Importance measure analysis with epistemic uncertainty and its moving least squares solution

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A B S T R A C T

For the structural systems with both epistemic and aleatory uncertainties, in order to analyze the effect of the epistemic uncertainty on the safety of the systems, a variance based importance measure of failure probability is constructed. Due to the large computational cost of the proposed measure, a novel moving least squares (MLS) based method is employed. By fitting the relationship of parameters and failure probability with moving least squares strategy, the conditional failure probability can be obtained conveniently, then the corresponding importance measure can be calculated. Compared with Sobol’s method for the variance based importance measure, the proposed method is more efficient with sufficient accuracy. The Ishigami function is used to test the efficiency of the proposed method. Then the proposed importance measure is used in two engineering applications, including a roof truss and a riveting process.

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1. Introduction

It is a common practice to analyze the impact of input uncertainty on the structural systems in reliability engineering. Generally, two different uncertainty sources: aleatory uncertainty and epistemic uncertainty [1–3] are involved. Aleatory uncertainty describes the inherent variability associated with a structural system, which referred to as irreducible, objective uncertainty. Epistemic uncertainty results from lack of knowledge of fundamental phenomena and is related to our ability to understand, measure, and describe the systems under study. It is important to distinguish between aleatory and epistemic uncertainties because different types of uncertainty may trigger different responses: a concrete action must be taken to circumvent the potentially dangerous effects of inherent variability, whereas the best decision for the presence of epistemic uncertainty is probably to try to reduce it by collecting more information. Historically, probability theory has provided the mathematical structure used to represent both epistemic uncertainty and aleatory uncertainty [3]. However, different theories have been used to handle epistemic uncertainty. The theories include non-probability theories such as evidence theory [4], possibility theory [5], and fuzzy set theory [6]. The introduction of these alternative uncertainty representations has been accompanied by a lively discussion of their attributes and usefulness, with some individuals maintaining that the use of these alternative representations is essential in situations of rare information and other individuals maintaining that probability theory is sufficient for the representation of uncertainty in all situations [7–9]. In our work, we primarily focus on the representation of the traditional probabilistic approach to represent the epistemic uncertainty.

In general, sensitivity analysis (SA) can be classified into two groups: local sensitivity analysis and global sensitivity analysis (or importance measure analysis, IM analysis) [10]. Local SA studies how small variations of parameters around a reference point change the value of the output. The main drawback of local SA is its local nature which depends on the choice of the reference point. Global SA takes into account all the variation range of the parameters, and apportions the
output uncertainty to the uncertainty of the input parameters, covering their entire range space [11]. At present, a number of measures have been suggested, such as, Helton and Salter proposed the nonparametric techniques (input–output correlation) [12,13], Sobol, Iman and Salter proposed a series of variance based importance measures [13–15], and Chun, Liu and Borgonovo proposed moment independent sensitivity indicators [16,17]. But those indicators are all proposed for the structural system with epistemic input uncertainty. Hofer and Krzykacz-Hausmann [8,9] investigated another situation that the input uncertainty of a model is only aleatory uncertainty described by the probability distribution and the distribution parameters of inputs which are subject to epistemic uncertainty, are not precisely known. In their studies, they proposed the variance based sensitivity measures in the presence of epistemic and aleatory uncertainties, which can be used to identify the most influential distribution parameters. Based on this idea, we proposed the variance based sensitivity measures of failure probability in the presence of epistemic and aleatory uncertainties, which can be used to identify the most influential distribution parameters on the safety of a system.

Following the variance based importance measure proposed by Sobol [14], this paper investigates the effect of the epistemic parameters uncertainties on the failure probability of structural systems, and proposes a variance based IM of failure probability. The key of calculating the variance based IM is to compute the conditional expectation of failure probability, which generally requires a large number of function evaluations and is impractical for most engineering practice [9]. Thus, a novel moving least squares (MLS) based method is employed to calculate the proposed IM. This method can be used to approximate the functional relationship between distribution parameters and failure probability [18,19]. When defining an affected region at a certain point in parameters space, the conditional expectation of failure probability at this point can be approximated by the weighted average of training failure probability in the region. Then, the variance of the conditional expectation can be simulated and the proposed IM can be calculated directly. It is noticed that the proposed MLS based method needs only a group of training parameters and failure probability samples to calculate the proposed variance based IM and is independent of the dimensionality of parameters. Compared with the Sobol’s method presented in the following, the proposed MLS based method can improve the computational efficiency remarkably.

The remainder of this paper is organized as follows: Section 2 analyzes the epistemic uncertainty and aleatory uncertainty in structural systems and proposes a variance based IM of failure probability. Section 3 first employs the Sobol’s method to compute the proposed IM, and then proposes a novel MLS based method. In Section 4, a numerical example is employed to validate the efficiency of the proposed MLS based method, then two engineering examples, including a roof truss and a riveting process, are employed to demonstrate the rationality of the proposed variance based IM. Finally, some conclusions are drawn in Section 5.

2. Importance measures analysis with parameter uncertainty

2.1. Description of epistemic and aleatory uncertainties

Being subject to both the epistemic and aleatory uncertainties, the performance function of a model can be generally given as:

\[ Y = g(X, \theta) \]  

where \( X \) is the vector of aleatory variables and \( \theta \) is the vector of epistemic parameters.

When fixing the epistemic parameters \( \theta \) at a reference value \( \theta^* \), the aleatory uncertainty of variables can be described by conditional probability density function (PDF) \( f_X(x|\theta^*) \) [8,9] and the output \( Y \) is a function of aleatory variables \( X \). Consequently, the statistic characteristics of \( Y \), such as expectation, variance, failure probability and so on, only depend on \( \theta \) and can be regarded as a function of \( \theta \). However, those functions are always complex and cannot be given analytically, whereas they can be expressed by numerical mapping or approximated by meta-modeling methods such as the response surface method [20], the neural network method [21], etc.

As mentioned in Ref. [8], when sufficient knowledge of the epistemic parameters is not available, an approximate distribution assumption can be made such that the uncertainties of statistic characteristics are completely determined by their two central moments (and this is the case for almost all standard parametric distributions). Furthermore, an individual may alternatively apply the maximum entropy principle to arrive at a distribution having the two approximated central moments but otherwise having maximum epistemic uncertainty associated with it. Especially, if no further information about this distribution is available, then it will be the normal distribution with the given mean and the given standard deviation. This assumption will be made in our examples to represent the epistemic uncertainty.

2.2. Importance measure of failure probability

In order to enhance the safety of structural systems, it is necessary to investigate how the uncertainties of epistemic parameters affect the uncertainty of failure probability. Thus, following the Sobol’s variance based IM [14], a novel variance based IM of failure probability in the presence of epistemic uncertainty is proposed.

Consider the function \( Y = g(X, \theta) \) with both aleatory and epistemic uncertainties. If an individual wants to investigate the influence of epistemic uncertainty \( \theta \) on the safety of a structural system, the functional relationship between failure
probability and epistemic parameters can be represented as \( P_f = \psi(\theta) \), where \( \theta = (\theta_1, \theta_2, \ldots, \theta_p) \) are \( p \)-dimensional independent parameters. The method of variance based IM developed by Sobol is based on ANOVA high dimensional model representation (HDMR) [14]. Based on this idea, considering the square integrable function \( \psi(\theta) \) defined in the hypercube parameters space \( H^p \), there exists the following unique decomposition:

\[
\psi(\theta) = \psi_0 + \sum_{i=1}^{p} \psi_i(\theta_i) + \sum_{1 \leq i < j \leq n} \psi_{ij}(\theta_i, \theta_j) + \cdots + \psi_{1,2,\ldots,p}(\theta_1, \theta_2, \ldots, \theta_p)
\]

where

\[
\psi_0 = E(P_f) \\
\psi_i = E(P_f | \theta_i) - E(P_f) \\
\psi_{ij} = E(P_f | \theta_i, \theta_j) - \psi_i - \psi_j - E(P_f)
\]

\( E(P_f) \) is the expectation of failure probability and \( E(P_f | \theta_i) \) is the conditional expectation of failure probability. The high order items can be obtained similarly. It is noticed that each terms in the expansion above has zero-mean, i.e. \( \int \psi_i(\theta_i) d\theta_i = 0 \).

The basic idea of Sobol’s measure is to decompose the model into terms of increasing dimensionality as in Eq. (2). Thus, the total variance of failure probability can be decomposed into:

\[
V = \sum_{i=1}^{p} V_i + \sum_{1 \leq i < j \leq n} V_{ij} + \cdots + V_{1,2,\ldots,p}
\]

where \( V \) is the total variance of failure probability,

\[
V_i = V(\psi_i) = V[E(P_f | \theta_i)] \\
V_{ij} = V(\psi_{ij}) = V[E(P_f | \theta_i, \theta_j)] - V[E(P_f | \theta_i)] - V[E(P_f | \theta_j)]
\]

are the first-order and the second-order variance contributions of parameters to the failure probability, respectively.

In this approach, the first-order sensitivity measure (or Importance Measure, IM) of failure probability can be defined as:

\[
S_{\theta_i} = \frac{V_{\theta_i}(E_{\theta_{-i}}(P_f | \theta_i))}{V(P_f)}.
\]

The first-order index \( S_{\theta_i} \) shows the effect of a single parameter \( \theta_i \) on the failure probability \( P_f \). According to the definition in Eq. (6), the difficulty of calculating the sensitivity measure \( S_{\theta_i} \) is to compute the variance of the conditional expectation of failure probability. This needs a “double-loop” sampling procedure to achieve, with an “inner-loop” for the distribution parameters \( \theta \) and an “outer-loop” for the inputs \( X \). If the failure probability is evaluated by the sampling based methods, the total procedure would increase to “triple-loop”, which would be unfeasible due to the large computational cost.

3. Solutions of importance measure of failure probability

3.1. Sobol’s method

Generally, the computation of the variance based IM can be a computationally expensive procedure. In order to calculate this measure efficiently, Sobol proposed a Monte Carlo based sampling method. This method can obtain the conditional expectation of the output by two groups of input samples, which is computationally cheap comparing with the nested “double-loop” procedure. Readers can refer to the Refs. [22,23] for further discussion of Sobol’s method. Thus, in this section, the Sobol’s method is firstly employed to calculate the proposed IM of failure probability and the detailed steps are given as follows.

1. According to the marginal probability density function (PDF) \( f_{\theta_i}(\theta_i) \) (i = 1, 2, ..., p), generate a \( N \times p \) sampling matrix \( U_{N \times p} \) of distribution parameters with each row a set of parameters samples. \( U_{N \times p} \) can be called the “sample” matrix:

\[
U_{N \times p} = \begin{bmatrix}
\theta_{11} & \cdots & \theta_{1i} & \cdots & \theta_{1p} \\
\theta_{21} & \cdots & \theta_{2i} & \cdots & \theta_{2p} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\theta_{N1} & \cdots & \theta_{Ni} & \cdots & \theta_{Np}
\end{bmatrix}
\]

(7)

2. Generate another \( N \times p \) sampling matrix \( W_{N \times p} \) by the bootstrapping method. \( W_{N \times p} \) can be called “re-sampled” matrix:

\[
W_{N \times p} = \begin{bmatrix}
\theta_{(N+1)1} & \cdots & \theta_{(N+1)i} & \cdots & \theta_{(N+1)p} \\
\theta_{(N+2)1} & \cdots & \theta_{(N+2)i} & \cdots & \theta_{(N+2)p} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\theta_{(2N)1} & \cdots & \theta_{(2N)i} & \cdots & \theta_{(2N)p}
\end{bmatrix}
\]

(8)
3. Define a matrix \( W'_N \times p \) formed by all columns of \( W_N \times p \) except the \( i \)th column obtained from the \( i \)th column of \( U_N \times p \):

\[
W'_N = \begin{bmatrix}
\theta_{(N+1)1} & \cdots & \theta_{(N+1)i} & \cdots & \theta_{(N+1)p} \\
\theta_{(N+2)1} & \cdots & \theta_{(N+2)i} & \cdots & \theta_{(N+2)p} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\theta_{2Ni1} & \cdots & \theta_{2Ni} & \cdots & \theta_{2Nip}
\end{bmatrix}
\tag{9}
\]

4. From each set of parameters samples of the matrix \( U_N \times p \) and \( W'_N \times p \), generate a matrix of input variables \( X_m = (X_{1m}, X_{2m}, \ldots, X_{Nm}) \) and \( X'_m = (X'_{1m}, X'_{2m}, \ldots, X'_{Nm}) \) \((m = 1, 2, \ldots, M)\) according to the conditional PDF \( f_X(X|\theta^*) \). Then we can get the output \( Y = (Y_1, Y_2, \ldots, Y_M) \) and \( Y' = (Y'_1, Y'_2, \ldots, Y'_M) \), and the corresponding failure probability of the model can be obtained conveniently. Repeating this procedure \( N \) times, we can obtain two vectors of failure probability of dimension \( N \times 1 \), which can be noted as:

\[
P^U_f = \psi(U_N \times p) \quad P^{W'}_f = \psi(W'_N \times p).
\tag{10}
\]

5. The IM \( S_0 \) is hence computed based on the obtained failure probability vectors:

\[
S_0 = \frac{P^U_f P^{W'}_f - \psi_0^2}{(1/N) \sum_{j=1}^N (P^U_f(j) - \psi_f(j))} = \frac{(1/N) \sum_{j=1}^N (P^W_f(j) - \psi_f(j))^2}{(1/N) \sum_{j=1}^N (P^U_f(j))^2 - \psi_0^2}
\tag{11}
\]

where \( \psi_0 = 1/N \sum_{j=1}^N P^U_f(j) \) is the expectation of failure probability.

Compared with the crude Monte Carlo method, which needs a total computational cost of \( M \times M \times N \times p \) \((p \) is the dimensionality of parameters\) runs of performance function evaluations, Sobol’s method needs \( (M \times N + M \times N) \times p \) performance function evaluations only and the accuracy of Sobol’s method has been validated in Refs. \([24,25]\). Thus, solutions of Sobol’s method can be used as the standard ones to validate the efficiency of other methods.

3.2. Moving least squares method

As mentioned in Section 2.1, the relationship between failure probability and epistemic parameters can be approximated by meta–modeling method \([26,27]\). Due to the strong flexibility of moving least squares approximation in nonlinear models, the MLS model is employed to fit the relationship between failure probability and epistemic parameters, and a novel MLS based method is proposed. This MLS based method normally calculates the proposed variance based IM with only a few thousand samples and is independent of the dimensionality of parameters.

3.3. Sampling strategy

The MLS approximation needs to know the information of a group of observation points (namely training points). Traditional least squares method treats all training points equally and it can be classified as global fitting method. But this method cannot reflect the highly nonlinear characteristic of a model. However, MLS method is a segmented local fitting method which can capture the severe change of a model \([18]\). By defining an affected domain at a test point, the weighted average of the training points located into the affected domain can be used to approximate the model value at the test point. It is noticed that if the training points can reflect the whole information of the parameters space, the approximation accuracy will be good enough. But this generally requires a great number of samples which will be computationally expensive. Thus the choice of training points is vital to the MLS approximation. If the traditional random sampling method is employed to obtain the training points, in order to achieve sufficient information about the parameter space to ensure the approximation accuracy, there always needs a large size of samples. To avoid that, low discrepancy sampling method, i.e. the Halton sequence \([28]\) is employed to obtain the training points, because this technique generates more uniform samples than random sampling method and has enjoyed increasing popularity in reliability analysis.

From the low discrepancy sequence proposed in Ref. \([28]\), a group of epistemic parameters samples \((\theta_1, \theta_2, \ldots, \theta_{N_T})\) can be obtained. Then the training failure probability samples can be obtained by a “double-loop” low discrepancy sampling procedure with an “inner-loop” for the parameters \( \theta \) and an “outer-loop” for the inputs \( X \), which can be represented as:

\[
P^T_f = \begin{bmatrix}
P_1 \\
P_2 \\
\vdots \\
P_{N_T}
\end{bmatrix} = \begin{bmatrix}
\psi(\theta_{11}, \theta_{21}, \ldots, \theta_{p1}) \\
\psi(\theta_{12}, \theta_{22}, \ldots, \theta_{p2}) \\
\vdots \\
\psi(\theta_{1N_T}, \theta_{2N_T}, \ldots, \theta_{pN_T})
\end{bmatrix}
\tag{12}
\]

where \( N_T \) is the number of training points.

From the training epistemic parameters and failure probability, MLS method can be used to fit the relationship between parameters and failure probability.
3.3.1. Basic theory

Denote the relationship between parameters and failure probability as an implicit function \( \psi(\theta) \), on a local affected region of parameters \( \theta \). The implicit function \( \psi(\theta) \) can be represented as:

\[
\psi(\theta) = \sum_{l=1}^{K} \alpha_l(\theta)q_l(\theta) = q^T(\theta)\alpha(\theta)
\]  

(13)

where \( \alpha(\theta) = [\alpha_1(\theta), \alpha_2(\theta), \ldots, \alpha_K(\theta)]^T \) are the coefficient vectors, which are the function of parameters vector \( \theta \). The key of the MLS approximation is to obtain this coefficient vector. In addition, \( q^T(\theta) = [q_1(\theta), q_2(\theta), \ldots, q_K(\theta)]^T \) are the basic function vectors and \( K \) is the dimensionality of them. Basic function can generally be taken as \( k \)-order polynomial which is commonly chosen as \[ 19,26 \]:

- **Linear**: \( q(\theta) = [1, \theta_1, \theta_2, \ldots, \theta_p]^T \)
- **Quadratic**: \( q(\theta) = [1, \theta_1, \theta_2, \ldots, \theta_p, \theta_1^2, \theta_2^2, \ldots, \theta_p^2]^T \)  

(14)

In order to obtain the coefficient vector \( \alpha(\theta) \), the error of the MLS approximation must to be minimized, which can be represented by the following weighted discrete paradigm:

\[
J = \sum_{l=1}^{N'} w(\theta - \theta_l)(\psi(\theta_l) - P_{l1})^2 = \sum_{l=1}^{N'} w(\theta - \theta_l)[q^T(\theta_l)\alpha(\theta_l) - P_{l1}]^2
\]  

(15)

where \( N'(N' < N_f) \) is the number of training points located into the affected region of parameters vector \( \theta_l \), \( P_{l1} \) is the training failure probability at point \( \theta_l \) and \( w(\theta - \theta_l) \) is the corresponding weighted function, which can be used to measure the influence of the training points on the value of fitting function. According to the analysis in Ref. [18], the weighted function can be taken as the following spline function:

\[
w(s) = \begin{cases} 
\frac{2}{3} - 4s^2 + 4s^3 & (s \leq \frac{1}{2}) \\
\frac{4}{3} - 4s + 4s^2 - \frac{4}{3}s^3 & \left( \frac{1}{2} < s \leq 1 \right) \\
0 & (s > 1)
\end{cases}
\]  

(16)

where \( s = \|\theta - \theta_l\|_2 / s_{\text{max}}, s_{\text{max}} \) is the maximum diameter of affected region at point \( \theta_l \).

The weighted function plays a crucial role in the MLS approximation, which is used to measure the influences of the test points on the value of the fitting function at the training point. Generally, the weighted function is positive in the affected domain (often radial) and zero out of the affected domain, and its value increases with the distance from the center. As presented in many Refs. [29,30], the spline functions, i.e. Eq. (16), are chosen to be the candidate of the weighted functions, which satisfy the properties of the weighted function. The maximum diameter \( s_{\text{max}} \) of the affected domain is another important factor for the goodness MLS approximation. If the size of affected domain is large, the accuracy will be good but the computational cost will be expensive. Since the scattered points are generally dense around the center and sparse in the peripheral domain, we can use the adaptive \( s_{\text{max}} \) to achieve the good approximation [31]. Around the center, the size of affected domain can be small due to the abundance of test point, while in the peripheral domain the size can be large due to the sparsity of the test point.

When the derivative of the paradigm in Eq. (15) is zero, the error of the moving least squares approximation is minimized:

\[
\frac{\partial J}{\partial \alpha} = A(\theta)\alpha(\theta) - B(\theta)P_f \approx 0
\]  

(17)

and the corresponding coefficient vector can be represented as:

\[
\alpha(\theta) = A(\theta)^{-1}B(\theta)P_f
\]  

(18)

where

\[
A(\theta) = \sum_{l=1}^{N'} w(\theta - \theta_l)q(\theta_l)q^T(\theta_l)
\]  

(19)

\[
B(\theta) = [w(\theta - \theta_1)q_1(\theta), w(\theta - \theta_1)q_2(\theta), \ldots, w(\theta - \theta_1)q_p(\theta)]
\]  

(20)

\[
P_f = [P_{f1}, P_{f2}, \ldots, P_{fn}]^T.
\]  

(21)

Substitute Eq. (18) into Eq. (13) and the implicit function \( \psi(\theta) \) can be rewritten as:

\[
\psi(\theta) = \sum_{l=1}^{N'} \xi_l^k(\theta)P_{l1} = \xi^k(\theta)P_f
\]  

(22)
where $\zeta^k(\theta)$ is the $k$-order shape function of parameter $\theta$, which can be represented as:

$$\zeta^k(\theta) = [\zeta_1^k, \zeta_2^k, \ldots, \zeta_N^k] = q^T(\theta)A(\theta)^{-1}B(\theta).$$  

(23)

It is noticed that Eq. (22) is the relationship between parameters and failure probability approximated by the MLS method. If a test point $\theta$ is given, it can be substituted into Eq. (22) to evaluate the corresponding failure probability conveniently.

3.3.2. Solution for the proposed measure

As mentioned above, the difficulty of calculating the proposed IM is to calculate the variance of the expectation of failure probability $\text{Var}[E[P_f|\theta_i]]$. This section employs the MLS method to approximate the conditional expectation of failure probability $E[P_f|\theta]$. According to the definition of the proposed IM, the total variance needs to be calculated first. Sampling a group of test points $(\theta_1, \theta_2, \ldots, \theta_p)(N > N_T)$ from the marginal PDF $f_{\theta_i}(\theta_i) (i = 1, 2, \ldots, p)$, the corresponding test failure probability can be calculated by a “double-loop” sampling procedure, which can be represented as:

$$\mathbf{P}_f = \begin{pmatrix} P_{f1} \\ P_{f2} \\ \vdots \\ P_{fN_T} \end{pmatrix} = \begin{pmatrix} \psi(\theta_{11}, \theta_{21}, \ldots, \theta_{p1}) \\ \psi(\theta_{12}, \theta_{22}, \ldots, \theta_{p2}) \\ \vdots \\ \psi(\theta_{1N_T}, \theta_{2N_T}, \ldots, \theta_{pN_T}) \end{pmatrix}.$$  

(24)

Then the total variance of failure probability can be calculated by Eq. (25):

$$\text{Var}(P_f) = \frac{1}{N-1} \sum_{j=1}^{N} (P_{fj} - \bar{P}_f)^2$$  

(25)

where $\bar{P}_f$ is the mean value of failure probability $P_f$.

In order to obtain the conditional expectation $E[P_f|\theta_i]$, the MLS method is employed to fit the relationship between the failure probability and the single parameter $\theta_i$ which can be represented as:

$$\tilde{P}_f^{(i)} = \zeta^k(\theta_i)P_f^{(0)} + e$$  

(26)

where $E[P_f|\theta_i]$ is denoted as $\tilde{P}_f^{(i)}$. $P_f^{(0)}$ is the failure probability of the training point located into the affected region of $\theta_i$ and $e$ is the approximate error.

Eq. (26) means that the weighted average of $P_f^{(0)}$ can be used to estimate the conditional expectation $E[P_f|\theta]$. Putting the samples of the $i$th parameter $(\theta_{1i}, \theta_{2i}, \ldots, \theta_{Ni}) (i = 1, 2, \ldots, p)$ into Eq. (26), the samples of conditional expectation of failure probability $(\tilde{P}_f^{(1)}, \tilde{P}_f^{(2)}, \ldots, \tilde{P}_f^{(p)})$ can be calculated, and then the corresponding variance can be represented as:

$$\text{Var}(E[P_f|\theta_i]) = \frac{1}{N-1} \sum_{j=1}^{N} (\tilde{P}_f^{(j)} - \bar{\tilde{P}}_f^{(i)})^2$$  

(27)

where $\bar{\tilde{P}}_f^{(i)}$ is the mean value of $\tilde{P}_f^{(i)}$.

According to the definition of the variance based IM of failure probability, Eqs. (25) and (27) can be used to estimate the IM $S_{\theta_i}$ conveniently. It can be seen that the total computational cost of MLS based method rests with the cost of the training failure probability and the test failure probability, which needs $M \times (N_T + N)$ performance function evaluations. In addition, the MLS based method is independent of the dimensionality of epistemic parameters. Compared with Sobol’s method which needs a computational cost of $(M \times N + M \times N) \times p$ performance function evaluations, the proposed method is much more efficient, let alone the crude Monte Carlo method.

4. Examples

In this section, both numerical and engineering examples are used to demonstrate the reasonability of the proposed variance based IM and the efficiency of the proposed MLS based method. To calculate the proposed IM, Sobol’s method and MLS based method are employed. Results of the proposed IM and the number of performance function evaluation (NPFE) are presented for comparison.

4.1. Ishigami test example

Ishigami function [32] is a commonly used example in importance measure analysis. It can be written as follows.

$$Y = \sin x_1 + a \sin^2 x_2 + bx_3^4 \sin x_1$$  

(28)
Table 1
Sensitivity results of the numerical example.

<table>
<thead>
<tr>
<th>Method</th>
<th>$S_{\mu_1}$</th>
<th>$S_{\mu_2}$</th>
<th>$S_{\mu_3}$</th>
<th>NPFE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sobol</td>
<td>0.5914</td>
<td>0.0040</td>
<td>0.3839</td>
<td>$50000 \times 50000 \times 2 \times 3$</td>
</tr>
<tr>
<td>MLS</td>
<td>0.5941</td>
<td>0.0004</td>
<td>0.3920</td>
<td>$5000 \times (5000 + 1000)$</td>
</tr>
</tbody>
</table>

Fig. 1. The curves of the importance measure $S_{\mu_i}$ with the increase of the number of failure samples computed by the MLS method.

where variables $x_1$, $x_2$ and $x_3$ are independent and satisfy the normal distribution $N(0, \pi/\sqrt{3})$. The values of constants $a$ and $b$ are 7 and 0.1, respectively. It is assumed that the means of inputs are subject to the epistemic uncertainty and satisfy the standard normal distribution. Sensitivity results of the Ishigami test example are listed in Table 1. Figs. 1 and 2 give the curves of the results computed by MLS method and Sobol’s method with the increase of the number of failure samples, respectively.

As revealed by Table 1, the results of Sobol’s method and MLS based method are almost the same and the ranking is $S_{\mu_1} > S_{\mu_3} > S_{\mu_2}$. For the Sobol’s method, 50,000 samples of parameters are generated in the “outer-loop” and 50,000 samples of variables with given distribution parameters are generated in the “inner-loop”. While for the MLS based method, 5000 and 1000 samples of parameters are generated for the test points and training points respectively in the “outer-loop”, and 5000 samples of variables are generated in the “inner-loop”. It can be seen that the computational cost of MLS method is much less than the Sobol’s method, which demonstrates the high efficiency of the MLS method.

It can be seen from Figs. 1 and 2 that results of the MLS method begin to get convergence at the level of 5000 points, while results of the Sobol’ method begin to get convergence at the level of 50,000 points. This illustrates that the MLS method has a fast rate of convergence. The reason is that the sample size of estimating the input/output probability relationship (global sensitivity) is larger than that of estimating the input/output functional relationship expressed by meta-modeling. Thus, the MLS method employed in our paper can improve the computational efficiency for the challenging model of computing the variance based sensitivity measure of failure probability.

4.2. Roof truss

A roof truss is shown in Fig. 3. The top boom and the compression bars are reinforced by concrete, and the bottom boom and the tension bars are steel. Assume that a uniformly distributed load $q$ is applied on the roof truss, and the uniformly distributed load can be transformed into the nodal load $P = ql/4$. Taking safety and applicability into account, the perpendicular deflection $\Delta_C$ of the peak of structure node C not exceeding 2.8 cm is taken as the constraint condition. The performance response function can be constructed by $g(x) = 0.028 - \Delta_C$, where $\Delta_C$ is a function of the basic random variables, $\Delta_C = \frac{ql^2}{2} \left( \frac{3.81}{A_C E_C} + \frac{1.13}{A_S E_S} \right)$, where $A_C$, $A_S$, $E_C$, $E_S$ and $l$, respectively, are sectional area, elastic modulus and length of the concrete and steel bars. The distribution parameters (mean and the ratio of standard deviation and mean) of these independent normal basic random variables are listed in Table 2. It is assumed that the mean of the variables is uncertain and satisfies another normal distribution with parameters listed in Table 3. The results for the roof truss are listed in Table 4 and a histogram of the results is given in Fig. 4 to illustrate the ranking conveniently.
Fig. 2. The curves of the importance measure $S_{\mu_i}$ with the increase of the number of failure samples computed by Sobol’s method.

Table 2
Distribution parameters of the input variables of a roof truss.

<table>
<thead>
<tr>
<th>Random variables</th>
<th>$q$ (N/m)</th>
<th>$l$ (m)</th>
<th>$A_S$ (m$^2$)</th>
<th>$A_C$ (m$^2$)</th>
<th>$E_S$ (N/m$^2$)</th>
<th>$E_C$ (N/m$^2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Code of random variable $x$</td>
<td>$x_1$</td>
<td>$x_2$</td>
<td>$x_3$</td>
<td>$x_4$</td>
<td>$x_5$</td>
<td>$x_6$</td>
</tr>
<tr>
<td>Mean $\mu_x$</td>
<td>$\mu_{x_1}$</td>
<td>$\mu_{x_2}$</td>
<td>$\mu_{x_3}$</td>
<td>$\mu_{x_4}$</td>
<td>$\mu_{x_5}$</td>
<td>$\mu_{x_6}$</td>
</tr>
<tr>
<td>Coefficient of variation $cov_x$</td>
<td>0.07</td>
<td>0.01</td>
<td>0.06</td>
<td>0.12</td>
<td>0.06</td>
<td>0.06</td>
</tr>
</tbody>
</table>

Table 3
Distribution parameters of mean.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$\mu_{x_1}$</th>
<th>$\mu_{x_2}$</th>
<th>$\mu_{x_3}$</th>
<th>$\mu_{x_4}$</th>
<th>$\mu_{x_5}$</th>
<th>$\mu_{x_6}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>20,000</td>
<td>12</td>
<td>$9.82 \times 10^{-4}$</td>
<td>$1 \times 10^{14}$</td>
<td>$2 \times 10^{10}$</td>
<td></td>
</tr>
<tr>
<td>Coefficient of variation $cov_{\mu}$</td>
<td>0.01</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 4 
Sensitivity results of a roof truss.

<table>
<thead>
<tr>
<th>Method</th>
<th>$S_{\mu_1}$</th>
<th>$S_{\mu_2}$</th>
<th>$S_{\mu_3}$</th>
<th>$S_{\mu_4}$</th>
<th>$S_{\mu_5}$</th>
<th>NPFE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sobol</td>
<td>0.1491</td>
<td>0.6117</td>
<td>0.0798</td>
<td>0.0095</td>
<td>0.0105</td>
<td>($10^5 \times 10^5 + 10^4 \times 10^5$) × 6</td>
</tr>
<tr>
<td>MLS</td>
<td>0.1431</td>
<td>0.6263</td>
<td>0.0837</td>
<td>0.0168</td>
<td>0.0709</td>
<td>10^4 × (10^5 + 10^4)</td>
</tr>
</tbody>
</table>

Fig. 4. Histogram of the sensitivity results for a roof truss.

As revealed by Table 4 and Fig. 4, for this engineering example, the results of MLS based method are in good agreement with Sobol’s method except that the least influential ones are lacking accuracy, namely $S_{\mu_4}$ and $S_{\mu_6}$, but this does not matter in identifying the importance ranking of parameters. It can be noticed that with the increase of the dimensionality, the MLS based method have the advantage over Sobol’s method on the computational cost.

It can be seen that the mean of length $l$ is the most influential parameter, the means of load $q$, sectional area $A_C$ and elastic modulus $E_C$ are less influential ones, whereas, the means of sectional area $A_S$ and elastic modulus $E_S$ are the least influential ones which can attract less attention. Thus, in the design and optimization of the roof truss, one need to pay more attention to collecting the information and improving the understanding of those important epistemic parameters to decrease their uncertainties, especially to the mean of the length $l$, then the failure uncertainty of the roof truss can be reduced to a maximum extent. Additionally, with the ranking of the epistemic parameters, one can neglect the epistemic parameters with low importance to reduce the dimensionality and simplify the analysis.

4.3. Riveting process

In aircraft industry, sheet metal parts are widely used and the most common method of assembling them is through riveting [33]. There are many factors associated with a riveting process that directly affect the quality of rivets, and one main factor is the squeeze stress. If the squeeze stress is too high, it may induce failure of the rivet. Hence, controlling the squeeze stress is of great significance for the safety of aircrafts.

The true riveting process is very complex. In this paper, we take the headless rivet for example and simply divide the riveting process into two stages as explained in Fig. 5. In stage I, the rivet is punched from state $A$ (the initial state of the rivet before impact, without any deformation) to state $B$ (an intermediate state of the rivet when the clearance between rivet and hole is zero), then in stage II the rivet is further punched from state $B$ to state $C$ (the final state of the rivet after impact, with rivet heads formed). Throughout the riveting process we assume that the hole diameter is not changed.

In order to establish a mathematical relationship between the squeeze stress and the geometric dimensions of a rivet, we need to assume some ideal conditions as follows.

- The hole is not enlarged in the riveting process.
- The change of the rivet volume can be neglected.
- After impact, the rivet driven head has a cylindrical shape.
- The material of the rivet is isotropic.
The initial volume $V_{0}$ of the rivet before impact (as shown in Fig. 4(a)) is given by:

$$V_{0} = \frac{\pi}{4} d^{2} h$$  \hspace{1cm} (29)$$

where $d$ and $h$ are the rivet diameter and rivet length in state $A$, respectively.

At the end of stage I, the volume $V_{1}$ of the rivet in state $B$ can be calculated as follows,

$$V_{1} = \frac{\pi}{4} D_{0}^{2} h_{1}$$  \hspace{1cm} (30)$$

where $D_{0}$ and $h_{1}$ are the rivet diameter and rivet length in state $B$, respectively.

After stage II, we assume the top and bottom heads of the formed rivet in state $C$ have the same dimensions, then the volume $V_{2}$ of the rivet in state $C$ can be written as:

$$V_{2} = \frac{\pi}{4} D_{0}^{2} t + 2 \times \frac{\pi}{4} D_{1}^{2} H$$  \hspace{1cm} (31)$$

where $t$ is the whole thickness of the two sheets, parameters $D_{1}$ and $H$ are the diameter and the height of the driven rivet head in state $C$, respectively.

According to the power hardening theory, the maximum squeeze stress $\sigma_{\text{max}}$ in the $y$-direction can be expressed as:

$$\sigma_{\text{max}} = K (\varepsilon_{y})_{\text{RSHE}}$$  \hspace{1cm} (32)$$

where $K$ is the strength coefficient, parameter $\eta_{\text{RSHE}}$ is the strain hardening exponent of the rivet material, and $\varepsilon_{y}$ is the true strain in the $y$-direction of the rivet head in its formation. In our model, the true strain $\varepsilon_{y}$ is composed of two parts: the strain $\varepsilon_{y1}$ caused in stage I and the strain $\varepsilon_{y2}$ caused in stage II, then the true strain $\varepsilon_{y}$ can be expressed as follows:

$$\varepsilon_{y} = \varepsilon_{y1} + \varepsilon_{y2}$$  \hspace{1cm} (33)$$

where $\varepsilon_{y1} = \ln \frac{h}{h_{1}}$ and $\varepsilon_{y2} = \ln \frac{h_{1} - t}{2H}$.

Combining Eqs. (29)–(33) and assuming that the volume change of the rivet can be neglected, one can obtain the maximum squeeze stress for a certain riveting process as follows:

$$\sigma_{\text{max}} = K \left( \ln \frac{d^{2} h - D_{0}^{2} t}{2Hd^{2}} \right)_{\text{RSHE}}$$  \hspace{1cm} (34)$$

In this paper, the material of the rivet is aluminum 2017-T4 and its strain hardening exponent is $\eta_{\text{RSHE}} = 0.15$. In State $C$, the top and bottom head of the rivet must be left a certain height to avoid damaging the sheet metal parts, so $H = 2.2$ mm. According to the material manual, the compressive strength of the given material is $\sigma_{\text{sq}} = 565$ MPa. If the maximum squeeze stress exceeds the squeeze strength, the rivet will fail, then, the performance function can be represented as:

$$g = \sigma_{\text{sq}} - \sigma_{\text{max}}$$  \hspace{1cm} (35)$$

From stage I to stage II, some factors are random variables and distributed as independent normal distribution with parameters listed in Table 5. It is assumed that the mean of variables is uncertain and distributed as another normal distribution with parameters listed in Table 6. Then, we can analyze the effect of the mean on the failure probability of rivet. The results are listed in Table 7 and a histogram of the results is given in Fig. 6 to illustrate the ranking conveniently.

As revealed by Table 7 and Fig. 6, the mean of strength coefficient $K$ is the most influential parameter on the failure probability in the process of riveting. The mean of diameter $d$ in State $A$ and $D_{0}$ in State $B$ is less influential parameter and the effect is almost the same. Because the volume of the headless rivet is assumed to be constant, if the variables $d$ and $D_{0}$ are increased, but the diameter $D_{1}$ in State $C$ is constant, the contact face in $y$-direction will be decreased which can induce large compressive stress and lead to failure of the rivet. Thus, the effect of the mean of $d$ and $D_{0}$ cannot be neglected and a proper match of rivet and sheet metal parts is important for safety. The means of variables $h$ and $t$ are the least influential parameters because the squeeze happens in the $y$-direction and the contact surface in $x$-direction will be the main effective factor. However, because the volume of the headless rivet is assumed to be constant, the influence of the means of variables $h$ and $t$ cannot be neglected.
Table 5  
Distribution parameters of the input variables of the headless rivet.

<table>
<thead>
<tr>
<th>Random variables</th>
<th>(d) (mm)</th>
<th>(h) (mm)</th>
<th>(K) (MPa)</th>
<th>(D_0) (mm)</th>
<th>(t) (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Code of random variable (x)</td>
<td>(x_1)</td>
<td>(x_2)</td>
<td>(x_3)</td>
<td>(x_4)</td>
<td>(x_5)</td>
</tr>
<tr>
<td>Mean (\mu_x)</td>
<td>(\mu_{x_1})</td>
<td>(\mu_{x_2})</td>
<td>(\mu_{x_3})</td>
<td>(\mu_{x_4})</td>
<td>(\mu_{x_5})</td>
</tr>
<tr>
<td>Coefficient of variation (\text{cov}_x)</td>
<td>0.01</td>
<td>0.015</td>
<td>0.01</td>
<td>0.002</td>
<td>0.002</td>
</tr>
</tbody>
</table>

Table 6  
Distribution parameters of the mean.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>(\mu_{x_1})</th>
<th>(\mu_{x_2})</th>
<th>(\mu_{x_3})</th>
<th>(\mu_{x_4})</th>
<th>(\mu_{x_5})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>5</td>
<td>20</td>
<td>547.2</td>
<td>5.1</td>
<td>5</td>
</tr>
<tr>
<td>Coefficient of variation (\text{cov}_\mu)</td>
<td>0.05</td>
<td>0.01</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
</tr>
</tbody>
</table>

Table 7  
Sensitivity results of the headless rivet.

<table>
<thead>
<tr>
<th>Method</th>
<th>(S_{\mu_{x_1}})</th>
<th>(S_{\mu_{x_2}})</th>
<th>(S_{\mu_{x_3}})</th>
<th>(S_{\mu_{x_4}})</th>
<th>(S_{\mu_{x_5}})</th>
<th>NPFE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sobol</td>
<td>0.1346</td>
<td>0.0291</td>
<td>0.6504</td>
<td>0.1331</td>
<td>0.0319</td>
<td>((10^4 \times 10^5 + 10^4 \times 10^5) \times 5)</td>
</tr>
<tr>
<td>MLS</td>
<td>0.1168</td>
<td>0.0229</td>
<td>0.6429</td>
<td>0.1262</td>
<td>0.0282</td>
<td>(10^4 \times (10^5 + 10^4))</td>
</tr>
</tbody>
</table>

Fig. 6. Histogram of the sensitivity results for the headless rivet.

5. Conclusions

For structural systems with both the epistemic and aleatory uncertainties, the effect of parameters on the failure probability is investigated. In order to identify the more influential parameters, the variance based IM of failure probability is proposed. By collecting the information and improving the understanding of those more influential parameters, the best decision for the failure of the structural system can be made ambitiously. Due to the large computational cost of the proposed IM, a novel MLS based method is employed. By fitting the relationship between parameters and failure probability, this method efficiently deals with nonlinear models regardless of the number of variables. In addition, the MLS based method can avoid the complex sampling procedure to calculate the conditional expectation of failure probability. Compared with Sobol’s method, the proposed MLS based method is much more efficient. The results of a numerical example and engineering examples validate the rationality of the proposed IM and the efficiency of the proposed MLS based method.

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References


