2 - y_1))(y - y_1) \) if \( y_1 \leq y < y_2 \); and \( \mu_s(y) = 1 \), if \( y_2 \leq y \leq y_M \).

The membership function in Fig. 2(a) reflects the customer’s perception of the fuzzy success event based on the performance variable \( Y \). More specifically, if \( y \in [0, y_1] \), the component is considered a success with degree 0; if \( y \in (y_1, y_2] \), the component exhibits a certain degree of success in the interval \((0, 1)\); and if \( y \in (y_2, y_M] \), the component is considered a success with degree 1. The multi-state approach (Fig. 2(b)) can classify the state of the component as state 2 if \( y \in (y_2, y_M] \), state 1 if \( y \in (y_1, y_2] \), and state 0 if \( y \in [0, y_1] \). Such a multi-state classification is an extension of binary state reliability modeling, and it approximates the continuous performance levels by simplifying and discretizing \( y \) with a finite number of states. The state representation under multi-state reliability modeling can be further expressed as

\[
\mu_S(y) = \begin{cases} 
0 & y \in [0, y_1], \\
1 & y \in (y_1, y_2], \\
2 & y \in (y_2, y_M]. 
\end{cases}
\]
and multi-state representations are a simplification and discretization of the continuous performance levels.

2.3. Fuzzy reliability modeling

A static fuzzy reliability model is developed in this section. The success and failure are treated as fuzzy events, which contain continuous performance characteristic values $y$ exhibiting different degrees of success or failure. As a natural extension of the traditional binary state reliability evaluation given by Equation (2), the fuzzy reliability of a component can be evaluated as the probability of the fuzzy event of success (Zadeh, 1968). Thus, fuzzy reliability is defined as

$$ R = \Pr[\text{fuzzy success}] = \int \mu_S(y) dF(y) = E[\mu_S(Y)], \quad (5) $$

where $F(y)$ is the cumulative distribution function of the performance characteristic variable $Y$. The fuzzy reliability definition in Equation (5) degenerates into the classic binary reliability model when the membership function of the fuzzy success event is substituted with the characteristic function of a crisp success event. This fuzzy reliability definition is also an analog to the system's performance measure based on the state expectation definition given by $E[X] = \sum_{i=0}^{M} i \times \Pr[X = i]$, which has been traditionally used for both binary and multi-state reliability modeling (Russell and Kapur, 1997).

The proposed fuzzy reliability model has advantages over the traditional reliability models. For example, given the probability distribution of the performance characteristic, for the traditional binary and multi-state reliability modeling the probability of a system being in the success state or one of the multi-states heavily depends on a single threshold value and is highly sensitive to this specified value. Such issues also exist for the reliability performance measures that utilize these probabilities. In addition, since the membership function incorporates customers’ utility with respect to the performance of a system, the fuzzy reliability measures are customer-centric. When the state indicator variables in the traditional binary and multi-state reliability models are consistently written as $\mu_S(y)$, the evolving relationship for reliability modeling from binary state to multi-state to fuzzy state is as shown in Fig. 3.

![Fig. 3. Evolving relationship for reliability modeling from binary state to multi-state to fuzzy state.](image-url)
2.4. Fuzzy unreliability—a different perspective

Fuzzy reliability can also be defined in a negative way. In other words, we can define the fuzzy unreliability in terms of the fuzzy failure event:

\[ \tilde{R} = \Pr [\text{fuzzy failure}] = \int \mu_F (y) \, dF (y) = E [\mu_F (Y)] , \]  

where \( \mu_F (y) \) is the membership function for the fuzzy failure event. If the membership function for fuzzy failure is assumed to be the standard complement of the membership function for fuzzy success, i.e., \( \mu_F (y) = 1 - \mu_S (y) \), the fuzzy reliability and fuzzy unreliability of a component/system sum to one. However, the standard complement relationship between fuzzy success and fuzzy failure is based on the bi-valued logic and is not necessary for our approach.

3. Dynamic fuzzy reliability modeling

3.1. Dynamic fuzzy reliability

The fuzzy reliability defined in Equation (4) can be easily extended for dynamic fuzzy reliability evaluation by considering time-dependent probability distributions for the performance characteristic variable. It is reasonable to assume that the success membership function defined over the performance characteristic \( Y \) does not change over time since customers’ perception of fuzzy success based on \( Y \) is relatively stable. Thus, dynamic fuzzy reliability is defined as

\[ R(t) = \Pr [\text{fuzzy success}] = \int_y \mu_S (y) \, dF (y(t)) = E [\mu_S (Y)] . \]  

where \( F (y(t)) \) is the cumulative probability distribution function for the performance characteristic variable at a given time point \( t \), and \( E [\mu_S (Y)] \) is a function of time \( t \). From Equation (7), it is noted that the probability distributions over time for the performance characteristic variable and the fuzzy success membership function \( \mu_S (y) \) suffice to evaluate the dynamic fuzzy reliability. To evaluate this dynamic fuzzy reliability, the performance degradation processes needs to be fully understood and modeled such that the probability distributions for the performance variable \( Y \) over time can be quantified and evaluated using probabilistic models.

The dynamic behavior for fuzzy reliability can also be investigated by evaluating the time to fuzzy failure, which is a fuzzy random variable. In the following section, the concept of a fuzzy random variable is introduced to model the time to fuzzy failure when the failure threshold is imprecise and no single threshold value can differentiate between success and failure.

3.2. Time to fuzzy failure modeled by a fuzzy random variable

In the traditional dynamic reliability model, for a given failure threshold value \( y_0 \), the time to failure is a random variable \( X(\omega) \) due to random variation in the degradation processes (Fig. 4). When the system state is considered as a fuzzy event, it exhibits different degrees of failure from \([0, 1]\) over the performance range of performance characteristic values from \( y_I \) to \( y_M \), where \( y_I \) is the initial/ideal state of the system and \( y_M \) is the minimum performance value for a decreasing degradation path model with larger-the-better performance characteristics. For a given degree of system failure \( \alpha \), the observed time to fuzzy failure is a random interval \([\tilde{X} - \alpha (\omega), \tilde{X} + \alpha (\omega)]\). Thus, the time to fuzzy failure is not only random but also fuzzy; i.e., a fuzzy random variable.

The concept of a fuzzy random variable \( \tilde{X} \) is used to model the time to fuzzy failure, and it can be formally defined over a probability space as follows (Kwakernaak, 1978; Liu et al., 1997): Let \( (\Omega, F, P) \) be a probability space. A function \( \tilde{X} : \Omega \rightarrow F_0 \) is called a fuzzy random variable

---

Fig. 4. Time to fuzzy failure: a fuzzy random variable (color figure provided online).
on $\Omega$, $\mathcal{F}$, $\mathcal{P}$), if for any $\alpha \in (0, 1]$ and $\omega \in \Omega$:

$$\tilde{X}_\alpha(\omega) = \{ x : x \in R, \tilde{X}(\omega)(x) \geq \alpha \} = \tilde{X}_\alpha^- \cup \tilde{X}_\alpha^+$$  \hspace{1cm} (8)

is a random interval; i.e., $\tilde{X}_\alpha^-$ and $\tilde{X}_\alpha^+$ are two random variables and measurable with respect to the $\sigma$-algebra $\mathcal{F}$, where $\mathcal{F}_0 (R)$ is a collection of all bounded and closed fuzzy random variables on $\mathbb{R}$. The fuzzy random variable degenerates to a random variable when $\alpha = 1$ since $\tilde{X}_\alpha^- = \tilde{X}_\alpha^+$.

The expectation of a fuzzy random variable $E[\tilde{X}]$ is a fuzzy number (Liu et al., 1997). For a given value $\alpha$, the expectation of a fuzzy random variable can be expressed as $E[\tilde{X}_\alpha] = (E[\tilde{X}_\alpha^-], E[\tilde{X}_\alpha^+])$. The membership function for the expectation of fuzzy random variables, e.g., the membership function for the expectation of the time to fuzzy failure, can be evaluated using the resolution identity (Zadeh, 1975a, 1975b, 1975c; Wu, 2001):

$$\mu_{E[\tilde{X}_\alpha]}(x) = \sup_{0 < \alpha \leq \alpha} \alpha \times 1(\varepsilon \tilde{X}_\alpha^-),$$  \hspace{1cm} (9)

$$= \max\{\alpha : 0 < \alpha \leq 1, x \in (E[\tilde{X}_\alpha^-], E[\tilde{X}_\alpha^+])\},$$  \hspace{1cm} (9)

where $x$ is a value in the support of the expectation of the fuzzy random variable and $1(\varepsilon \tilde{X}_\alpha^-)$ is the indicator function. Given the underlying degradation mechanism, the membership function of $E[\tilde{X}]$ in Equation (9) can be evaluated by solving the following optimization problem for each fixed $x_0$ over all $\alpha \in (0, 1)$:

$$\max \alpha,$$

subject to

$$E[\tilde{X}_\alpha^-] \leq x_0 \leq E[\tilde{X}_\alpha^+]$$

$$0 < \alpha \leq 1.$$  \hspace{1cm} (10)

In Equation (10), the expectations of $E[\tilde{X}_\alpha^-]$ and $E[\tilde{X}_\alpha^+]$ can be evaluated based on stochastic degradation processes. The stochastic performance degradation processes and the methods for evaluating $E[\tilde{X}_\alpha^-]$ and $E[\tilde{X}_\alpha^+]$ are investigated in the following sections.

3.3. Stochastic performance degradation model

The stochastic performance degradation process can be modeled in different ways, such as physics-based models, data-driven models, and sometimes a combination of the physics of failure with data-driven methods. Generally, the performance characteristic of interest, such as the amount of degradation, can be measured even though the underlying failure mechanism cannot be accurately modeled. The gamma stochastic process can capture both the unit-by-unit uncertainty for degradation and the temporal uncertainty over time (Noortwijk, 2009), thus in this article we use the gamma process to model the stochastic performance degradation for the performance variable over time.

In mathematical terms, a stochastic process $Y_t$ is a gamma process such that for all $0 \leq s < t$, the random increment, $Y_{t-s} = Y_t - Y_s$, has gamma probability density function with a shape parameter $r(t-t_s) > 0$ and a scale parameter $\lambda > 0$ as

$$f_{(t-t_s),\lambda}(y) = \frac{\lambda}{\Gamma(r(t-t_s))} (\lambda y)^{r(t-t_s)-1} \exp(-\lambda y), \ y > 0,$$

where $\Gamma(.)$ is the gamma function, $\Gamma(r) = \int_0^\infty y^{r-1} e^{-y} dy$.

Suppose $Y_{t=0} = 0$, the probability density function of the increment $Y_t$ at time $t$ in Equation (11) reduces to

$$f_{r(t),\lambda}(y) = \frac{\lambda}{\Gamma(r(t))} (\lambda y)^{r(t)-1} \exp(-\lambda y), \ y > 0.$$  \hspace{1cm} (12)

Under the traditional binary state reliability model with a single failure threshold value $y_0$, the reliability function can be derived using Equation (12). Thus, the expectations for $E[\tilde{X}_\alpha^-]$ and $E[\tilde{X}_\alpha^+]$ in Equation (9) can be further evaluated. The following section elaborates on how the membership function for the expectation of time to fuzzy failure can be evaluated. In addition, the probability density function in Equation (12) can be used for the dynamic fuzzy reliability evaluation in Equation (7).

3.4. Membership function evaluation for the expectation of the time to fuzzy failure

To evaluate the membership function for the expectation of time to fuzzy failure, the gamma stochastic process is used to model the continuous performance degradation process. For the decreasing path degradation process (Fig. 4) with an initial performance level of $Y_1$, the associated cumulative failure time distribution function $F(t)$ for any given performance level $y_0$ can be derived as

$$F(t) = P(T \leq t) = P(Y_t \geq y_1 - y_0)$$

$$= \int_{y_1-y_0}^{y_1} f_{(t),\lambda}(y) \ dy$$

$$= \int_{y_1-y_0}^{y_1} \frac{\lambda}{\Gamma(r(t))} (\lambda y)^{r(t)-1} \exp(-\lambda y) \ dy$$

$$= \frac{1}{\Gamma(r(t))} \int_{y_1-y_0}^{y_1} \lambda (y)^{r(t)-1} \exp(-\lambda y) \ d(\lambda y)$$

$$\times (\text{by change of variable, } z = \lambda y)$$

$$= \frac{1}{\Gamma(r(t))} \int_{y_1-y_0}^{y_1} \frac{\lambda(\lambda y - y_0)}{\Gamma(r(t))} \exp(-z) \ dz$$

$$= \frac{\Gamma(r(t), \lambda(y_1-y_0))}{\Gamma(r(t))},$$

where $\Gamma(\cdot, \cdot)$ is the upper incomplete gamma function given by $\Gamma(r, y) = \int_y^\infty u^{r-1} e^{-u} du, y > 0, r > 0$.

For any degree of failure $\alpha$, the performance characteristic value $y = \mu^{-1}(\alpha)$ can be calculated using the membership function for the fuzzy failure event. Hence, we can
evaluate $E[\tilde{X}_a]$ as

$$E[\tilde{X}_a] = \int_0^\infty (1 - F(t))dt$$

$$= \int_0^\infty \left(1 - \frac{\Gamma \left( r(t), \lambda(y_1 - \mu^{-1}(a)) \right)}{\Gamma (r(t))} \right) dt,$$ (14)

and

$$E[\tilde{X}_a^+] = \int_0^\infty \left(1 - \frac{\Gamma \left( r(t), \lambda(y_1 - \mu^{-1}(1)) \right)}{\Gamma (r(t))} \right) dt,$$ (15)

The minimum value $t_{\text{min}}$ and maximum value $t_{\text{max}}$ for the expectation of the time to fuzzy failure can be computed by setting $\alpha = 0$ and 1, respectively. Thus,

$$t_{\text{min}} = \int_0^\infty \left(1 - \frac{\Gamma \left( r(t), \lambda(y_1 - \mu^{-1}(0)) \right)}{\Gamma (r(t))} \right) dt$$

$$= \int_0^\infty \left(1 - \frac{\Gamma \left( r(t), 0 \right)}{\Gamma (r(t))} \right) dt = 0,$$ (16)

and

$$t_{\text{max}} = \int_0^\infty \left(1 - \frac{\Gamma \left( r(t), \lambda(y_1 - \mu^{-1}(1)) \right)}{\Gamma (r(t))} \right) dt$$

$$= \int_0^\infty \left(1 - \frac{\Gamma \left( r(t), \lambda(y_1 - \mu^{-1}) \right)}{\Gamma (r(t))} \right) dt.$$ (17)

For any fixed $t_0 \in [t_{\text{min}}, t_{\text{max}}]$, the membership value can be evaluated by solving the optimization problem in Equation (10):

$$\max \alpha,$$

subject to

$$E[\tilde{X}_a] = \int_0^\infty \left(1 - \frac{\Gamma \left( r(t), \lambda(y_1 - \mu^{-1}(a)) \right)}{\Gamma (r(t))} \right) dt \leq t_0,$$

$$t_0 \leq E[\tilde{X}_a^+] = \int_0^\infty \left(1 - \frac{\Gamma \left( r(t), \lambda(y_1 - \mu^{-1}(1)) \right)}{\Gamma (r(t))} \right) dt,$$

$$0 < \alpha \leq 1.$$ (18)

For each value $t_0 \in [t_{\text{min}}, t_{\text{max}}]$, the corresponding membership value in Expression (18) can be solved using MATLAB. By choosing various values for $t_0 \in [t_{\text{min}}, t_{\text{max}}]$, the membership function for the expectation of the time to fuzzy failure can be evaluated.

4. Performance measures for dynamic fuzzy reliability

This section first introduces the methods for comparing fuzzy numbers; e.g., the expectation of the time to fuzzy failure. Then we propose two customer-centric reliability performance measures; i.e., the Normalized Integrated Expected State Value (NIESV) and the utility/disutility-based reliability performance measures for dynamic fuzzy reliability performance evaluation (Shu et al., 2010). These measures can capture customers’ cumulative experience with the system performance behavior over a mission time and are thus more comprehensive than the traditional measures such as the Mean Time To Failure (MTTF) and the reliability function $R(t)$.

4.1. Methods for comparing fuzzy numbers

Different system/product designs may result in different membership functions for the expectation of the time to fuzzy failure. In other words, the membership functions of the fuzzy numbers resulting from various system designs need to be compared for performance evaluation. The cardinality, degree of inclusion, energy measure, and entropy measure are some metrics for fuzziness measures (Pedrycz and Gomide, 2007). The cardinality of a fuzzy set $A$ defined over a universal set of $X$ can be expressed as $\text{Card}(A) = \sum_{x \in X} \mu_A(x)$ or $\int_X^{\mu_A}(x)dx$ for discrete and continuous membership functions, respectively. The degree of inclusion of a fuzzy number $A$ in $B$, denoted as $||A \subset B||$, is defined as follows:

$$||A \subset B|| = \frac{1}{\text{Card}(X)} \int_X (A \subset B) dx,$$ (19)

where $A \subset B = 1$, if $\mu_A(x) \leq \mu_B(x)$; otherwise, $A \subset B = 1 - \mu_A(x) + \mu_B(x)$. The degree of inclusion indicates the extent that one fuzzy number is contained in the other. For example, if $\mu_A(x) \leq \mu_B(x)$ for all $x \in X$, the degree of inclusion of $A$ in $B$ is equal to one; i.e., $||A \subset B|| = 1$. Both the cardinality and the degree of inclusion of fuzzy numbers are utilized later to compare the membership functions for the expectations of the time to fuzzy failure and hence for comparing different system designs.

4.2. Reliability performance measure using the NIESV metric

For dynamic reliability performance evaluation, the reliability metric $R(t)$ alone may not be sufficient to differentiate and compare different system designs. For instance, in Fig. 5, $R_B(T_\neq) < R_A(T_\neq)$, but it is hard to say that design B is worse than design A over some target life $T_\neq$ since design B sustains a higher reliability level for most of the time in $[0, T_\neq]$ and thus the customer will experience higher cumulative value over the period $[0, T_\neq]$.

The reliability performance measure NIESV for the system over some target time duration $T_\neq$ can be used for system performance evaluation (Liu and Kapur, 2006, 2008). For the fuzzy reliability model, the NIESV can be expressed
as follows:

\[
NIESV(T^*) = \frac{1}{T^*} \int_0^{T^*} R(t) \, dt \\
= \frac{1}{T^*} \int_0^{T^*} E[\mu_S(y)] \, dt \\
= \frac{1}{T^*} \int_0^{T^*} \int_{\gamma_1}^{\gamma_M} \mu_S(y) \, f(y(t)) \, dy \, dt. \quad (20)
\]

The metric \(NIESV(T^*)\) essentially represents, the ratio of the area for the shaded part in Fig. 5 to the ideal or the maximum value of the area, which is \(1 \times T^*\). A higher value of the \(NIESV(T^*)\) indicates that the system stays longer at high performance levels over the time period \([0, T^*]\). Thus, \(NIESV(T^*)\) is an effective metric in evaluating system designs, especially when other reliability metrics such as \(R(t)\) and MTTF are not able to differentiate between different system designs.

4.3. Reliability performance measure based on the disutility function of customers

Another method for dynamic fuzzy reliability evaluation is to introduce the utility or disutility function for the membership values for the expectation of time to fuzzy failure. It is reasonable to assume that customers’ utility would be lower if the membership value for the expectation of time to fuzzy failure deviates more from zero. The difference between the membership value and zero is denoted by \(d(t)\), which is equal to the membership value \(\mu(t)\), where \(t\) is one possible value for the expectation of time to fuzzy failure. The system that gives the customer the smaller expected loss \(E[d(t)]\) should be considered as more reliable and a better system. A common utility function that considers the effect of variance is the exponential utility function of the form \(U(x) = 1 - \exp(-\frac{x}{\gamma})\), where \(r\) is a scale parameter (Nicholson, 1998). We use the similar model for the disutility function with bigger \(x\) indicating greater disutility:

\[
DU(d(t)) = 1 - \exp\left(-\frac{d(t)}{r}\right) = 1 - \exp\left(-\frac{\mu(t)}{r}\right). \quad (21)
\]

Instead of using the disutility of a single membership value for the expectation of the time to fuzzy failure, the accumulated disutility for the membership values of all possible expectation values of the time to fuzzy failure is proposed for the evaluation of system performance, which can be expressed as

\[
\int_0^t DU(d(\tau)) \, d\tau = \int_0^t \left[1 - \exp\left(-\frac{\mu(\tau)}{r}\right)\right] \, d\tau. \quad (22)
\]

This cumulative disutility measure for any given value \(t\) for the life of the system is used to evaluate the system’s dynamic reliability performance in the following numerical examples.

5. Numerical results

In this section, numerical examples show how the proposed generalized/fuzzy reliability models can be applied for reliability performance evaluation and system design comparison or selection using the proposed customer-centric reliability performance evaluation metrics.

5.1. Dynamic fuzzy reliability evaluation under a gamma degradation process

First, we present an example that compares traditional binary state reliability with the proposed dynamic fuzzy reliability measures. For this analysis, the gamma degradation process is characterized by the following parameters: a linear shape parameter \(r(t) = 2.5t\), and a scale parameter \(\lambda = 5\), and the performance characteristic is smaller-the-better with values from \(\gamma_1 = 0\) to \(\gamma_M = 6\). Through interaction and communication with customers and system designers, the success membership function was estimated as \(\mu_S(y) = 1/(1 + 0.8y^2)\), which is a monotonically decreasing function over the performance characteristic variable \(y\). The probability density functions for the performance characteristic random variable \(Y\) at the different time values \([t = 1, t = 2, \ldots]\) were evaluated using Equation (11) and are plotted in Fig. 6, which also shows the success membership function.

Based on the above information about the degradation mechanism and the success membership function, the fuzzy dynamic reliability can be evaluated using Equations (7) and (11) as

\[
R(t) = E[\mu_S(Y)] = \int_y^{\mu_S} f_Y(y, \lambda) \, dy \\
= \int_0^6 \frac{1}{1 + 0.8y^2} \times \frac{5}{\Gamma(2.5)} (5y)^{2.5-1} \exp(-5y) \, dy.
\]

No closed-form solution exists for \(R(t)\) in Equation (23), and numerical evaluation was implemented using MATLAB. Under the binary state reliability model, a
Failure states.

Hence, the dynamic fuzzy reliability is more meaningful, especially when it is unrealistic to use one single failure threshold value to distinguish between success and failure states. Therefore, the dynamic fuzzy reliability integrates all failure modes, customer’s perception, and the success definition and it integrates customers’ preferences. Hence, the dynamic fuzzy reliability is a comprehensive reflection of the customers’ perception and is more meaningful, especially when it is unrealistic to use one single failure threshold value to distinguish between success and failure states.

![Fig. 6. The probability density functions for the performance characteristic random variable over time.](image)

![Fig. 7. Dynamic fuzzy reliability and binary reliability.](image)

| Table 1. Fuzzy dynamic reliability for two systems with gamma degradation processes $r(t) = 2.5t, \lambda = 5$ and $r(t) = 10t, \lambda = 20$ |
|---|---|---|---|---|---|---|---|
| $t$ | $R_1(t)$ | $R_2(t)$ | $t$ | $R_1(t)$ | $R_2(t)$ | $t$ | $R_1(t)$ | $R_2(t)$ |
| 0.0 | 1.0000 | 1.0000 | 4.0 | 0.2736 | 0.2473 | 8.5 | 0.0719 | 0.0667 |
| 0.1 | 0.9913 | 0.9961 | 4.5 | 0.2288 | 0.2058 | 9.0 | 0.0634 | 0.0598 |
| 0.5 | 0.9324 | 0.9457 | 5.0 | 0.1929 | 0.1732 | 9.5 | 0.0557 | 0.0537 |
| 1.0 | 0.8233 | 0.8294 | 5.5 | 0.1640 | 0.1473 | 10.0 | 0.0486 | 0.0480 |
| 1.5 | 0.7019 | 0.6930 | 6.0 | 0.1407 | 0.1256 | 11.0 | 0.0357 | 0.0355 |
| 2.0 | 0.5860 | 0.5645 | 6.5 | 0.1216 | 0.1096 | 12.0 | 0.0244 | 0.0200 |
| 2.5 | 0.4843 | 0.4559 | 7.0 | 0.1058 | 0.0958 | 13.0 | 0.0151 | 0.0071 |
| 3.0 | 0.3991 | 0.3688 | 7.5 | 0.0927 | 0.0844 | 14.0 | 0.0084 | 0.0014 |
| 3.5 | 0.3296 | 0.3005 | 8.0 | 0.0815 | 0.0748 | 15.0 | 0.0041 | 0.0002 |

Now we demonstrate how the proposed $NIESV(T^*)$ reliability performance metric can be applied to evaluate and compare system designs based on a dynamic fuzzy reliability model for a given mission time $T^*$ from the viewpoint of a customer.

In addition to the previous system, called design 1, suppose we have another system design (design 2) and it can be characterized by a different gamma degradation process with a linear shape parameter $r(t) = 10t$ and a scale parameter $\lambda = 20$. The fuzzy dynamic reliability for both designs over the time period $[0, 15]$ was evaluated (Table 1) using Equations (7) and (12). The two gamma processes have the same value for the mean; that is, $2.5t = \frac{10t}{20} = 0.5t$.

It is observed that design 1 has lower reliability than design 2 over the time period $[0, 1.5]$ and design 1 has higher reliability than design 2 over $[1.5, 15]$. Thus, it is hard to compare design 1 and design 2. Instead, the normalized integrated/accumulated expected state value, i.e., $NIESV(T^*)$, is more meaningful for system design evaluation. Suppose that the mission time for the desired system is $T^* = 4$. Then

$$NIESV(T^* = 4) = \frac{1}{4} \int_0^4 \int_0^6 \mu_S(y) f(y(t)) dy \, dt,$$

can be evaluated for design 1 and design 2 and the values are 0.6094 and 0.5974, respectively. In terms of the NIESV metric, design 1 is better than design 2 for the given mission time $T^* = 4$.

5.2. Reliability performance evaluation based on the time to fuzzy failure

Now we show the application of the dynamic behavior of the time to fuzzy failure to evaluate two designs where the performance characteristic is larger-the-better type with values from $y_1 = 10$ to $y_M = 2$. The first design has a gamma degradation process characterized by a linear shape parameter $r(t) = 2t$ and a scale parameter $\lambda = 0.5$. Each performance level of $y \in (2, 10)$ exhibits...
Table 2. Membership values for the expectation of time to fuzzy failure for design 1 \( r(t) = 2t, \lambda = 0.5 \)

<table>
<thead>
<tr>
<th>( t ) (s)</th>
<th>( \mu(t) )</th>
<th>( t ) (s)</th>
<th>( \mu(t) )</th>
<th>( t ) (s)</th>
<th>( \mu(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>9.0063e-5</td>
<td>0.40</td>
<td>0.1115</td>
<td>0.75</td>
<td>0.4237</td>
</tr>
<tr>
<td>0.10</td>
<td>0.0034</td>
<td>0.45</td>
<td>0.1406</td>
<td>0.80</td>
<td>0.5048</td>
</tr>
<tr>
<td>0.15</td>
<td>0.0125</td>
<td>0.50</td>
<td>0.1733</td>
<td>0.85</td>
<td>0.6078</td>
</tr>
<tr>
<td>0.20</td>
<td>0.0262</td>
<td>0.55</td>
<td>0.2103</td>
<td>0.90</td>
<td>0.7492</td>
</tr>
<tr>
<td>0.25</td>
<td>0.0431</td>
<td>0.60</td>
<td>0.2525</td>
<td>0.95</td>
<td>0.9803</td>
</tr>
<tr>
<td>0.30</td>
<td>0.0630</td>
<td>0.65</td>
<td>0.3009</td>
<td>1.00</td>
<td>1.0000</td>
</tr>
<tr>
<td>0.35</td>
<td>0.0860</td>
<td>0.70</td>
<td>0.3571</td>
<td>1.24</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

Table 3. Membership values for the expectation of time to fuzzy failure for design 2 \( r(t) = 2.7t, \lambda = 0.6 \)

<table>
<thead>
<tr>
<th>( t ) (s)</th>
<th>( \mu(t) )</th>
<th>( t ) (s)</th>
<th>( \mu(t) )</th>
<th>( t ) (s)</th>
<th>( \mu(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>5.1621e-5</td>
<td>0.40</td>
<td>0.1142</td>
<td>0.75</td>
<td>0.5770</td>
</tr>
<tr>
<td>0.10</td>
<td>0.0025</td>
<td>0.45</td>
<td>0.1485</td>
<td>0.80</td>
<td>0.7736</td>
</tr>
<tr>
<td>0.15</td>
<td>0.0103</td>
<td>0.50</td>
<td>0.1887</td>
<td>0.82</td>
<td>0.9058</td>
</tr>
<tr>
<td>0.20</td>
<td>0.0229</td>
<td>0.55</td>
<td>0.2364</td>
<td>0.85</td>
<td>1.0000</td>
</tr>
<tr>
<td>0.25</td>
<td>0.0397</td>
<td>0.60</td>
<td>0.2937</td>
<td>0.90</td>
<td>1.0000</td>
</tr>
<tr>
<td>0.30</td>
<td>0.0603</td>
<td>0.65</td>
<td>0.3641</td>
<td>0.95</td>
<td>1.0000</td>
</tr>
<tr>
<td>0.35</td>
<td>0.0851</td>
<td>0.70</td>
<td>0.4539</td>
<td>0.965</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

a fuzzy failure membership value \( \alpha \), which is given by \( \mu_F(y) = (10 - y)/8 \), a monotonic fuzzy failure membership function defined over the performance characteristic variable \( y \). The upper limit of the time to fuzzy failure is

\[
t_{\text{max}} = \int_0^\infty \left(1 - \frac{r(2t)}{r(2t)}\right) dt = 1.2362.
\]

The non-linear optimization problem of Expression (18) was solved for various values of \( l_0 \in [0, 1.2362] \) and the membership function of the expectation of the time to fuzzy failure is given in Table 2.

The second system degrades over time with a different shape parameter \( r(t) = 2.7t \) and a scale parameter \( \lambda = 0.6 \). The membership function for the expectation of the time to fuzzy failure of this system is given in Table 3.

For comparison, the membership functions of the expectation of the time to fuzzy failure of the two gamma degradation processes are plotted in Fig. 8. It is observed that the membership values for the expectation of the time to fuzzy failure for the degradation process with \( r(t) = 2.7t, \lambda = 0.6 \) (design 2) are smaller than for the degradation process with \( r(t) = 2t, \lambda = 0.5 \) (design 1) over \([0, 0.4]\) and vice versa over the interval \([0.4, 1]\).

The performance of the two system designs are evaluated using the reliability measures developed in Section 4.1 and 4.3; i.e., the methods for fuzzy number comparison and the disutility-based metric. Since the cardinality of a fuzzy number measures the magnitude of that fuzzy number, the cardinality value of a fuzzy number is an indication of a customer’s perception of the fuzzy event. In our case, a larger cardinality value implies worse system performance/design since the fuzzy number is the expectation of the time to fuzzy failure. The cardinality values for design 1 and design 2 using the equation given in Section 4.1 are 0.4885 and 0.5547, respectively. Thus, design 1 is better than design 2 in terms of the cardinality measure.

The degree of inclusion for the two systems was evaluated using Equation (19), and it was found that \( \| A \subset B \| = 0.959 \), which is close to one. The high degree of inclusion indicates that most membership values for design 1 are lower than that for design 2. Thus, we conclude that design 1 is better than design 2 in terms of the degree of inclusion.

System performance evaluation that incorporates a customer’s utility/disutility is more customer-centric since the customer’s and designer’s preferences for system behavior are considered. The accumulated disutility of the two systems was evaluated based on Equation (22) with a scale parameter of \( r = 2 \). The cumulative disutility values of design 1 and design 2 are 0.1502 and 0.1741, respectively. Based on the disutility values, we also conclude that design 1 is better. The dynamic fuzzy reliability of the two systems was evaluated from various aspects using three metrics, and the same conclusion was reached for this example.

6. Conclusions

This article presents the use of the theory and methods of fuzzy sets to model the reliability behavior for a component or system with continuous performance levels. The acceptable level of functionality for a system is related to a performance or performance characteristic variable. We consider the linguistic word \( \text{success} \) or \( \text{failure} \) as fuzzy sets from the viewpoint of the customer, and the membership function for a fuzzy success/failure event is defined over the performance characteristic variable. It is shown that fuzzy...
modeling is more realistic for systems with continuous performance levels. We demonstrate the evolving relationship from the binary reliability model to multi-state reliability model to fuzzy reliability measures. Dynamic fuzzy reliability is evaluated based on the gamma degradation process, and the time to fuzzy failure, which is modeled as a fuzzy random variable, is evaluated by solving a sequence of non-linear optimization problems. This article also develops some reliability performance metrics that are able to capture the cumulative experiences of the customers with the system. The numerical examples on dynamic fuzzy reliability evaluation show the benefits of fuzzy reliability modeling over traditional reliability modeling for system performance evaluation and decision making. Fuzzy reliability evaluation for systems by considering system structure and element-level performance variables will be investigated in our future research.

References
Biographies

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