Super-connected arc-transitive digraphs

Jixiang Meng, Zhao Zhang

College of Mathematics and System Sciences, Xinjiang University, Urumqi, Xinjiang, 830046, People’s Republic of China

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1. Introduction

By a digraph $D = (V(D), A(D))$, we mean a directed graph without loops and multiple arcs. A digraph $D$ is said to be an oriented graph if there are no directed cycles of length 2 in $D$. A simple undirected graph $G = (V(G), E(G))$ can be viewed as a digraph by replacing each edge by two arcs with opposite directions.

The connectivity $\kappa(D)$ of a digraph $D$ is the minimum number of vertices the deletion of which makes the remaining digraph no longer strongly connected. In designing networks, high connectivity is desirable since such a digraph is more reliable. It is well known that $\kappa(D) \leq \delta(D)$, where $\delta(D) = \min\{\delta^+(D), \delta^-(D)\}$ is the minimum degree of $D$ (the symbols $\delta^+(D)$ and $\delta^-(D)$ are the minimum out-degree and the minimum in-degree respectively). Hence a digraph $D$ with $\kappa(D) = \delta(D)$ is said to be maximally vertex connected. A maximally edge-connected digraph is defined similarly by requiring the edge connectivity $\lambda(D) = \delta(D)$.

To design more reliable networks, besides the requirement of maximal vertex connectivity, it is also desirable that the number of minimum vertex cuts is as small as possible. For this purpose, Boesch [2] proposed the concept of super-connectedness, which is a special kind of conditional connectivity [9]. A digraph $D$ is super-connected if for any minimum vertex cut $C$, there exists a vertex $x$ such that $C = N^+(x)$ or $N^-(x)$, where $N^+(x)$ and $N^-(x)$ are the out-neighbor set and the in-neighbor set of $x$ in $D$ respectively. A minimum vertex cut of the form $N^+(x)$ or $N^-(x)$ is said to be trivial. When there are edge failures, the concept of super-arc-connected digraph is defined similarly. Boesch also proposed the concept of hyper-connectedness [2]. A digraph $D$ is hyper-connected if the removal of any minimum vertex cut of $D$ results in exactly two strongly connected components one of which is a singleton. Another concept which is very close to hyper-connected is Vosperian used in [7]. The difference between these two concepts is little. In fact, it can be proved that a digraph $D$ is Vosperian if and only if either $D$ is hyper-connected or $|V(D)| = \kappa(D) + 3$ (see [12] for the proof when $D$ is undirected; when $D$ is directed, the proof is similar). Clearly, a hyper-connected digraph is also super-connected. Furthermore, hyper-connectedness is more favorable than super-connectedness since the number of components in the remaining digraph measures the seriousness of the damage to the network.
This paper studies super-connectedness of arc-transitive digraphs. Transitive graphs and digraphs are favorable in network designs since they have many desirable properties such as high symmetry, easy routing, low transmission delay, and high reliability etc. [17]. In particular, high reliability is closely related with high connectivity. A digraph $D$ is said to be vertex-transitive if the automorphism group $\text{Aut}(D)$ acts transitively on $V(D)$, and is arc-transitive if $\text{Aut}(D)$ acts transitively on $A(D)$. An important class of transitive digraphs is the Cayley digraph. For a group $G$ and a subset $S \subseteq G$, the Cayley digraph $\text{Cay}(G, S)$ is a digraph with vertex set $G$ and arc set $\{(g, gs) \mid g \in G, s \in S\}$. In particular, if $G$ is Abelian, then $\text{Cay}(G, S)$ is an Abelian Cayley digraph. A Cayley digraph is always vertex transitive.

It is known that connected (strongly connected) vertex-transitive graphs (digraphs) are maximally edge (arc) connected [5,11]; connected (strongly connected) edge-transitive (arc-transitive) graphs (digraphs) are maximally vertex connected [13,14]. In [8], Hamidoune and Tindell gave a nice characterization of super-edge-connected (super-arc-connected) vertex-transitive graphs (digraphs). In [12], Meng showed that with the exception of two classes of well-defined graphs, all vertex- and edge-transitive graphs are super-connected (see Theorem 4.1 of this paper). In [7], Hamidoune et al. obtained complete characterizations for Vosperian and super-connected Abelian Cayley digraphs [7]. For more results on connectivities of transitive graphs and digraphs, we cite [1,4,6,10,16,18,19] for references.

As far as we know, there are no results on super-connected arc-transitive digraphs, which is the motivation of this paper. To state our results, we need more concepts.

A digraph $D$ is reducible if there are two vertices $x, y \in V(D)$ such that $N^+(x) = N^+(y)$ or $N^-(x) = N^-(y)$. Reducibility in undirected graphs was used to characterize non-super-connected vertex- and edge-transitive graphs in [12]. In this paper, we propose a new class of super-connected digraphs: a super-connected digraph $D$ is bi-super-connected, if there are two vertices $x, y \in V(D)$ such that $N^+(x) = N^-(y)$. In the class of super-connected digraphs with the same number of vertices, a bi-super-connected digraph or a reducible super-connected digraph often has less minimum vertex cuts than those which are not. For example, in a vertex-transitive super-connected digraph, every vertex $x$ corresponds to two minimum vertex cuts, namely $N^+(x)$ and $N^-(x)$. If many of the trivial vertex cuts coincide (in the form $N^+(x) = N^+(y)$ or $N^+(x) = N^-(y)$ or $N^-(x) = N^-(y)$), then the total number of minimum vertex cuts may be reduced.

In this paper, we prove that a strongly connected arc-transitive oriented graph is either reducible, or super-connected. Furthermore, if it is super-connected, then it is either hyper-connected or bi-super-connected. In the special case that the digraph is an Abelian Cayley digraph, we show that it is super-connected. For terminologies not defined here, we follow [3] for references.

2. Fragments and superatoms

In this section, we study the properties of superatoms in strongly connected arc-transitive digraphs. The concept of fragments and atoms were first proposed by Mader [11] and Watkins [15]. Their variations have been used as a powerful tool in studying various kinds of connectivities. We introduce the concepts of strict fragments and superatoms for a digraph in the following.

Let $D$ be a strongly connected digraph with connectivity $\kappa = \kappa(D)$ and $F \subseteq V(D)$. Set

$$\begin{align*}
N^+(F) &= \{x \in V(D) \setminus F \mid \exists y \in F \text{ such that } (y, x) \in A(D)\}, \\
C^+(F) &= F \cup N^+(F), \\
R^+(F) &= V(D) \setminus C^+(F), \\
N^-(F) &= \{x \in V(D) \setminus F \mid \exists y \in F \text{ such that } (x, y) \in A(D)\}, \\
C^-(F) &= F \cup N^-(F), \\
R^-(F) &= V(D) \setminus C^-(F).
\end{align*}$$

A set of vertices $F$ is called a positive (negative) fragment if $N^+(F) \cap N^-(F)$ is a minimum vertex cut of $D$. A fragment $F$ with $2 \leq |F| \leq |V(D)| - \kappa - 2$ is called a strict fragment. Strict fragments with minimum cardinality are called superatoms (the minimum is taken over all positive and negative fragments). The cardinality of a superatom is denoted by $\omega(D)$. A superatom $A$ is called a positive (negative) superatom if $A$ is a positive (negative) fragment of $D$. Note that a strongly connected digraph does not necessarily contain a superatom (for example, in a super-connected digraph). Furthermore, if it contains superatoms, it does not necessarily contain both positive and negative superatoms. The digraph in Fig. 1 has a negative superatom $\{u, v\}$ but does not have positive super-atoms. Clearly, if $F$ is a positive (negative) fragment, then $R^+(F) \cap R^-(F)$ is a negative (positive) fragment. If $F$ is a strict positive (negative) fragment, then $R^+(F) \setminus R^-(F)$ is a strict negative (positive) fragment.

The strict fragment and superatom defined by Hamidoune in [6] is different from ours’ in that he only defined in fact strict positive fragment, and the superatom appeared in his paper is then the strict positive fragment of least cardinality. But in his theorems, he always assumed that $D \cong D^\sim$, the reverse digraph of $D$. Clearly, for this kind of digraphs, his superatom is the same as ours’. A generalization of strict fragment and superatom called $\eta$-fragment and $\eta$-atom can be found in [1], which is used to study $\eta$-extraconnectivity of a digraph.

A digraph $D$ is $k$-regular if for any $x \in V(D)$, $d^+(x) = d^-(x) = k$. In what follows, we always assume that $D$ is a strongly connected $k$-regular digraph with $\kappa(D) = k$. From the following lemma, arc-transitive digraphs always satisfy this assumption.
Lemma 2.1. If $D$ is a strongly connected arc-transitive digraph, then it is vertex- and edge-transitive. Furthermore, $\kappa(D)$ attains the regularity of $D$.

Proof. The edge transitivity is obvious. To show the vertex transitivity, let $x, y$ be two vertices in $V(D)$. Since $D$ is strongly connected, we have $d^+(x) > 0$ and $d^+(y) > 0$. Let $e_1 = (x, x_1)$ and $e_2 = (y, y_1)$ be two arcs of $D$. Then there exists an automorphism $\sigma$ such that $\sigma(e_1) = e_2$, whence $\sigma(x) = y$. Since a strongly connected arc-transitive digraph is maximally vertex connected [13, 14], the conclusion about $\kappa(D)$ holds. □

The following two results were proved in [6] which play an important role in our deduction.

Lemma 2.2 ([6]). Let $D$ be a $k$-regular vertex-transitive digraph with $\kappa(D) = k$. Let $A$ be a positive (negative) superatom of $D$ and let $F$ be a strict positive (negative) fragment such that $A \cap F = \emptyset$ and $A \not\subseteq F$. Then

(i) $|A \cap F| = 1$ and $A \cup F$ is a positive (negative) fragment;
(ii) $C^+(A \cap F) = C^+(A) \cap C^+(F)$ ($C^-(A \cap F) = C^-(A) \cap C^-(F)$).

Lemma 2.3 ([6]). Let $D$ be a $k$-regular strongly connected vertex-transitive digraph with $\kappa(D) = k$ and $\omega(D) \geq 3$. Then the intersection of three distinct positive (negative) superatoms is empty.

From Lemma 2.2, we can prove the following:

Lemma 2.4. Let $D$ be a strongly connected arc-transitive digraph with $\omega(D) \geq 3$. Then $D$ has both positive and negative superatoms.

Proof. By Lemma 2.1, we may assume that $\kappa(D) = k$, where $k$ is the regularity of $D$.

Since $\omega(D) \geq 3$, the subdigraphs induced by superatoms are not empty. In fact, if a superatom $A$ is an independent set, then for any $x$ and $y$ in $A$ we have $N^+(x) = N^+(A) = N^+(y)$. Thus $\{x, y\}$ is a strict fragment, contradicting that $\omega(D) \geq 3$. By arc transitivity of $D$, each arc is contained in a superatom.

Suppose $D$ has only positive superatoms. We shall prove that for any integer $n = 1, 2, \ldots$, there are $n$ distinct superatoms $A_1, A_2, \ldots, A_n$ of $D$ such that $A_1 \cup A_2 \cup \cdots \cup A_n$ is a strict positive fragment of $D$, thus contradicting the finiteness of $D$. This is proved by induction on $n$. The case $n = 1$ is obvious. Suppose $n \geq 2$ and $A_1, \ldots, A_{n-1}$ are $n - 1$ distinct superatoms of $D$ such that $F_{n-1} \supseteq A_1 \cup \cdots \cup A_{n-1}$ is a strict positive fragment. Let $e$ be an arc with tail in $F_{n-1}$ and head in $V \setminus F_{n-1}$, and $A_n$ be a superatom containing $e$. Then $A_n \not\subseteq F_{n-1}$, $A_n \cap F_{n-1} \neq \emptyset$, and $A_n$ is a ‘positive’ superatom by our assumption that $D$ has only positive superatoms. It follows from Lemma 2.2 that $F_{n-1} \cup A_n = A_1 \cup A_2 \cup \cdots \cup A_n$ is a strict positive fragment. This finishes the induction step.

The case that $D$ has only negative superatoms can be proved similarly. □

A desirable property of any type of atoms is that, if non-trivial, they form imprimitive blocks for the automorphism group of a graph or a digraph. To be more precise, an imprimitive block of a digraph $D$ is a proper non-trivial vertex subset $A$ of $V(D)$ such that for any $\sigma \in \text{Aut}(D)$, either $\sigma(A) = A$ or $\sigma(A) \cap A = \emptyset$. An imprimitive block of a graph can be defined similarly. The next result cited from [14] indicates why imprimitivity is so useful.

Lemma 2.5 ([14]). Let $D$ be a graph or a digraph and let $X$ be the subgraph or subdigraph induced by an imprimitive block $A$ of $D$.

(i) If $D$ is vertex-transitive, then so is $X$;
(ii) if $D$ is a strongly connected arc-transitive digraph or a connected edge-transitive graph and $A$ is a proper subset of $V$, then $A$ is an independent subset of $D$.

Next, we study the structures of superatoms in strongly connected arc-transitive digraphs.

Lemma 2.6. Let $D$ be a strongly connected arc-transitive digraph with $\omega(D) \geq 3$.

(i) The subdigraphs induced by superatoms are non-empty and arc-transitive;
(ii) the subdigraphs induced by positive (negative) superatoms are isomorphic.
Lemma 2.7. Let $D$ be a strongly connected arc-transitive digraph with $\omega(D) \geq 3$. Then the subdigraphs induced by superatoms are strongly connected.

Proof. Let $A$ be a superatom of $D$, say, a positive one. Let $X = D[A]$. Then $X$ is not empty since $\omega(D) \geq 3$. Let $e_1$ and $e_2$ be two arcs in $X$ (if they exist). Then there exists an automorphism $\sigma$ of $D$ such that $\sigma(e_1) = e_2$. Since $|\sigma(A) \cap A| \geq 2$, we have $\sigma(A) = A$ by Lemma 2.2. Thus the restriction of $\sigma$ on $A$ is an automorphism of $X$ mapping $e_1$ to $e_2$, hence (i) follows. By a similar argument, (ii) can be proved. \[\square\]

Lemma 2.8. Let $D$ be a strongly connected arc-transitive digraph with $\omega(D) \geq 3$ and regularity $k$. Then $k$ is even and the subdigraphs induced by superatoms are $k/2$-regular strongly connected arc-transitive digraphs.

Proof. We only prove the result for positive superatoms. If any two distinct positive superatoms are disjoint, then positive superatoms are imprimitive blocks of $D$ and are therefore, by Lemma 2.7, independent subsets, contradicting Lemma 2.7. Suppose that $A$ and $B$ are two distinct positive superatoms with $A \cap B \neq \emptyset$. Then $|A \cap B| = 1$ by Lemma 2.2. Let $A \cup B = [a]$. Then any arc $e$ with tail $a$ is either in $D[A]$ or in $D[B]$. Otherwise, by arc transitivity of $D$, $e$ is contained in the subdigraph induced by a third positive superatom $C$. But then $a \in A \cap B \cap C$, contradicting Lemma 2.3. By arc transitivity and Lemma 2.6, $D[A]$ and $D[B]$ are $k/2$-regular. \[\square\]

In particular, for oriented graph, we have the following two results.

Lemma 2.9. Let $D$ be a strongly connected arc-transitive oriented graph which has superatoms. Then $\omega(D) = 2$.

Proof. Assume by contradiction that $\omega(D) \geq 3$. Then, by the proof of Lemma 2.8, there exist two superatoms $A$ and $B$, say positive ones, such that $A \cap B \neq \emptyset$. Then $|A \cap B| = 1$. Write $X = D[A]$, $Y = D[B]$ and $A \cap B = \{a\}$. Then both $X$ and $Y$ are $k/2$-regular, where $k$ is the regularity of $D$. By (ii) of Lemma 2.2, we have

\[N^+(a) = (A \cap N^+(B)) \cup (B \cap N^+(A)) \cup (N^+(A) \cap N^+(B)).\]

Let $T_1 = N^+(A) \cap B$, $T_2 = N^+(B) \cap A$, $S_1 = N^+(A) \setminus T_1$ and $S_2 = N^+(B) \setminus T_2$. Then $|T_1|, |T_2| \geq k/2$. It follows that $|T_1| = |T_2| = k/2, N^+(A) \cap N^+(B) = \emptyset$, and $|S_1| = |S_2| = k/2$.

For every $y \in A \setminus \{a\}$, by the structure of superatoms (Lemma 2.8) and the vertex transitivity of $D$, there exists a superatom $A_y$ other than $A$ such that $y \in A_y$. Since $|A_y \cap A| = 1, |A_y \cap B| \leq 1$, and $|A_y| \geq 3$, we have $A_y \not\subseteq A \cup B$. By Lemma 2.1, $A \cup B$ is a positive fragment. Hence $|A_y \cap (A \cup B)| = 1$. That is, $A_y \cap A \cup B = \{y\}$, and $A_y \cap B = \emptyset$. Combining this with $N^+_y(y) \subseteq A \cup B$ and $N^+_y(y) \cap N^+_y(y) \subseteq S_1$. Since $N^+_y(y) \setminus N^+_y(y) = k/2 = |S_1|$, we have $N^+_y(y) \cap N^+_y(y) = S_1$. It follows that $S_1 \subseteq A_y$. When $y$ runs over all vertices in $A \setminus \{a\}$, we have $S_1 \subseteq \bigcap_{y \in A \setminus \{a\}} A_y$. Then $|A \setminus \{a\}| \leq 2$ since no three distinct superatoms may have non-empty intersection. So $\omega(D) = 3$. Write $A = \{a, y_1, y_2\}$, where $y_1$ and $y_2$ are the out-neighbor and the in-neighbor of $a$ in $X$, respectively. Since the subdigraphs of $D$ induced by superatoms are strongly connected and arc-transitive, $X$ is a directed cycle of length $3$. Hence $y_1, y_2$ is an arc. By $S_1 \subseteq A_1 \cap A_2$, we have $|S_1| = 1$, and thus $k = 2$. Write $S_1 = \{s\}$. Then the two ends of the arc $y_1, y_2$ have a common out-neighbor $s$. By arc-transitivity of $D$, $a$ and $y_1$ must have a common out-neighbor too. Since $y_2$ and $s$ are the only two out-neighbors of $y_1$, and $s \not\in N^+(a)$, we see that $a, y_2$ is also an arc, contradicting that $D$ is an oriented graph. \[\square\]

Lemma 2.10. Let $D$ be a strongly connected arc-transitive oriented graph which has superatoms. If the regularity $k \geq 2$, then any superatom of $D$ is an independent subset of cardinality $2$.

Proof. Let $A$ be a superatom of $D$, say, a positive one. By Lemma 2.9, $|A| = \omega(D) = 2$. Write $A = \{x, y\}$. Suppose $x$ is adjacent to $y$. Since $|N^+(x) \cup N^+(y)| = |N^+(A) \cup \{y\}| = k + 1$, we have $|N^+(x) \cap N^+(y)| = k - 1 \geq 1$. For each $z \in N^+(x) \cap N^+(y)$, by arc transitivity of $D$, we have $|N^+(z) \cap N^+(x)| = k - 1$. But then $z$ must be adjacent to $y$, contradicting the assumption that $D$ is an oriented graph. \[\square\]

Under the condition of Lemma 2.10, we see that $D$ is reducible. In fact, suppose $A$ is a superatom of $D$ which is, say, positive. Then by Lemma 2.10, we have $A = \{x, y\}$ such that $N^+(x) = N^+(A) = N^+(y)$. The reducibility of $D$ follows from the vertex-transitivity of $D$. 

3. Main results

Now, we are ready to prove our main results.

**Theorem 3.1.** Let $D$ be a strongly connected arc-transitive oriented graph. Then, $D$ is either reducible, or hyper-connected, or bi-super-connected.

**Proof.** Suppose $D$ is $k$-regular. If $k = 1$, then $D$ is a directed cycle, and is clearly bi-super-connected. So, suppose $k \geq 2$ in the following.

We first show that if $D$ is not super-connected, then $D$ is reducible. In this case, $D$ has superatoms. In fact, suppose $C$ is a minimum vertex cut of $D$ which is neither $N^+(x)$ nor $N^-(x)$ for any vertex $x$. Let $F_1$, $F_2$ be the vertex sets of two strongly connected components of $D - C$ such that $N^+_D(F_1) = N^+_D(F_2) = \emptyset$. Then $|F_1| \geq 2$ and $|F_2| \geq 2$. It follows that $F_1$ is a strict positive fragment and $F_2$ is a strict negative fragment. Taking the minimum of all strict positive and negative fragments, the existence of superatoms is proved. Then, by the discussion after Lemma 2.10, $D$ is reducible.

Next, suppose $D$ is super-connected but not hyper-connected. Let $C$ be a minimum vertex cut of $D$ such that $D - C$ has at least three strongly connected components. Since $D$ is super-connected, there is a vertex $u$ in $V(D)$ with, say, $N^-(u) = C$. Let $F$ be the vertex set of a strongly connected component of $D - C$ such that $N^+_D(F) = \emptyset$. If $|F| = 1$, then $D$ is bi-super-connected. So, suppose $|F| \geq 2$. Then $F$ is a strict positive fragment of $D$ (note that $|V(D)| - |F| - |C| \geq 2$ since $D - C - F$ has at least two strongly connected components), and thus $D$ has superatoms. It follows from Lemma 2.10 that $D$ is reducible. □

In particular, if $D$ is Abelian and Cayley, we can prove the following:

**Theorem 3.2.** Any strongly connected arc-transitive oriented Abelian Cayley digraph is super-connected.

**Proof.** Suppose $D = \text{Cay}(G,S)$ is a strongly connected arc-transitive oriented Abelian Cayley digraph which is not super-connected. Then by the proof of Theorem 3.1, $D$ has superatoms, say, positive ones, which are independent sets of sets of cardinality 2 by Lemma 2.10. Clearly, the two vertices in a positive superatom have the same out-neighbor set. By vertex transitivity of $D$, every vertex shares its out-neighbor set with another vertex. Define an equivalence relation $R$ on the vertex set $V(D)$ of $D$: for $v_1$ and $v_2$ in $V(D)$, $v_1 R v_2$ if and only if $N^+_D(v_1) = N^+_D(v_2)$. According to this equivalence, $V(D)$ is partitioned into non-empty subsets $A_1$, $A_2$, ..., $A_p$. Since $D$ is vertex-transitive, we have, by the above argument, that $|A_i| \geq 2$ ($1 \leq i \leq p$). Clearly, for $a \in A_i$, $A_i$ is the set of vertices with the same out-neighbor set as $a$. By the vertex transitivity of $D$, $|A_i|$ is independent of $i$. So, $|A_i| = |V(D)|/p$ ($1 \leq i \leq p$) is a constant, denoted by $\alpha(D)$.

The key observation here is the following: vertices in a same equivalence class also have the same in-neighbor set. This follows easily from the fact that in an Abelian Cayley digraph, if $x + s_1 = y + s_2$ for $x, y \in G$ and $s_1, s_2 \in S$, then $x - s_2 = y - s_1$.

Define a quotient digraph $Q(D)$ with vertex set $\{A_i \mid 1 \leq i \leq p\}$. There is an arc from $A_i$ to $A_j$ in $Q(D)$ if and only if there exist $a \in A_i$ and $b \in A_j$ such that $a$ is adjacent to $b$ in $D$. Clearly, $A_1, A_2, ..., A_p$ are imprimitive blocks of $\text{Aut}(D)$. It follows that $Q(D)$ is a strongly connected arc-transitive oriented graph with degree $k/\alpha(D)$, where $k = |S|$ is the regularity of $D$ (here the key observation is used). By the key observation, vertices in a same equivalence class play the same role in connecting vertices. So, every minimum vertex cut of $D$ must be a union of some equivalence classes, and for every minimum vertex cut $C$ of $D$, the vertex set of each non-trivial strongly connected component in $D - C$ must be a union of some equivalence classes.

As $D$ is not super-connected, there exists a minimum vertex cut $C$ such that each strongly connected component of $D - C$ is non-trivial. Let $\hat{C}$ be the corresponding vertex subset of $V(Q(D))$. Then $\hat{C}$ is a vertex cut of $Q(D)$ with cardinality $k/\alpha(D)$. Since $Q(D)$ is a strongly connected arc-transitive digraph, its connectivity equals its regularity. Thus $\hat{C}$ is a minimum vertex cut of $Q(D)$. Noting that each strongly connected component of $Q(D) - \hat{C}$ is non-trivial, we see that $Q(D)$ has superatoms. It is easy to see that if $Q(D)$ has positive (negative) superatoms, then $D$ has positive (negative) superatoms. Thus we may assume, without loss of generality that $Q(D)$ has positive superatoms. By Lemma 2.10, let $A = \{A_i, A_j\}$ be a positive superatom of $Q(D)$. Then $A_i$ and $A_j$ have the same out-neighbor sets in $Q(D)$. Therefore, the vertices in $A_i \cup A_j$ have the same out-neighbor sets in $D$, which is a contradiction to the definition of quotient digraphs. □

4. Discussion

For vertex- and edge-transitive undirected graphs, Meng obtained the following characterization.

**Theorem 4.1 (\cite{12}).** Let $G$ be a connected vertex- and edge-transitive graph. Then $G$ is not super-connected if and only if $G \cong C_n(N_m)$ ($n \geq 6$ and $m \geq 1$), the lexicographic product of a $n$-cycle $C_n$ of length $n$ by an empty graph $N_m$, or $G \cong L(Q_3)(N_m)$ ($m \geq 1$), the lexicographic product of the line graph of the 3-cube $Q_3$ by $N_m$.

Since an arc-transitive digraph is either an oriented one or an undirected one, in view of Theorem 4.1, to give a complete characterization of super-connected arc-transitive digraphs, it remains to consider oriented graphs. Our original goal is to show that except for a few special cases, every strongly connected arc-transitive oriented graph is super-connected. Theorem 3.1 gives a partial solution to it, and Theorem 3.2 gives a confirmatory answer to a special case when the digraph is Abelian and Cayley. The difficulty for the general case lies in the following: for an undirected graph, reducible means that
the graph has two vertices having the same in-and-out-neighborhood; while in a digraph, two vertices might have the same in-neighborhood but different out-neighborhood. We have tried to find out whether the latter case is possible for strongly connected arc-transitive oriented digraphs, but in vain. Hence, to completely characterize super-connected arc-transitive digraphs is still open.

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References