In [1], we established the mathematical programming model:

Minimize : \[ \sum_{k=1}^{p} (\hat{\lambda}(t_k))^2 \]

Subject to : \[ \sum_{k=1}^{p} (p-k)\hat{\lambda}(t_k) = \bar{I} \]
\[ \sum_{k=1}^{p} \hat{\lambda}(t_k) = 1, \hat{\lambda}(t_k) \geq 0, \quad k = 1, 2, \ldots, p. \]

By solving the model, we got
\[ \hat{\lambda}(t_k) = (12I - 6k + 6(p-k) + 4(p-1)^2 - 6\bar{I}(p-1)) / p\bar{I}(p-1), \quad k = 1, 2, \ldots, p. \] (1)

However, there exists an error in (1), in the following we will improve it and give a detailed solution process.

We first construct the Lagrange function:

\[ L(\hat{\lambda}(t), \eta_1, \eta_2) = \sum_{k=1}^{p} (\hat{\lambda}(t_k))^2 - 2\eta_1 \left( \sum_{k=1}^{p} (p-k)\hat{\lambda}(t_k) - \bar{I} \right) - 2\eta_2 \left( \sum_{k=1}^{p} \hat{\lambda}(t_k) - 1 \right), \] (2)

where \( \hat{\lambda}(t) = (\hat{\lambda}(t_1), \hat{\lambda}(t_2), \ldots, \hat{\lambda}(t_p))^T \). \( \eta_1 \) and \( \eta_2 \) are the Lagrange multipliers.

Differentiating (2) with respect to \( \hat{\lambda}(t_k) (k = 1, 2, \ldots, p) \), \( \eta_1 \) and \( \eta_2 \), and setting these partial derivatives equal to zero, the following set of equations is obtained:

\[ \frac{\partial L(\hat{\lambda}(t), \eta_1, \eta_2)}{\partial \hat{\lambda}(t_k)} = 2\hat{\lambda}(t_k) - 2\eta_1 (p-k) - 2\eta_2 = 0, \] (3)
\[ \frac{\partial L(\hat{\lambda}(t), \eta_1, \eta_2)}{\partial \eta_1} = -2 \left( \sum_{k=1}^{p} (p-k)\hat{\lambda}(t_k) - \bar{I} \right) = 0, \] (4)
\[ \frac{\partial L(\hat{\lambda}(t), \eta_1, \eta_2)}{\partial \eta_2} = -2 \left( \sum_{k=1}^{p} \hat{\lambda}(t_k) - 1 \right) = 0. \] (5)
Simplifying (3)–(5), we have

\[ \lambda(t_k) = (p - k)\eta_1 + \eta_2, \]  

\[ \sum_{k=1}^{p} (p - k)\lambda(t_k) = \bar{r}, \]  

\[ \sum_{k=1}^{p} \lambda(t_k) = 1. \]

Combining (6) and (7), (6) and (8), it follows that

\[ \eta_1 \sum_{k=1}^{p} (p - k)^2 + \eta_2 \sum_{k=1}^{p} (p - k) = \bar{r}, \]  

\[ \eta_1 \sum_{k=1}^{p} (p - k) + \eta_2 p = 1. \]

Since

\[ \sum_{k=1}^{p} (p - k)^2 = \frac{1}{6}p(p + 1)(2p + 1) - p^2 \]

and

\[ \sum_{k=1}^{p} (p - k) = \frac{1}{2}p(p - 1), \]

then by solving (9) and (10), we have

\[ \eta_1 = \frac{12\bar{r} - 6(p - 1)}{p(p - 1)(p + 1)}, \]  

\[ \eta_2 = \frac{4p - 2 - 6\bar{r}}{p(p + 1)}. \]

and thus, by (6), it yields

\[ \lambda(t_k) = \frac{(12\bar{r} - 6p + 6)(p - k) + (4p - 2 - 6\bar{r})(p - 1)}{p(p - 1)(p + 1)}, \quad k = 1, 2, \ldots, p. \]

Since \( \lambda(t_k) \geq 0 \) for all \( k \), then

\[ \frac{(12\bar{r} - 6p + 6)(p - k) + (4p - 2 - 6\bar{r})(p - 1)}{p(p - 1)(p + 1)} \geq 0, \quad k = 1, 2, \ldots, p, \]

i.e.,

\[ (3p - 6k + 3)\bar{r} \geq (p - 1)(p - 3k + 1), \quad k = 1, 2, \ldots, p, \]

thus,

(i) If \( 3p - 6k + 3 = 0 \), i.e., \( k = \frac{3p-1}{2} \), then (17) holds, for all \( \bar{r} \).

(ii) If \( 3p - 6k + 3 > 0 \), i.e., \( k < \frac{3p-1}{2} \), then (17) holds, for \( \bar{r} \geq \frac{p-2}{3} \).

(iii) If \( 3p - 6k + 3 < 0 \), i.e., \( k > \frac{3p-1}{2} \), then (17) holds, for \( \bar{r} \leq \frac{2p-1}{3} \).

Therefore, we can obtain the weights \( \lambda(t_k) \) \( (k = 1, 2, \ldots, p) \) by using (15) with the following condition:

\[ \frac{p - 2}{3} \leq \bar{r} \leq \frac{2p - 1}{3}. \]

If let

\[ g(x) = \frac{(12\bar{r} - 6p + 6)(p - x) + (4p - 2 - 6\bar{r})(p - 1)}{p(p - 1)(p + 1)}, \]

then

\[ g'(x) = -\frac{12\bar{r} - 6p + 6}{p(p - 1)(p + 1)}. \]
thus,

(i) If \( \frac{p}{2} < t < \frac{p}{2} \), then \( g'(x) > 0 \), i.e., \( g(x) \) is a strictly monotonic increasing function.

(ii) If \( t = \frac{p}{2} \), then \( g'(x) = 0 \), i.e., \( g(x) \) is a constant function.

(iii) If \( \frac{p-1}{2} < t < \frac{p-1}{2} \), then \( g'(x) < 0 \), i.e., \( g(x) \) is a strictly monotonic decreasing function.

Therefore, by (15), we have

(i) If \( \frac{p}{2} < t < \frac{p}{2} \), then \( k(t_k) > k(t_k) \), \( k = 1, 2, \ldots, p - 1 \), i.e., the sequence \( \{k(t_k)\} \) is a monotonic increasing sequence. Also since

\[
\frac{\lambda(t_{k+1}) - \lambda(t_k)}{p(p-1)(p+1)} = \frac{(12t - 6p + 6)(p - (k + 1)) + (4p - 2 - 6t)(p - 1)}{p(p-1)(p+1)}
\]

\[
= \frac{-(12 - 6p + 6)}{p(p-1)(p+1)} > 0, \quad k = 1, 2, \ldots, p - 1,
\]

then the sequence \( \{\lambda(t_k)\} \) is an increasing arithmetic sequence.

(ii) If \( t = \frac{p}{2} \), then

\[
\frac{\lambda(t_k) - \lambda(t_k)}{p(p-1)(p+1)} = \frac{1}{p}, \quad k = 1, 2, \ldots, p.
\]

thus \( \lambda(t) = (1/p, 1/p, \ldots, 1/p)^T \).

(iii) If \( \frac{p-1}{2} < t < \frac{p-1}{2} \), then \( k(t_{k+1}) < k(t_k) \), \( k = 1, 2, \ldots, p - 1 \), i.e., the sequence \( \{k(t_k)\} \) is a monotonic decreasing sequence. Similar to (21), we have

\[
\frac{\lambda(t_{k+1}) - \lambda(t_k)}{p(p-1)(p+1)} < 0, \quad k = 1, 2, \ldots, p - 1,
\]

thus the sequence \( \{\lambda(t_k)\} \) is a decreasing arithmetic sequence.

Reference