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WEAK INJECTIVE AND WEAK FLAT MODULES

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Let R be a ring. A left R-module M (resp., right R-module N) is called weak injective (resp., weak flat) if $\mathrm{Ext}_R^1(F, M) = 0$ (resp., $\mathrm{Tor}_1^R(N, F) = 0$) for every super finitely presented left R-module F. By replacing finitely presented modules by super finitely presented modules, we may generalize many results of a homological nature from coherent rings to arbitrary rings. Some examples are given to show that weak injective (resp., weak flat) modules need not be FP-injective (resp., not flat) in general. In addition, we introduce and study the super finitely presented dimension (denote by l.sp.gldim(R)) of R that are defined in terms of only super finitely presented left R-modules. Some known results are extended.

Key Words: Super finitely presented dimension; Super finitely presented module; Weak flat module; Weak flat preenvelope; Weak injective module.

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1. INTRODUCTION

Throughout this paper, R denotes an associative ring with identity and all modules are unitary. For any left R-module M, the character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is denoted by M^+. We use w.gl.dim(R) to stand for the weak global dimension of a ring R. For unexplained concepts and notations, we refer the reader to [1, 8, 19].

It is well known that coherent rings have been characterized in various ways. Many nice properties were obtained for this class of rings. For instance, Chase proved in [3] that a ring R is left coherent if and only if any direct product of flat right R-modules is flat. Stenström showed in [21] that the class of FP-injective modules plays an important role in characterizing coherent rings. Recall a left R-module M is called *FP-injective* if $\mathrm{Ext}_R^1(F, M) = 0$ for any finitely presented left R-module F. Accordingly, the *FP-injective dimension* of M, denoted by $\text{FP-id}_R(M)$, is defined to be the smallest $n \geq 0$ such that $\mathrm{Ext}_R^{n+1}(F, M) = 0$ for all finitely presented left R-modules F (if no such n exists, set $\text{FP-id}_R(M) = \infty$), and $l.\text{FP-dim}(R)$ is

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defined as $\sup\{\text{FP-id}_R(M) \mid M$ is a left R-module$\}$. It was proved in [21, Theorem 3.3] that the equality $\text{w.gl.dim}(R) = l.\text{FP-dim}(R)$ holds when R is a left coherent ring. In particular, a ring R is von Neumann regular if and only if $\text{w.gl.dim}(R) = 0$ if and only if $l.\text{FP-dim}(R) = 0$ ([21, Proposition 3.6]). Enochs in [9, Proposition 5.1] proved that a ring R is left coherent if and only if every right R-module has a flat preenvelope. In [12], Glaz gave a systematic study for commutative coherent rings.

For a non-negative integer n, Costa in [7] introduced the concept of n-coherent rings. Following [7], an R-module M is said to be *n-presented* if it has a finite n-presentation, i.e., there is an exact sequence $F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$, where each F_i is finitely generated and projective. A ring R is called *left n-coherent* in [7] if every n-presented left R-module is $(n+1)$-presented. In [6], Chen and Ding introduced the notion of n-FP-injective and n-flat modules, and they showed that there are many similarities between coherent rings and n-coherent rings. Since then, various generalizations of coherent rings were given (see, for example, [14, 17, 23]).

A natural question is whether there is a reasonable concept such that some results of a homological nature may be generalized from coherent rings to arbitrary rings. In this paper, we find that the notion of super finitely presented modules plays a crucial role in this process. A left R-module F is said to be *super finitely presented* [10] if there exists an exact sequence of left R-modules: $\cdots \to P_n \to \cdots \to P_1 \to P_0 \to F \to 0$, where each P_i is finitely generated and projective. It is clear that every super finitely presented R-module is finitely presented, but the inverse is not true in general. In fact, one easily checks that every finitely presented R-module is super finitely presented if and only if R is a coherent ring. In this paper, we study the weak injectivity and weak flatness and generalize some principal results of coherent rings to arbitrary rings. This paper is organized as follows.

In Section 2, we introduce the concepts of weak injective and weak flat modules and present some of their general properties. An example is given to show that weak injective (resp., weak flat) modules need not be FP-injective (resp., not flat) in general. For any ring R, we prove as follows: (a) the class of weak injective (weak flat) modules is closed under pure submodules; (b) a left R-module M is weak injective if and only if M^+ is weak flat; (c) any direct product of weak flat right R-modules is weak flat; and (d) every right R-module has a weak flat preenvelope.

In Section 3, we shall investigate a homological dimension $l.\text{sp.gldim}(R)$ of a ring R, called left super finitely presented dimension, that is defined in terms of only super finitely presented left R-modules. For this, it is convenient to introduce the notion of weak injective and weak flat dimensions for modules, denoted by $\text{wid}_R(-)$ and $\text{wfd}_R(-)$, respectively. For any ring R, we prove that (Theorem 3.8)

$$\begin{aligned} l.\text{sp.gldim}(R) &= \sup\{\text{pd}_R(M) \mid M \text{ is a super finitely presented left } R\text{-module}\} \\ &= \sup\{\text{wid}_R(M) \mid M \text{ is a left } R\text{-module}\} \\ &= \sup\{\text{wfd}_R(N) \mid N \text{ is a right } R\text{-module}\}. \end{aligned}$$

As applications, some earlier results in [18, 21] are obtained as corollaries if R is left coherent.

2. WEAK INJECTIVE AND WEAK FLAT MODULES

In this section we give a treatment of weak injective and weak flat modules. Some general properties of these modules are discussed, and many known results are developed.

Weak-injective modules were introduced by S. B. Lee in [15]. An R-module D is called weak-injective if $\mathrm{Ext}^1_R(M, D) = 0$ for all R-modules M of weak dimension ≤ 1. It was shown that direct products and summands of weak-injective R-modules are again weak-injective (see [15]). In what follows, from another point of view, we introduce the notion of weak injective and weak flat modules in terms of super finitely presented modules. In our study, the weak injectivity is different from the one in [15].

Definition 2.1. A left R-module M is called *weak injective* if $\mathrm{Ext}^1_R(F, M) = 0$ for any super finitely presented left R-module F. A right R-module N is called *weak flat* if $\mathrm{Tor}^R_1(N, F) = 0$ for any super finitely presented left R-module F.

Remark 2.2.

(1) It is clear that every FP-injective left R-module is weak injective, and every flat right R-module is weak flat. If R is a left coherent ring, then weak injective left R-modules and weak flat right R-modules coincide with FP-injective left R-modules and flat right R-modules, respectively.
(2) A right R-module M is weak flat if and only if M^+ is weak injective by the standard isomorphism: $\mathrm{Ext}^1_R(N, M^+) \cong \mathrm{Tor}^R_1(M, N)^+$ for any left R-module N.

Using Definition 2.1, we immediately get the following results.

Proposition 2.3.

(1) *Let $(M_i)_{i\in I}$ be a family of left R-modules. Then $\prod M_i$ (resp., direct sums $\oplus M_i$) is weak injective if and only if each M_i is weak injective.*
(2) *Let $(N_i)_{i\in I}$ be a family of right R-modules. Then $\oplus N_i$ is weak flat if and only if each N_i is weak flat.*

The following two lemmas are useful in this section.

Lemma 2.4 ([2, Exercise 3, p. 187]). *Let M be an R-module. For a positive integer $n \geq 1$, the following assertions are equivalent:*

(1) *M is n-presented*;
(2) *For every direct system $(N_j)_{j\in J}$ of R-modules over a directed index set J, the canonical homomorphism*

$$\varinjlim \mathrm{Ext}^i_R(M, N_j) \to \mathrm{Ext}^i_R(M, \varinjlim N_j)$$

is bijective for every $i < n$.

Lemma 2.5 ([4, Lemma 1]). *Let I be any index set and $(A_i)_{i\in I}$ be any family of left (right) R-modules.*

(1) $\oplus A_i$ *is a pure submodule of* $\prod A_i$.
(2) *If for each* $i \in I$, B_i *is a pure submodule of* A_i, *then* $\prod B_i$ *is a pure submodule of* $\prod A_i$.

Proposition 2.6. *The class of weak injective left R-modules is closed under direct limit.*

Proof. This result is a direct consequence of Lemma 2.4. □

Proposition 2.7. *Let M be an R-module.*

(1) *If* M_m *is a weak injective* R_m*-module for every prime ideal m of R, then M is weak injective.*
(2) *Suppose that, for a multiplicative set S, any super finitely presented* R_S*-module is a localization of a super finitely presented R-module. Then* M_S *is a weak injective* R_S*-module if M is weak injective.*

Proof. (1) Assume that M_m is a weak injective R_m-module for every prime ideal m of R. Let N be a super finitely presented R-module. Then N_m is a super finitely presented R_m-module. Thus the desired result follows by the following isomorphism:

$$\mathrm{Ext}^1_R(N, M)_m \cong \mathrm{Ext}^1_{R_m}(N_m, M_m) = 0.$$

(2) This is a consequence of the third isomorphism in [12, Theorem 1.3.11].

□

Recall from [16] that, for a ring A and an A-module E, $B = A\alpha E$ is the set of pairs (a, e) with pairwise addition and multiplication given by $(a, e)(a', e') = (aa', ae' + a'e)$. This is called *the trivial extension* of A by E. Now we give an example to show that, in general, weak injective (resp., weak flat) modules need not be FP-injective (resp., not flat).

Example 2.8. Let K be a field and E be a K-vector space with infinite rank. Set $R = K\alpha E$ to be the trivial extension of K by E. Then, there exists a weak injective (resp., weak flat) R-module which is not FP-injective (resp., not flat).

Proof. By [16, Theorem 3.4], it is easy to see that every super finitely presented R-module is projective. Thus every R-module is weak injective (resp., weak flat). Since the ring R is not von Neumann regular, it follows that there exists an R-module which is not FP-injective (resp., not flat). □

Proposition 2.9.

(1) *Every pure submodule of a weak flat right R-module is weak flat.*
(2) *Every pure submodule of a weak injective left R-module is weak injective.*

Proof. (1) Let N be a weak flat right R-module, and let A be a pure submodule of N. There exists a pure exact sequence $0 \to A \to N \to N/A \to 0$, which gives rise to a split exact sequence $0 \to (N/A)^+ \to N^+ \to A^+ \to 0$. By Remark 2.2(2), N^+ is a weak injective left R-module. Since A^+ is isomorphic to a direct summand of N^+, A^+ is weak injective by Proposition 2.3. Therefore, A is weak flat by Remark 2.2(2) again.

(2) Let M_1 be a pure submodule of a weak injective left R-module M. Then there exists a pure exact sequence $0 \to M_1 \to M \to M/M_1 \to 0$. For every super finitely presented left R-module F, we get the exactness of $\mathrm{Hom}_R(F, M) \to \mathrm{Hom}_R(F, M/M_1) \to \mathrm{Ext}^1_R(F, M_1) \to 0$. Since M_1 is a pure submodule of M, $\mathrm{Hom}_R(F, M) \to \mathrm{Hom}_R(F, M/M_1) \to 0$ is exact by [8, Definition 5.3.6]. Thus $\mathrm{Ext}^1_R(F, M_1) = 0$, and hence M_1 is weak injective. □

Theorem 2.10. *A left R-module M is weak injective if and only if M^+ is weak flat.*

Proof. For any super finitely presented left R-module F, there exists an exact sequence $0 \to N \to F_0 \to F \to 0$, where F_0 is finitely generated projective. Then N is super finitely presented by [13, Lemma 2.3]. We consider the natural homomorphism

$$\eta : \mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z}) \otimes_R X \to \mathrm{Hom}(\mathrm{Hom}_R(X, M), \mathbb{Q}/\mathbb{Z}).$$

Then η is an isomorphism when $X = F_0$ or N by [19, Lemma 3.60] since N and F_0 are finitely presented. Now consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{Tor}_1^R(M^+, F) & \longrightarrow & M^+ \otimes_R N & \longrightarrow & M^+ \otimes_R F_0 \\ & \downarrow & & \downarrow \cong & & \downarrow \cong \\ 0 \longrightarrow & \mathrm{Ext}^1_R(F, M)^+ & \longrightarrow & \mathrm{Hom}_R(N, M)^+ & \longrightarrow & \mathrm{Hom}_R(F_0, M)^+ \end{array}$$

Then we have $\mathrm{Tor}_1^R(M^+, F) \cong \mathrm{Ext}^1_R(F, M)^+$, and so the desired result follows. □

Proposition 2.11. *A left R-module M is weak injective if and only if M^{++} is weak injective.*

Proof. Let M be a weak injective left R-module. Then M^+ is weak flat by Theorem 2.10. Thus M^{++} is weak injective by Remark 2.2(2). Conversely, if M^{++} is a weak injective left R-module, then M, as a pure submodule of M^{++} (see [22, Exercise 41, p. 48]), is weak injective by Proposition 2.9. □

Proposition 2.12. *A right R-module M is weak flat if and only if M^{++} is weak flat.*

Proof. If M is a weak flat right R-module, then M^+ is a weak injective left R-module, whence M^{+++} is weak injective by Proposition 2.11. Thus M^{++} is weak flat by Remark 2.2(2). Conversely, if M^{++} is a weak flat right R-module, then M, being a pure submodule of M^{++}, is weak flat by Proposition 2.9. □

Theorem 2.13. *Any direct product of weak flat right R-modules is weak flat.*

Proof. Let $\{M_i\}_{i\in I}$ be a set of weak flat right R-modules. Then $\oplus_{i\in I}M_i$ is weak flat by Proposition 2.3. From [19, Theorem 2.4], one gets the isomorphism $(\oplus_{i\in I}M_i)^+ \cong \prod_{i\in I} M_i^+$, and so $(\prod_{i\in I} M_i^+)^+ \cong (\oplus_{i\in I}M_i)^{++}$ is weak flat by Proposition 2.12. Since each M_i^+ is weak injective, it follows from Proposition 2.3 that $\prod_{i\in I} M_i^+$ is weak injective. But $\oplus_{i\in I}M_i^+$ is a pure submodule of $\prod_{i\in I} M_i^+$ by Lemma 2.5, and so $(\prod_{i\in I} M_i^+)^+ \to (\oplus_{i\in I}M_i^+)^+ \to 0$ splits. Thus $(\oplus_{i\in I}M_i^+)^+$ is weak flat by Theorem 2.10. We note that $\prod_{i\in I} M_i^{++} \cong (\oplus_{i\in I}M_i^+)^+$, it follows that $\prod_{i\in I} M_i^{++}$ is a weak flat right R-module. Since M_i is a pure submodule of M_i^{++}, $\prod_{i\in I} M_i$ is a pure submodule of $\prod_{i\in I} M_i^{++}$ by Lemma 2.4. So the desired result follows by Proposition 2.9. □

Corollary 2.14. *Any direct product of R is a weak flat right R-module.*

Proof. This is a simple consequence of Theorem 2.13. □

It is well known that R is a left coherent ring if and only if every right R-module has a flat preenvelope (see [9, Proposition 5.1]). Now we conclude this section with the following results which are of independent interest.

Theorem 2.15. *Let R be a ring. Then every right R-module has a weak flat preenvelope.*

Proof. Let M be any right R-module. By [8, Lemma 5.3.12], there is an infinite cardinal number $\aleph_\alpha$ such that for any R-homomorphism $f: M \to L$ with L weak flat, there is a pure submodule Q of L such that $Card(Q) \leq \aleph_\alpha$ and $f(M) \subseteq Q$. Note that Q is weak flat by Proposition 2.9(1); it follows that M has a weak flat preenvelope by [8, Proposition 6.2.1] and Theorem 2.13. □

Remark 2.16. By Theorem 2.15 and [5, Lemma 1], we obtain that the class of weak flat right R-modules is closed under direct products.

In what follows, we discuss when every right R-module has a monic weak flat preenvelope and when every right R-module has an epic weak flat (pre)envelope.

Proposition 2.17. *The following statements are equivalent for a ring R:*

(1) *R is weak injective as a left R-module*;
(2) *Every right R-module has a monic weak flat preenvelope*;
(3) *Every injective right R-module is weak flat*;
(4) *Every flat left R-module is weak injective.*

Proof. (1) ⇒ (2) Let M be a right R-module. By Theorem 2.15, M has a weak flat preenvelope $f: M \to F$. Since $({}_RR)^+$ is a cogenerator in the category of right R-modules, there is an exact sequence $0 \to M \to \Pi({}_RR)^+$. Note that $({}_RR)^+$ is a weak flat right R-module by assumption and Theorem 2.10. It follows that $\Pi({}_RR)^+$ is weak flat by Theorem 2.13. Thus f is monic, and so (2) follows.

(2) ⇒ (3) Let I be an injective right R-module. By (2), there exists an exact sequence $0 \to I \to F \to N \to 0$, where $I \to F$ is a weak flat preenvelope with F

weak flat. Then this short exact sequence is split since I is injective. Thus I is weak flat as a direct summand of F by Proposition 2.3. Hence (3) holds.

(3) $\Rightarrow$ (4) Let M be a flat left R-module. Then M^+ is injective by [19, Theorem 3.52], and so M^+ is weak flat by assumption. So M is weak injective by Theorem 2.10, as desired.

(4) $\Rightarrow$ (1) is obvious. □

Proposition 2.18. *The following statements are equivalent for a ring R:*

(1) *Every right R-module has an epic weak flat envelope;*
(2) *Every left R-module has a monic weak injective cover;*
(3) *Every submodule of any weak flat right R-module is weak flat;*
(4) *Every quotient of any weak injective left R-module is weak injective.*

Proof. (1) $\Leftrightarrow$ (3) follows immediately from [5, Theorem 2].

(2) $\Leftrightarrow$ (4) Because the class of weak injective left R-modules is closed under direct sums by Proposition 2.3, this result holds by [11, Proposition 4].

(3) $\Rightarrow$ (4) Let M be a weak injective left R-module and N any submodule of M. There exists an exact sequence $0 \to N \to M \to M/N \to 0$, which induces the exactness of $0 \to (M/N)^+ \to M^+ \to N^+ \to 0$. Since M^+ is weak flat by Theorem 2.10, it follows from (3) that $(M/N)^+$ is weak flat. Thus M/N is weak injective by Theorem 2.10 again.

(4) $\Rightarrow$ (3) Let A be any submodule of a weak flat right R-module B. Then the exactness of $0 \to A \to B \to B/A \to 0$ induces an exact sequence $0 \to (B/A)^+ \to B^+ \to A^+ \to 0$ by [19, Lemma 3.51]. Note that B^+ is weak injective by Remark 2.2(2); we have A^+ is weak injective by assumption. Therefore, A is weak flat by Remark 2.2(2) again. □

3. SUPER FINITELY PRESENTED DIMENSION OF RINGS

The left super finitely presented dimension, $l.\mathrm{sp.gldim}(R)$, of R is defined as $l.\mathrm{sp.gldim}(R) = \sup\{\mathrm{pd}_R(M) \mid M$ is a super finitely presented left R-module$\}$. In this section, we investigate this global dimension of R, and some principal results of [21] are generalized.

We start with the following proposition.

Proposition 3.1.

(1) *If M is a weak injective left R-module, then $\mathrm{Ext}^n_R(F, M) = 0$ for all $n \geq 1$ and all super finitely presented left R-modules F.*
(2) *If N is a weak flat left R-module, then $\mathrm{Tor}^R_n(N, F) = 0$ for all $n \geq 1$ and all super finitely presented left R-modules F.*

Proof. (1) Let N be a weak injective left R-module. For every super finitely presented left R-module F, we have an exact sequence of left R-modules

$$0 \to G \to P_{n-2} \to \cdots \to P_1 \to P_0 \to F \to 0,$$

where each P_i is finitely generated and projective. Then G is super finitely presented by [13, Lemma 2.3], and so $\mathrm{Ext}_R^1(G, M) = 0$. It follows that $\mathrm{Ext}_R^n(F, M) = 0$ for all $n \geq 1$.

(2) Similar to the proof of (1). □

Definition 3.2. For any left R-module M, the *weak injective dimension* of M, denoted by $\mathrm{wid}_R(M)$, is defined to be the smallest $n \geq 0$ such that $\mathrm{Ext}_R^{n+1}(F, M) = 0$ for all super finitely presented left R-modules F. If no such n exists, set $\mathrm{wid}_R(M) = \infty$.

For any right R-module N, the *weak flat dimension* of N, denoted by $\mathrm{wfd}_R(N)$, is defined to be the smallest $n \geq 0$ such that $\mathrm{Tor}^R_{n+1}(N, F) = 0$ for all super finitely presented left R-modules F. If no such n exists, set $\mathrm{wfd}_R(N) = \infty$.

Proposition 3.3. *The following statements are equivalent for a left R-module M:*

(1) $\mathit{wid}_R(M) \leq n$;
(2) $\mathrm{Ext}_R^{n+1}(F, M) = 0$ *for all super finitely presented left R-modules F*;
(3) $\mathrm{Ext}_R^{n+j}(F, M) = 0$ *for all super finitely presented left R-modules F and all $j \geq 1$*;
(4) *There exists an exact sequence* $0 \to M \to E_0 \to E_1 \to \cdots \to E_n \to 0$, *where each E_i is weak injective*;
(5) *If the sequence* $0 \to M \to E_0 \to E_1 \to \cdots \to E_n \to 0$ *is exact with* $E_0, \cdots, E_{n-1}$ *weak injective, then also E_n is weak injective.*

Proof. (3) ⇒ (1) ⇒ (2) are obvious.

(2) ⇒ (3) For any super finitely presented left R-module F, there is a short exact sequence $0 \to N \to P \to F \to 0$, where P is finitely generated projective. Then, by [13, Lemma 2.3], N is super finitely presented, and so the sequence

$$0 = \mathrm{Ext}_R^{n+1}(N, M) \to \mathrm{Ext}_R^{n+2}(F, M) \to \mathrm{Ext}_R^{n+2}(P, M) = 0$$

is exact. Thus $\mathrm{Ext}_R^{n+2}(F, M) = 0$, and (3) follows by induction.

(1) ⇔ (4) is straightforward.

(3) ⇒ (5) Let $0 \to M \to E_0 \to E_1 \to \cdots \to E_{n-1} \to C_n \to 0$ be exact with $E_0, \cdots, E_{n-1}$ weak injective. Let $L_0 = M$, $L_i = \mathrm{Im}(E_{i-1} \to E_i)$ for $1 \leq i \leq n-1$ and $L_n = C_n$. Then

$$0 \to L_i \to E_i \to L_{i+1} \to 0 \quad \text{for } 0 \leq i \leq n-1.$$

are exact. For every super finitely presented left R-module F and for all $j \geq 1$, it follows from (3) that

$$\mathrm{Ext}_R^j(F, C_n) \cong \mathrm{Ext}_R^{j+1}(F, L_{n-1}) \cong \mathrm{Ext}_R^{j+2}(F, L_{n-2}) \cong \cdots \cong \mathrm{Ext}_R^{j+n}(F, M) = 0.$$

Therefore, C_n is weak injective.

(5) $\Rightarrow$ (3) Let $0 \to M \to E_0 \to E_1 \to \cdots$ be an injective resolution of M, and L_n the nth cosyzygy. Then we obtain an exact sequence

$$0 \to M \to E_0 \to E_1 \to \cdots \to E_{n-1} \to L_n \to 0.$$

It follows from (5) that L_n is weak injective. For any super finitely presented left R-module F, we have $\mathrm{Ext}_R^{n+j}(F, M) \cong \mathrm{Ext}_R^{j}(F, L_n) = 0$ for all $j \geq 1$. Thus (3) follows. □

Similar to Proposition 3.3, we have

Proposition 3.4. *The following statements are equivalent for a right R-module N:*

(1) $wfd_R(N) \leq n$;
(2) $\mathrm{Tor}_{n+1}^R(N, F) = 0$ *for all super finitely presented left R-modules F*;
(3) $\mathrm{Tor}_{n+j}^R(N, F) = 0$ *for all super finitely presented left R-modules F and all* $j \geq 1$;
(4) *There is an exact sequence* $0 \to F_n \to \cdots \to F_1 \to F_0 \to N \to 0$, *where each* F_i *is weak flat*;
(5) *If the sequence* $0 \to F_n \to \cdots \to F_1 \to F_0 \to N \to 0$ *is exact with* $F_0, \cdots, F_{n-1}$ *weak flat, then also* F_n *is weak flat.*

Proposition 3.5. *The following statements are equivalent for any* $n \geq 0$:

(1) $pd_R(F) \leq n$ *for all super finitely presented left R-modules F*;
(2) $fd_R(F) \leq n$ *for all super finitely presented left R-modules F*;
(3) $\mathrm{Ext}_R^{n+1}(F, M) = 0$ *for any super finitely presented left R-module F and any left R-module M*;
(4) $\mathrm{Tor}_{n+1}^R(N, F) = 0$ *for any super finitely presented left R-module F and any right R-module N.*

Proof. (1) $\Leftrightarrow$ (2) Since every super finitely presented module is finitely presented, this follows from the fact that a finitely presented module is flat if and only if it is projective.

(1) $\Rightarrow$ (3) and (2) $\Rightarrow$ (4) are obvious.

(3) $\Rightarrow$ (1) follows from [19, Theorem 9.5].

(4) $\Rightarrow$ (2) holds by [19, Theorem 9.13]. □

We now generalize a homological dimension introduced and studied by McRae in [18] defined as $\text{l.f.p.gl.dim}(R) = \sup\{\mathrm{pd}_R(M) \mid M \text{ is a finitely presented left } R\text{-module}\}$.

Definition 3.6. Let R be a ring. Define

$$\text{l.sp.gldim}(R) = \sup\{\mathrm{pd}_R(M) \mid M \text{ is a super finitely presented left } R\text{-module}\}.$$

Remark 3.7.

(1) It is clear that if R is left coherent, then $\text{l.sp.gldim}(R) = \text{l.f.p.gl.dim}(R)$.

(2) In general, $\mathrm{l.sp.gldim}(R) \neq \mathrm{l.f.p.gl.dim}(R)$, as shown by the example in [18, p. 71]. Indeed, Small in [20] gave an example of a ring T which is right coherent but not left coherent and for which $\mathrm{w.gl.dim}(T) = \mathrm{r.f.p.gl.dim}(T) = \mathrm{r.gl.dim}(T) = 1$, and $\mathrm{l.f.p.gl.dim}(T) = \mathrm{l.gl.dim}(T) = 3$. By Theorem 3.8(1), we can conclude that $\mathrm{l.sp.gldim}(T) \leq \mathrm{w.gl.dim}(T) = 1 < \mathrm{l.f.p.gl.dim}(T) = 3$.
(3) Using [18, Proposition 1.1] and Theorem 3.8(1) below, we can conclude that if R is any ring, then $\mathrm{l.sp.gldim}(R) \leq \mathrm{w.gl.dim}(R) \leq \mathrm{l.f.p.gl.dim}(R)$. Also, the left inequality may be strict by Remark 3.11(2).

Theorem 3.8. *Let R be any ring. Then we have as follows*:

(1) *$l.sp.gldim(R) \leq w.gl.dim(R)$, with equality $l.sp.gldim(R) = w.gl.dim(R)$ when R is a left coherent ring*;
(2)

$$\begin{aligned} l.sp.gldim(R) &= sup\{wid_R(M) \mid M \text{ is a left } R\text{-module}\} \\ &= sup\{wfd_R(N) \mid N \text{ is a right } R\text{-module}\}. \end{aligned}$$

Proof. (1) Assume that $\mathrm{w.gl.dim}(R) = n < \infty$. Let M be a super finitely presented left R-module. Then there is an exact sequence of left R-modules

$$0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0,$$

where $P_0, P_1, \cdots, P_{n-1}$ are finitely generated projective and P_n is flat. Since P_n is super finitely presented by [13, Lemma 2.3], it is finitely presented. It follows that P_n is projective. Thus $\mathrm{pd}_R(M) \leq n$, and hence $l.\mathrm{sp.gldim}(R) \leq n$, as desired.

The second assertion follows from [18, Proposition 1.1(ii)] and the fact that R is a left coherent ring if and only if all finitely presented left R-modules are super finitely presented.

(2) This follows immediately by Proposition 3.5. □

Since over a coherent ring the class of weak injective (resp. flat modules) coincides with the class of FP-injective modules (resp. flat modules), the following result of Stenström follows immediately from Theorem 3.8.

Corollary 3.9 ([21, Theorem 3.3]). *The following integers are identical when R is a left coherent ring*:

(1) *$w.gl.dim(R)$*;
(2) *$l.FP\text{-}dim(R)$*;
(3) *$sup\{pd_R(F) \mid F$ is a fintiely presented left R-module$\}$*.

Based on Theorem 3.8, it is easy to get the following result.

Corollary 3.10. *The following statements are equivalent for any ring R*:

(1) *$l.sp.gldim(R) = 0$*;
(2) *Every left R-module is weak injective*;

(3) *Every right R-module is weak flat*;
(4) *Every super finitely presented R-module is projective.*

Remark 3.11.

(1) Corollary 3.10 shows that if R is a left coherent ring, then the condition $l.\text{sp.gldim}(R) = 0$ just gives a characterization of von Neumann regular rings.
(2) There are many examples to show that the left inequality in Remark 3.7(3) may be strict, that is, $\text{l.sp.gldim}(R) \neq \text{w.gl.dim}(R)$ in general. In fact, recall from [7] that a ring R is called an (n, d)*-ring* if every R-module having a finite n-presentation has projective dimension at most d. In [16], Mahdou gave an example of $(2,0)$-ring B which is not $(1,0)$-ring (see [16, Theorem 3.4]), for which $\text{l.sp.gldim}(B) = 0$, but $\text{w.gl.dim}(B) \neq 0$. Also, Costa in [7] gave examples of (2, 1)-domains R which are not (1, 1)-domains (Prüfer), and for such rings R, we have $\text{l.sp.gldim}(R) = 1$, but $\text{w.gl.dim}(R) \neq 1$.

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