Journal of Algebra and Its Applications
Vol. 12, No. 2 (2013) 1250165 (15 pages)

DOI: 10.1142/S0219498812501654

World Scientific
www.worldscientific.com

WEAK GORENSTEIN PROJECTIVE, INJECTIVE AND FLAT MODULES

ZENGHUI GAO
College of Mathematics
Chengdu University of Information Technology
Sichuan 610225, P. R. China
gaozenghui@cuit.edu.cn

Received 7 November 2011
Accepted 7 May 2012
Published 11 December 2012

Communicated by P. Ara

In this paper, weak Gorenstein projective, injective and flat modules are introduced and investigated. A left R-module M is called weak Gorenstein projective if there exists an exact sequence $\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$ of left R-modules such that: (1) all P_i and P^i are projective; and (2) $M \cong \ker(P^0 \to P^1)$. The weak Gorenstein injective and flat modules are defined similarly. Several well-known classes of rings are characterized in terms of weak Gorenstein projective, injective and flat modules. We also give a partial answer to Holm's question ([Question C, on p. 114 in Gorenstein projective, injective and flat modules, MSc thesis, Institute for Mathematical Sciences, University of Copenhagen (2000)]).

Keywords: Weak Gorenstein projective module; weak Gorenstein flat module; weak Gorenstein injective module; n-Gorenstein ring; n-FC ring.

Mathematics Subject Classification: 18G05, 18G25, 16D40

1. Introduction

Throughout this paper, R is an associative ring with identity and all modules are unitary. For any left R-module M, $\mathrm{id}_R(M)$ and $\mathrm{fd}_R(M)$ will denote the injective and flat dimensions of M, respectively. The character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is denoted by M^+. We use $D(R)$ (respectively, $wD(R)$) to stand for the global dimension (respectively, weak global dimension) of a ring R. General background materials can be found in [10, 17, 21, 24].

It is well known that the projective, injective and flat modules are important and fundamental research objects in classical homological algebra. As a generalization of the notion of projective dimensions, Auslander and Bridger [2] introduced the notion of G-dimension for finitely generated modules over a two-sided Noetherian ring. Several decades later, Enochs, Jenda and Torrecillas in [9, 14] extended the ideas of Auslander and Bridger and introduced the notions of Gorenstein projective, injective and flat modules, which have been studied extensively by many authors (see, for example, [3, 4, 7, 10, 16, 17, 24]). In the recent years, Gorenstein homological algebra has become a vigorously active area of research, and various generalizations of Gorenstein homological modules have been given over specific rings (see [11, 13, 23]). More recently, the authors [15] introduced a concept of Gorenstein FP-injective modules and characterized self-FP-injective coherent rings in terms of this class of modules. A left R-module M is called Gorenstein FP-injective if there exists an exact sequence $\cdots \to E_1 \to E_0 \to E^0 \to E^1 \to \cdots$ of FP-injective left R-modules with $M = \ker(E^0 \to E^1)$ and such that $\mathrm{Hom}_R(P, -)$ leaves the sequence exact whenever P is a finitely presented left R-module with $\mathrm{pd}_R(P) < \infty$. The class of Gorenstein FP-injective modules contains all FP-injective modules and all Gorenstein injective modules ([15, Proposition 2.5]). This class of modules has nice properties when the ring in question is left coherent. It has been proven in [15, Theorem 2.3] that if R is a left coherent ring and M is a left R-module, then M is Gorenstein FP-injective if and only if there exists an exact sequence $\cdots \to E_1 \to E_0 \to E^0 \to E^1 \to \cdots$ of FP-injective left R-modules such that $M = \ker(E^0 \to E^1)$.

Holm in [16] raised an open question: Is it true that every Gorenstein projective module also is Gorenstein flat? He proved that if R is right coherent with finite left finitistic projective dimension (or R is a commutative Noetherian ring with finite Krull dimension), then every Gorenstein projective left R-module is Gorenstein flat (see [16, Proposition 5.5] and [17, Proposition 3.4]). Enochs and Jenda in [10, Proposition 10.3.2] showed that if R is left and right coherent, then every finitely presented Gorenstein projective R-module is Gorenstein flat.

The purpose of this paper is to introduce and study weak Gorenstein projective, injective and flat modules. Inspired by [15], in Sec. 2, we give the definition of weak Gorenstein projective and injective modules and show some general properties of them. The relations between the weak Gorenstein projective (respectively, injective) modules and other modules are discussed. Then in Sec. 3, the definition and some general results of weak Gorenstein flat modules are given. Let R be any ring. We prove that if every injective right R-module has a finite flat dimension, then every weak Gorenstein flat left R-module is Gorenstein flat. As an immediate consequence, we give an affirmative answer to Holm' question ([16, Question C, p. 119]) for any ring R with $r \cdot \mathrm{IFD}(R) < \infty$ (Corollary 3.7). We also discuss some connections between weak Gorenstein injective and weak Gorenstein flat modules.

2. Weak Gorenstein Projective and Injective Modules

In this section, we give a treatment of weak Gorenstein projective and injective modules, the relations between the weak Gorenstein projective (respectively, injective) modules and other modules are discussed.

Recall that a left R-module M is called *Gorenstein projective* [9] if there is an exact sequence $\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$ of projective left R-modules with $M = \ker(P^0 \to P^1)$ such that $\mathrm{Hom}_R(-, Q)$ leaves the sequence exact whenever Q is a projective left R-module. The Gorenstein injective modules are defined dually.

Definition 2.1. A left R-module M is called *weak Gorenstein projective* if there exists an exact sequence of left R-modules

$$\mathbf{P} = \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

such that: (1) All P_i and P^i are projective; (2) $M = \ker(P^0 \to P^1)$. The exact sequence $\mathbf{P}$ is called a *weak complete projective resolution*.

The weak Gorenstein injective modules are defined dually.

Remark 2.2. (1) It is clear that every Gorenstein projective (respectively, injective) left R-module is weak Gorenstein projective (respectively, injective).

(2) We note that a sum (respectively, product) of weak complete projective (respectively, injective) resolutions is also a weak complete projective (respectively, injective) resolution, it follows that the class of weak Gorenstein projective (respectively, injective) left R-modules is closed under direct sums (respectively, direct products).

(3) If $\mathbf{L} = \cdots \to L_1 \to L_0 \to L^0 \to L^1 \to \cdots$ is a weak complete projective (respectively, injective) resolution, then by symmetry, all the kernels, the images and the cokernels of $\mathbf{L}$ are weak Gorenstein projective (respectively, injective).

Proposition 2.3. *The following are equivalent for a left R-module M:*

(1) *M is weak Gorenstein projective.*

(2) *There exists an exact sequence of left R-modules $0 \to M \to P^0 \to P^1 \to \cdots$ with each P^i projective.*

(3) *There exists a short exact sequence of left R-modules $0 \to M \to P \to N \to 0$, where P is projective and N is weak Gorenstein projective.*

Proof. (1) ⇒ (2), (1) ⇒ (3) follow from Definition 2.1. Next we will prove that (3) ⇒ (2) ⇒ (1).

(3) ⇒ (2) Suppose that there exists a short exact sequence of left R-modules

$$(\alpha) = 0 \to M \to P \to N \to 0$$

with P projective and N weak Gorenstein projective. Since N is weak Gorenstein projective, we have an exact sequence of left R-modules

$$(\beta) = 0 \to N \to P^{0'} \to P^{1'} \to \cdots$$

with all $P^{i'}$ projective. Assembling the sequences (α) and (β), we obtain the exactness of $0 \to M \to P \to P^{0'} \to P^{1'} \to \cdots$, where P and $P^{i'}$ are projective. Thus (2) follows.

(2) ⇒ (1) Let M be a left R-module satisfying (2) and let $\cdots \to P_1 \to P_0 \to M \to 0$ be a projective resolution of M. Then we get an exact sequence of projective left R-modules

$$\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

with $M = \ker(P^0 \to P^1)$. It follows that M is weak Gorenstein projective. □

Let R be a commutative Noetherian ring. For any R-module M we set $M^* = \mathrm{Hom}_R(M, R)$. An R-module M is said to be *reflexive* if it is finitely generated and the canonical map $M \to M^{**}$ is bijective. In the main theorem from the introduction of [19], D. Jorgensen and L. Sega gave a family of weak Gorenstein projective modules which are not Gorenstein projective.

Theorem 2.4 ([19]). *There exists a commutative Artinian local ring R, and a family $\{M_s\}_{s\geq 1}$ of reflexive R-modules such that*:

(1) $\mathrm{Ext}^i_R(M_s, R) = 0$ *if and only if* $1 \leq i \leq s-1$;
(2) $\mathrm{Ext}^i_R(M_s^*, R) = 0$ *for all* $i > 0$.

Example 2.5. Consider a commutative Artinian local ring R and a family of reflexive R-modules $\{M_s\}_{s\geq 1}$ satisfying the conditions in Theorem 2.4. Then the family $\{M_s\}_{s\geq 1}$ are weak Gorenstein projective, but not Gorenstein projective.

Proof. Let $\cdots \to P_1 \to P_0 \to (M_s)^* \to 0$ be a resolution of $(M_s)^*$ by finitely generated projective modules. Then the sequence $0 \to (M_s)^{**} \to (P_0)^* \to (P_1)^* \to \cdots$ is exact by Theorem 2.4(2), and each $(P_i)^*$ is projective by [1, p. 202, Exercise 8]. Hence $(M_s)^{**}$ is weak Gorenstein projective by Proposition 2.3. However, $(M_s)^{**} = M_s$ since M_s is reflexive. By Theorem 2.4(1), the module M_s cannot be Gorenstein projective. □

Recall that a ring R is said to be *n-Gorenstein* [10] if R is left and right Noetherian with $\mathrm{id}({}_RR) \leq n$ and $\mathrm{id}(R_R) \leq n$ for an integer $n \geq 0$.

Proposition 2.6. *If M is a Gorenstein projective left R-module, then M is weak Gorenstein projective. The converse to this statement holds if the ring R is n-Gorenstein for some n.*

Proof. By Remark 2.2, it suffices to show that if R is n-Gorenstein, then all weak Gorenstein projective left R-modules are Gorenstein projective. Let M be a weak Gorenstein projective left R-module. Then there is an exact sequence

$$\mathbf{P} = \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

of projective left R-modules with $M = \ker(P^0 \to P^1)$. Next we will show that $\mathrm{Hom}_R(-, Q)$ leaves the sequence $\mathbf{P}$ exact for every projective left R-module Q. In order to do so, we first prove that if N is a left R-module with $\mathrm{id}_R(N) = n < \infty$, then $\mathrm{Hom}_R(\mathbf{P}, N)$ is exact. We proceed by induction on n. The case $n = 0$ is trivial. Now let $\mathrm{id}_R(N) = n$ with $1 \leq n < \infty$. Then there exists an exact sequence

$$0 \to N \to E \to L \to 0,$$

where E is injective and $\mathrm{id}_R(L) \leq n-1$. Thus we get an exact sequence of complexes

$$0 \to \mathrm{Hom}_R(\mathbf{P}, N) \to \mathrm{Hom}_R(\mathbf{P}, E) \to \mathrm{Hom}_R(\mathbf{P}, L) \to 0.$$

It is clear that $\mathrm{Hom}_R(\mathbf{P}, E)$ is exact, and $\mathrm{Hom}_R(\mathbf{P}, L)$ is exact by induction. Therefore, $\mathrm{Hom}_R(\mathbf{P}, N)$ is exact by [21, Theorem 6.3]. Note that R is n-Gorenstein for some n, it follows that $\mathrm{id}_R(Q) \leq n$ by [10, Theorem 9.1.10]. Thus $\mathrm{Hom}_R(\mathbf{P}, Q)$ is exact, and so M is Gorenstein projective. □

Recall that a ring R is said to be *quasi-Frobenius* (*QF-ring* for short) if R is left Noetherian and ${}_RR$ is injective. The following gives a characterization of such rings:

Lemma 2.7 ([1, Theorem 3.19]). *The following are equivalent for a ring R:*

(1) *R is quasi-Frobenius*;
(2) *Every projective left* (*right*) *R-module is injective*;
(3) *Every injective left* (*right*) *R-module is projective.*

Theorem 2.8. *The following assertions are equivalent for a ring R:*

(1) *Every left R-module is weak Gorenstein projective.*
(2) *Every left R-module is weak Gorenstein injective.*
(3) *Every weak Gorenstein injective left R-module is weak Gorenstein projective.*
(4) *Every weak Gorenstein projective left R-module is weak Gorenstein injective.*
(5) *Every Gorenstein injective left R-module is weak Gorenstein projective.*
(6) *Every Gorenstein projective left R-module is weak Gorenstein injective.*
(7) *Every injective left R-module is weak Gorenstein projective.*
(8) *Every projective left R-module is weak Gorenstein injective.*
(9) *R is quasi-Frobenius.*

Proof. (1) ⇒ (3) ⇒ (5) ⇒ (7) and (2) ⇒ (4) ⇒ (6) ⇒ (8) are trivial. (9) ⇒ (1) follows from [3, Proposition 2.6] and the fact that every Gorenstein projective module is weak Gorenstein projective. Similarly, (9) ⇒ (2) follows. Next it suffices to show that (7) ⇒ (9) and (8) ⇒ (9).

(7) ⇒ (9) Suppose the condition (7) holds. Let M be an injective left R-module. Then M is weak Gorenstein projective. There exists a short exact sequence $0 \to M \to P \to L \to 0$ with P projective by Proposition 2.3. Then the sequence splits since M is injective. Therefore M is projective as a direct summand of P, and so R is quasi-Frobenius by Lemma 2.7.

(8) ⇒ (9) Let N be a projective left R-module. Then N is weak Gorenstein injective by the hypothesis, and we get the exactness of $0 \to K \to E \to N \to 0$

with E injective by dual of Proposition 2.3. Note that N is projective, it follows that $0 \to K \to E \to N \to 0$ splits. Hence N is injective, which shows that R is quasi-Frobenius. □

Following [7], a ring is said to be an *n-FC ring* if R is a left and right coherent ring with $\text{FP-id}({}_RR) \leq n$ and $\text{FP-id}(R_R) \leq n$ for an integer $n \geq 0$. Clearly every n-Gorenstein ring is n-FC, but the converse is not true in general. Ding and Chen in [7, Proposition 12] proved that if R is an n-FC ring and M is a Gorenstein injective right R-module, then M^+ is a Gorenstein flat left R-module. The next result gives a generalization of [10, Corollary 10.3.9] and [7, Proposition 12].

Proposition 2.9. *Let R be left coherent and suppose that every injective left R-module has finite flat dimension. If M is a weak Gorenstein injective left R-module, then M^+ is a Gorenstein flat right R-module.*

Proof. Suppose that M is a weak Gorenstein injective left R-module, then there exists an exact sequence of injective left R-modules

$$\mathbf{E} = \cdots \to E_1 \to E_0 \to E^0 \to E^1 \to \cdots$$

with $M = \ker(E^0 \to E^1)$. Then

$$\tilde{\mathbf{E}} = \cdots \to (E^1)^+ \to (E^0)^+ \to (E_0)^+ \to (E_1)^+ \to \cdots$$

is exact such that $M^+ \cong \ker((E_0)^+ \to (E_1)^+)$ and all $(E^i)^+, (E_i)^+$ are flat right R-modules. Next we will show $\tilde{\mathbf{E}} \otimes I$ is exact for every injective left R-module I. We first prove that if N is a left R-module with $\text{fd}_R(N) = m < \infty$, then $\tilde{\mathbf{E}} \otimes N$ is exact. We proceed by induction on m. The case $m = 0$ is clear. Let $\text{fd}_R(N) = m$ with $1 \leq m < \infty$. Then there exists an exact sequence

$$0 \to F_1 \to F \to N \to 0,$$

where F is flat and $\text{fd}_R(F_1) \leq m-1$. Thus we obtain an exact sequence of complexes

$$0 \to \tilde{\mathbf{E}} \otimes_R F_1 \to \tilde{\mathbf{E}} \otimes_R F \to \tilde{\mathbf{E}} \otimes_R N \to 0.$$

Note that $\tilde{\mathbf{E}} \otimes_R F$ is exact, and $\tilde{\mathbf{E}} \otimes_R F_1$ is exact by induction. It follows from [21, Theorem 6.3] that $\tilde{\mathbf{E}} \otimes_R N$ is exact. Now for every injective left R-module I, $\text{fd}_R(I) < \infty$ by hypothesis. Therefore $\tilde{\mathbf{E}} \otimes_R I$ is exact. This shows that M^+ is Gorenstein flat. □

Proposition 2.10. *If M is a projective left R-module, then M is weak Gorenstein projective. The converse to this statement holds if the ring R has finite global dimension.*

Proof. It is enough to prove that if $D(R) < \infty$, then all weak Gorenstein projective left R-modules are projective. Let $D(R) = n < \infty$. If $n = 0$, then the desired result follows. Next we assume that $n \geq 1$. Let M be a weak Gorenstein projective left

R-module. Then, by Proposition 2.3, there exists an exact sequence $0 \to M \to P^0 \to P^1 \to \cdots$ with all P^i projective. Let $L = \mathrm{Im}(P^{n-1} \to P^n)$, then $0 \to M \to P^0 \to P^1 \to \cdots \to P^{n-1} \to L \to 0$ is exact, and hence M is projective because $\mathrm{pd}_R(L) \leq n$. □

Remark 2.11. We point out the dual versions on weak Gorenstein injectivity in Propositions 2.3, 2.6 and 2.10 also hold true by using completely dual arguments.

3. Weak Gorenstein Flat Modules

In this section, we introduce and study weak Gorenstein flat modules. We also study the connection between weak Gorenstein injective and flat modules, and characterize FC rings in terms of weak Gorenstein flat modules.

Recall that a left R-module M is said to be *Gorenstein flat* [14] if there exists an exact sequence $\cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$ of flat left R-modules with $M = \ker(F^0 \to F^1)$ such that $E \otimes_R$-leaves the sequence exact whenever E is an injective right R-module.

Definition 3.1. A left R-module M is called *weak Gorenstein flat* if there exists an exact sequence of left R-modules

$$\mathbf{F} = \cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$$

such that: (1) All F_i and F^i are flat; and (2) $M = \ker(F^0 \to F^1)$. The exact sequence $\mathbf{F}$ is called a weak complete flat resolution.

Remark 3.2. (1) It is clear that every Gorenstein flat left R-module is weak Gorenstein flat, and all (weak) Gorenstein projective modules are weak Gorenstein flat.

(2) Because a sum of weak complete flat resolutions is also a weak complete flat resolution, one easily checks that the class of weak Gorenstein flat left R-modules is closed under direct sums.

(3) If $\mathbf{F} = \cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$ is a weak complete flat resolution, then by symmetry, all the kernels, the images and the cokernels of $\mathbf{F}$ are weak Gorenstein flat.

The following result gives some characterizations of weak Gorenstein flat modules.

Proposition 3.3. *The following assertions are equivalent for a left R-module M:*

(1) *M is weak Gorenstein flat.*

(2) *There is an exact sequence $0 \to M \to F^0 \to F^1 \to \cdots$ of left R-modules with each F^i flat.*

(3) *There is a short exact sequence $0 \to M \to F \to H \to 0$ of left R-modules, where F is flat and H is weak Gorenstein flat.*

Proof. The proof is similar to that of Proposition 2.3 and omitted. □

Proposition 3.4. *Let* $0 \to N \to M \to F \to 0$ *be an exact sequence of left* R*-modules. If* N *is weak Gorenstein flat and* F *is flat, then* M *is weak Gorenstein flat.*

Proof. Since N is weak Gorenstein flat, there exists an exact sequence $0 \to N \to F' \to L \to 0$ of left R-modules with F' flat and L weak Gorenstein flat by Proposition 3.3. Now we consider the following pushout diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & N & \longrightarrow & M & \longrightarrow & F & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & F' & \longrightarrow & D & \longrightarrow & F & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & L & = & L & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

From the middle horizontal sequence $0 \to F' \to D \to F \to 0$ with F' and F flat, it follows that D is flat. Therefore, by the middle vertical sequence and Proposition 3.3, we get M is weak Gorenstein flat. □

Proposition 3.5. *Let* R *be a left coherent ring. Then the class of weak Gorenstein flat right* R*-modules is closed under arbitrary direct products.*

Proof. Let $M = \prod_{i \in I} M_i$, where each M_i is weak Gorenstein flat right R-module. We will prove that M is also weak Gorenstein flat. Since every M_i is weak Gorenstein flat right R-module, there exists an exact sequence

$$\mathbb{F}_i = \cdots \to F_{1i} \to F_{0i} \to F_i^0 \to F_i^1 \to \cdots$$

such that $M_i = \ker(F_i^0 \to F_i^1)$ for each $i \in I$. Then

$$\prod_{i \in I} \mathbb{F}_i = \cdots \to \prod_{i \in I} F_{1i} \to \prod_{i \in I} F_{0i} \to \prod_{i \in I} F_i^0 \to \prod_{i \in I} F_i^1 \to \cdots$$

is exact such that $M \cong \ker(\prod_{i \in I} F_i^0 \to \prod_{i \in I} F_i^1)$ and $\prod_{i \in I} F_{ji}, \prod_{i \in I} F_i^j$ are flat for $j = 0, 1, \ldots$ since R is left coherent. So the desired result follows. □

Theorem 3.6. *Let* R *be any ring and suppose that every injective right* R*-module has a finite flat dimension. Then every weak Gorenstein flat left* R*-module is Gorenstein flat.*

Proof. Let M be a weak Gorenstein flat left R-module. Then there exists an exact sequence of flat left R-modules

$$\mathbf{F} = \cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$$

with $M = \ker(F^0 \to F^1)$. It suffices to show that $I \otimes_R$-leaves the sequence exact for every injective right R-module I. By the hypothesis, $\mathrm{fd}_R(I) = n < \infty$. Similar to the proof of Proposition 2.6, one easily checks that $I \otimes_R \mathbf{F}$ is exact by induction on n, and so M is Gorenstein flat. □

Using Remark 3.2 and Theorem 3.6, we immediately get the next result, which gives a partial answer to Holm's question (see [16, Question C, p. 119]). Before that, recall the dimension $r \cdot \mathrm{IFD}(R)$ of a ring R was defined in [6] as $r \cdot \mathrm{IFD}(R) = \sup\{\mathrm{fd}_R(M) | M$ is an injective right R-module$\}$. It is clear that $r \cdot \mathrm{IFD}(R) = 0$ if and only if R is a right IF ring [5].

Corollary 3.7. *Let R be any ring with $r \cdot \mathrm{IFD}(R) < \infty$. Then every Gorenstein projective left R-module is Gorenstein flat.*

Recall a left R-module M is called *strongly copure flat* [8] if $\mathrm{Tor}_i^R(E, M) = 0$ for all $i \geq 1$ and all injective right R-modules E.

Proposition 3.8. *Let R be an n-FC ring with $n \geq 0$. Then the following are equivalent for a left (right) R-module M.*

(1) *M is weak Gorenstein flat.*
(2) *M is Gorenstein flat.*
(3) *M is strongly copure flat.*
(4) *M^+ is Gorenstein injective.*

Proof. (1) ⇒ (2) follows from Theorem 3.6 and [6, Theorem 3.8].
(2) ⇒ (1) is trivial.
(2) ⇔ (3) by [7, Corollary 11].
(2) ⇔ (4) holds by [17, Theorem 3.6]. □

Recall that a left R-module M is called *FP-injective* [22] if $\mathrm{Ext}_R^1(F, M) = 0$ for any finitely presented left R-module F; a left R-module C is said to be *cotorsion* [24] if $\mathrm{Ext}_R^1(N, C) = 0$ for all flat left R-modules N.

Corollary 3.9. *The following are equivalent for a ring R:*

(1) *Every R-module (left and right) is weak Gorenstein flat.*
(2) *Every Gorenstein injective R-module (left and right) is weak Gorenstein flat.*
(3) *Every injective R-module (left and right) is weak Gorenstein flat.*
(4) *Every cotorsion R-module (left and right) is weak Gorenstein flat.*
(5) *Every FP-injective R-module (left and right) is weak Gorenstein flat.*
(6) *Every finitely presented R-module (left and right) is weak Gorenstein flat.*

(7) *Every R-module (left and right) is Gorenstein flat.*
(8) *R is an FC ring.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3), (1) $\Rightarrow$ (4) $\Rightarrow$ (3), (1) $\Rightarrow$ (5) $\Rightarrow$ (3) and (7) $\Rightarrow$ (1) $\Rightarrow$ (6) are trivial. (7) $\Leftrightarrow$ (8) by [7, Theorem 6]. Next it remains to show (3) $\Rightarrow$ (8) and (6) $\Rightarrow$ (8).

(3) $\Rightarrow$ (8) Assume that the condition (3) holds and let M be an injective left R-module. Then M is weak Gorenstein flat. There exists an exact sequence $0 \to M \to F \to N \to 0$ with F flat by Proposition 3.3. The sequence $0 \to M \to F \to N \to 0$ splits since M is injective. Thus M is isomorphic to a direct summand of F, and hence M is flat. So R is left IF. Similarly, R is right IF. Consequently R is FC by [6, Corollary 3.14].

(6) $\Rightarrow$ (8) Let M be a finitely presented left R-module. Then M is weak Gorenstein flat by (6). There exists an exact sequence $0 \to M \to F^0$ with F^0 flat. From the flatness of F^0, it follows that $M \to F^0$ can be factored $M \to P \to F^0$ where P is a finitely generated free module, and hence M can be embedded in P. So R is left IF by [5, Theorem 1]. Similarly, R is right IF. It follows from [6, Corollary 3.14] that R is an FC ring. □

Holm in [17, Theorem 3.6] proved that if R is right coherent, then M is a Gorenstein flat left R-module if and only if M^+ is a Gorenstein injective right R-module. We write ${}_R\mathcal{M}$ for the category of left R-modules.

Proposition 3.10. *Let R be a commutative ring. We consider the following conditions for an R-module M:*

(1) *M is weak Gorenstein flat.*
(2) $\mathrm{Hom}_R(M, E)$ *is weak Gorenstein injective for every injective R-module E.*
(3) $\mathrm{Hom}_R(M, E)$ *is weak Gorenstein injective for any injective cogenerator E for ${}_R\mathcal{M}$.*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3). *If R is an n-Gorenstein ring, then also* (3) $\Rightarrow$ (1).

Proof. (1) $\Rightarrow$ (2) Let M be a weak Gorenstein flat R-module. There exists an exact sequence of flat R-modules

$$\cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$$

such that $M = \ker(F^0 \to F^1)$. Then, for every injective R-module E, we have

$$\cdots \to \mathrm{Hom}_R(F^1, E) \to \mathrm{Hom}_R(F^0, E) \to \mathrm{Hom}_R(F_0, E) \to \mathrm{Hom}_R(F_1, E) \to \cdots$$

is exact and $\mathrm{Hom}_R(F^i, E)$ and $\mathrm{Hom}_R(F_i, E)$, for $i = 0, 1, \ldots$, are injective R-modules by [21, Theorem 3.44]. The exact sequence $\cdots \to F_1 \to F_0 \to M \to 0$

gives rise to the exactness of

$$0 \to \mathrm{Hom}_R(M, E) \to \mathrm{Hom}_R(F_0, E) \to \mathrm{Hom}_R(F_1, E) \to \cdots.$$

Therefore $\mathrm{Hom}_R(M, E) \cong \ker(\mathrm{Hom}_R(F_0, E) \to \mathrm{Hom}_R(F_1, E))$, and so $\mathrm{Hom}_R(M, E)$ is a weak Gorenstein injective R-module.

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1) Suppose R is an n-Gorenstein ring. Since $M^+ \cong \mathrm{Hom}_R(M, R^+)$ is a Gorenstein injective R-module by the dual of Proposition 2.6, we have M is a Gorenstein flat left R-module by [17, Theorem 3.6]. Thus the desired result follows. □

When R is a right coherent ring and M is a right R-module, one has the isomorphism: $\mathrm{Ext}^n_R(F, M)^+ \cong \mathrm{Tor}^R_n(F, M^+)$ for all $n \geq 1$ and all finitely presented right R-modules F by [22]. It follows immediately that a right R-module M is FP-injective if and only if M^+ is flat.

We shall say a left R-module M *weak Gorenstein FP-injective* if there exists an exact sequence of left R-modules $\mathbf{E} = \cdots \to E_1 \to E_0 \to E^0 \to E^1 \to \cdots$ such that: (1) All E_i and E^i are FP-injective; (2) $M = \ker(E^0 \to E^1)$. The exact sequence $\mathbf{E}$ is called a *weak complete FP-injective resolution*. Because every injective module is FP-injective, it follows that all (weak) Gorenstein injective left R-modules are weak Gorenstein FP-injective. If R is a Noetherian ring, then the class of weak Gorenstein injective left R-modules coincides with the class of weak Gorenstein FP-injective left R-modules.

Theorem 3.11. *Let R be a left coherent ring. If M is a weak Gorenstein FP-injective left R-module, then M^+ is a weak Gorenstein flat right R-module.*

Proof. Let M be a weak Gorenstein FP-injective left R-module. There exists an exact sequence of FP-injective left R-modules

$$\mathbb{E} = \cdots \to E_1 \to E_0 \to E^0 \to E^1 \to \cdots$$

such that $M = \ker(E^0 \to E^1)$. Then

$$\mathbb{E}^+ = \cdots \to E^{1+} \to E^{0+} \to E_0^+ \to E_1^+ \to \cdots$$

is exact with $M^+ = \mathrm{coker}(E^{1+} \to E^{0+})$ and all the E^{i+}, E_i^+ are flat right R-modules since R is left coherent. Thus M^+ is a weak Gorenstein flat right R-module, as desired. □

Proposition 3.12. *If R is an n-FC ring with $n \geq 0$ and M is a weak Gorenstein FP-injective left R-module, then M^+ is a Gorenstein flat right R-module.*

Proof. Since the class of weak Gorenstein flat modules coincides with the class of Gorenstein flat modules over an n-FC ring by Proposition 3.8, this result follows directly from Theorem 3.11. □

As application of Proposition 3.12, some known results are obtained as corollaries.

Corollary 3.13 ([10, Corollary 10.3.9]). *If R is n-Gorenstein and G is a Gorenstein injective left (right) R-module, then G^+ is a Gorenstein flat right (left) R-module.*

Corollary 3.14 ([7, Proposition 12]). *Let R be an n-FC ring with $n \geq 0$. If G is a Gorenstein injective left R-module, then G^+ is Gorenstein flat.*

Proposition 3.15. *Let R be a commutative ring and Q a flat R-module.*

(1) *If M is a weak Gorenstein flat R-module, then $M \otimes_R Q$ is a weak Gorenstein flat R-module.*
(2) *If M is a weak Gorenstein injective R-module, then $M \otimes_R Q$ is a weak Gorenstein injective R-module.*

Proof. (1) Let M be a weak Gorenstein flat R-module. There exists an exact sequence

$$\mathbb{F} = \cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$$

of flat R-modules with $M \cong \ker(F^0 \to F^1)$. Then

$$\mathbb{F} \otimes_R Q = \cdots \to F_1 \otimes_R Q \to F_0 \otimes_R Q \to F^0 \otimes_R Q \to F^1 \otimes_R Q \to \cdots$$

is exact such that $M \otimes_R Q \cong \ker(F^0 \otimes_R Q \to F^1 \otimes_R Q)$ and all the $F_i \otimes_R Q$, $F^i \otimes_R Q$ are flat for $i = 0, 1, \ldots$. Consequently, $M \otimes_R Q$ is a weak Gorenstein flat R-module.

(2) By analogy with the proof of (1). □

Proposition 3.16. *Let R be commutative and P a finitely generated projective R-module. If M is a weak Gorenstein flat R-module, then $\mathrm{Hom}_R(P, M)$ is a weak Gorenstein flat R-module.*

Proof. Let P be a finitely generated projective R-module and Q a flat R-module. We note the fact that P is a direct summand of R^n for some n, it follows that $\mathrm{Hom}_R(P, Q)$ is flat. Since M is weak Gorenstein flat, there exists an exact sequence

$$\mathbb{F} = \cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$$

of flat R-modules with $M \cong \ker(F^0 \to F^1)$. Then

$$\begin{aligned}\mathrm{Hom}_R(P, \mathbb{F}) &= \cdots \to \mathrm{Hom}_R(P, F_1) \to \mathrm{Hom}_R(P, F_0) \to \mathrm{Hom}_R(P, F^0) \\ &\to \mathrm{Hom}_R(P, F^1) \to \cdots\end{aligned}$$

is exact such that $\mathrm{Hom}_R(P, M) \cong \ker(\mathrm{Hom}_R(P, F^0) \to \mathrm{Hom}_R(P, F^1))$ and $\mathrm{Hom}_R(P, F_i)$, $\mathrm{Hom}_R(P, F^i)$ are flat for $i = 0, 1, \ldots$. Therefore $\mathrm{Hom}_R(P, M)$ is weak Gorenstein flat. □

Recall a ring is called *n-perfect* [12] if every flat R-module has projective dimension less than or equal to n. Now we have the following proposition.

Proposition 3.17. *Let R be a commutative n-perfect ring. If M is a weak Gorenstein injective R-module, then $\mathrm{Hom}_R(Q, M)$ is weak Gorenstein injective for any flat R-module Q.*

Proof. Let M be a weak Gorenstein injective R-module. Then there exists an exact sequence of injective R-modules

$$\mathbb{E} = \cdots \to E_1 \to E_0 \to E^0 \to E^1 \to \cdots$$

such that $M \cong \ker(E^0 \to E^1)$. Next we will show that $\mathrm{Hom}_R(Q, \mathbb{E})$ is exact for every flat R-module Q. By hypothesis, $\mathrm{pd}_R(Q) = n < \infty$. We proceed by induction on n. The case $n = 0$ is clear. Let $n \geq 1$. There exists an exact sequence $0 \to L \to P \to Q \to 0$ with P projective and $\mathrm{pd}_R(L) \leq n - 1$. Then we have a short exact sequence of complexes

$$0 \to \mathrm{Hom}_R(Q, \mathbb{E}) \to \mathrm{Hom}_R(P, \mathbb{E}) \to \mathrm{Hom}_R(L, \mathbb{E}) \to 0.$$

It is clear that $\mathrm{Hom}_R(P, \mathbb{E})$ is exact, and $\mathrm{Hom}_R(L, \mathbb{E})$ is exact by induction. It follows from [21, Theorem 6.3] that $\mathrm{Hom}_R(Q, \mathbb{E})$ is exact. Thus we get the following exact sequence

$$\cdots \to \mathrm{Hom}_R(Q, E_1) \to \mathrm{Hom}_R(Q, E_0) \to \mathrm{Hom}_R(Q, E^0) \to \mathrm{Hom}_R(Q, E^1) \to \cdots$$

such that $\mathrm{Hom}_R(Q, M) \cong \ker(\mathrm{Hom}_R(Q, E^0) \to \mathrm{Hom}_R(Q, E^1))$ and all $\mathrm{Hom}_R(Q, E_i)$, $\mathrm{Hom}_R(Q, E^i)$ are injective for $i = 0, 1, \ldots$ by [21, Theorem 3.44]. Hence $\mathrm{Hom}_R(Q, M)$ is weak Gorenstein injective. This completes the proof. □

Remark 3.18. It is worth noting that R is commutative n-perfect, e.g. when R is commutative, Noetherian with Krull dimension $\dim(R) \leq n$. This follows from results in [18, 20].

Proposition 3.19. *If M is a flat left R-module, then M is weak Gorenstein flat. The converse to this statement holds if the ring R has finite weak global dimension.*

Proof. It suffices to show that if $wD(R) < \infty$, then every weak Gorenstein flat left R-module is flat. Let $wD(R) = m < \infty$. If $m = 0$, then the desired result is clear. Next we suppose that $m \geq 1$. For every weak Gorenstein flat left R-module M, there is an exact sequence

$$0 \to M \to E^0 \to E^1 \to \cdots$$

with each E^i flat. Let $K = \mathrm{Im}(E^{m-1} \to E^m)$. Then

$$0 \to M \to E^0 \to E^1 \to \cdots \to E^{m-1} \to K \to 0$$

is exact and hence M is flat since $\mathrm{fd}_R(K) \leq m$, as desired. □

Acknowledgments

I would like to thank the referee and Professor Pere Ara for many considerable inputs and suggestions, which have improved this paper. This research was partially supported by NSFC (No. 11171240 and 11226057), and the Scientific Research Foundation of CUIT (No. KYTZ201201).

References

[1] F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules*, Graduate Text in Mathematics, Vol. 13, 2nd edn. (Springer-Verlag, New York, 1992).

[2] M. Auslander and M. Bridger, *Stable Module Theory*, Memoirs of the American Mathematical Society, Vol. 94 (American Mathematical Society, Providence, RI, 1969).

[3] D. Bennis and N. Mahdou, Global Gorenstein dimensions, *Proc. Amer. Math. Soc.* **138** (2010) 461–465.

[4] L. W. Christensen, *Gorenstein Dimensions*, Lecture Notes in Mathematics, Vol. 1747 (Springer-Verlag, Berlin, 2000).

[5] R. R. Colby, Rings which have flat injective modules, *J. Algebra* **35** (1975) 239–252.

[6] N. Q. Ding and J. L. Chen, The flat dimensions of injective modules, *Manuscripta Math.* **78** (1993) 165–177.

[7] N. Q. Ding and J. L. Chen, Coherent rings with finite self-FP-injective dimension, *Commun. Algebra* **24** (1996) 2963–2980.

[8] E. E. Enochs and O. M. G. Jenda, Copure injective resolutions, flat resolvents and dimensions, *Comment. Math. Univ. Carolin.* **34** (1993) 203–211.

[9] E. E. Enochs and O. M. G. Jenda, Gorenstein injective and projective modules, *Math. Z.* **220** (1995) 611–633.

[10] E. E. Enochs and O. M. G. Jenda, *Relative Homological Algebra* (Walter de Gruyter, Berlin, 2000).

[11] E. E. Enochs and O. M. G. Jenda, Ω-Gorenstein projective and flat covers and Ω-Gorenstein injective envelopes, *Commun. Algebra* **32** (2004) 1453–1470.

[12] E. E. Enochs, O. M. G. Jenda and J. A. López-Ramos, Dualizing modules and n-perfect rings, *Proc. Edinb. Math. Soc. (2)* **48** (2005) 75–90.

[13] E. E. Enochs, O. M. G. Jenda and J. A. López-Ramos, Covers and envelopes by V-Gorenstein modules, *Commun. Algebra* **33** (2005) 4705–4717.

[14] E. E. Enochs, O. M. G. Jenda and B. Torrecillas, Gorenstein flat modules, *Nanjing Daxue Xuebao Shuxue Bannian Kan* **10** (1993) 1–9.

[15] Z. H. Gao and F. G. Wang, Coherent rings and Gorenstein FP-injective modules, *Commun. Algebra.* **40** (2012) 1669–1679.

[16] H. Holm, Gorenstein projective, injective and flat modules, MSc thesis, Institute for Mathematical Sciences, University of Copenhagen (2000).

[17] H. Holm, Gorenstein homological dimensions, *J. Pure Appl. Algebra* **189** (2004) 167–193.

[18] C. U. Jensen, On the vanishing of $\varprojlim^{(i)}$, *J. Algebra* **15** (1970) 151–166.

[19] D. A. Jorgensen and L. M. Sega, Independence of the total reflexivity conditions for modules, *Algebr. Represent. Theory* **9** (2006) 217–226.

[20] M. Raynaud and L. Gruson, Critères de platitude et de projectivité, techniques de "platification" d'un module, *Invent. Math.* **13** (1971) 1–89.

[21] J. J. Rotman, *An Introduction to Homological Algebra* (Academic Press, New York, 1979).

[22] B. Stenström, Coherent rings and FP-injective modules, *J. London Math. Soc.* **2** (1970) 323–329.
[23] J. Q. Wei, ω-Gorenstein modules, *Commun. Algebra* **36** (2008) 1817–1829.
[24] J. Z. Xu, *Flat Covers of Modules*, Lecture Notes in Mathematics, Vol. 1634 (Springer-Verlag, Berlin, 1996).