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Abstract

A class of continuous-time systems with periodic coefficients is analysed and controlled by robust linear controllers. Time varying parameters are considered as perturbations of a nominal time-invariant linear system. The robust control synthesis is based on general solutions of Diophantine equations in the ring of proper and Hurwitz stable rational functions \( \mathbb{R}_{PS} \) and the Youla-Kučera parameterization of controllers is utilized. Perturbations and robustness of proposed algorithms are studied through the infinity norms (\( H_{\infty} \)). Resulting control laws for first order systems are of a generalized PI type and a scalar parameter \( m > 0 \) is introduced for tuning and influencing of control responses. A Matlab + Simulink program system for automatic design and simulation has been developed.

1 Introduction

Analysis and control of periodic systems have been widely studied (e.g. [3]) but there are still many methods and approaches worthy to investigate. This contribution deals with robust linear control of a class of first order systems with periodic varying parameters. A nominal transfer function is considered as a linear one with constant parameters and periodic changes of parameters are assumed as perturbations. There are many available design and tuning methods, especially for stable systems (e.g. [2], [7]). The most popular and widely applied controllers have a PI or PID structure. Extensions of this structure can be found in [1], [9]. Modifications for unstable and time delay systems are available e.g. in [10], [12].

The proposed control design in this contribution is based on algebraic approach, see [5], [6], [9], [11]. The control design is performed in the ring of proper and Hurwitz stable rational functions \( \mathbb{R}_{PS} \) and the \( H_{\infty} \) norm serves as a tool for perturbation evaluation. The proposed methodology yields a general PI controller with a scalar parameter \( m > 0 \) for control and robust tuning.

2 System Description

Any transfer function \( G(s) \) of a (continuous-time) linear system has been traditionally expressed as a ratio of two polynomials in \( s \). For the purposes of this contribution it is necessary to express the transfer functions as a ratio of two elements of \( \mathbb{R}_{PS} \). It can be easily performed by dividing, both the polynomial denominator and numerator by the same stable polynomial of the order of the original denominator. Moreover, a scalar parameter \( m > 0 \) seems to be a suitable „tuning knob” influencing control behaviour as well as robustness of the closed loop system. Then all transfer functions could be described by

\[
G(s) = \frac{b(s)}{a(s)} = \frac{b(s)}{(s + m)^n} = \frac{B(s)}{A(s)},
\]

\[
n = \max(\text{deg}(a), \text{deg}(b)), \quad m > 0
\]

The simplest description for a process with periodic varying parameters can be expressed by a transfer function with one pole...
The value of the parameter $a_0 > 0$ represents stable systems, $a_0 < 0$ represents unstable ones and $a_0 = 0$ is an integrator. Time varying parameters in (2) are governed by relations:

$$b_0(t) = \beta_0(1 + \lambda_1 \sin \omega_1 t)$$
$$a_0(t) = \alpha_0(1 + \lambda_2 \sin \omega_2 t)$$

where $\alpha_0, \beta_0$ are real constants. The choice $\lambda_1 = \lambda_2 = 0$ represents a time invariant linear system. The time step responses of periodic system (1) are depicted in Figs. 1 - 3. Fig. 1 shows the step response of the time invariant system

$$G(s) = \frac{1}{s + 2}$$

and the responses of perturbations in $b_0$ and $a_0$ separately. Fig. 2 represents a simultaneous perturbations of both parameters with the same frequency. Step responses for $\omega_1 \neq \omega_2$ are depicted in Fig. 3

The traditional engineering design approach of PID like controllers was performed either in the frequency domain or in polynomial representation (see e.g. [3]). However, the fractional approach developed by Vidyasagar in [14], or by Kučera in [8] enables a deeper insight into control tuning and a more elegant expression of all suitable controllers. The situation and details for time-delay free systems can be found in [7] or for more sophisticated problems in [8], [10].
Suppose a 2DOF structure with the control law governed by
\[ P(s)u(t) = R(s)q(t) - Q(s)y(t) \]  
(5)
Basic relations following from Fig. 3 give
\[ y = \frac{B}{A} u + v \quad u = \frac{R}{P} w - \frac{Q}{P} y \]  
(6)
and \( w, v \) are independent inputs into the closed loop system.

Further, the following equations hold:
\[ w = \frac{AP}{AP + BQ} F_v + \frac{BR}{AP + BQ} G_w \]  
(7)
\[ e = w - y = \frac{AP}{AP + BQ} G_v + \left(1 - \frac{BR}{AP + BQ}\right) G_w \]  
(8)
For the structure FB (\( R=Q \)) the last relation gives the form:
\[ e = \frac{AP}{AP + BQ} G_v + \frac{AP}{AP + BQ} G_w \]  
(9)
The first step of the control design is to stabilize the system by a proper feedback loop. It can be formulated in an elegant way in \( \mathbb{R}_{\mathbb{P}}(s) \) by the Diophantine equation:
\[ AP + BQ = 1 \]  
(10)
with a general solution \( P=P_0+BT, Q=Q_0-AT \); where \( T \) is free in \( \mathbb{R}_{\mathbb{P}} \) and \( P_0, Q_0 \) is a pair of particular solutions (Youla – Kucera parameterization of all stabilizing controllers).

Details and proofs can be found e.g. in [5], [6], [9], [11]. Then equation (8) takes the form:
\[ e = \frac{AP}{F_v} G_v + \left(1 - \frac{BR}{F_w}\right) \frac{G_w}{F_w} \]  
(11)
Now, it is necessary to solve both structures separately.
The tracking error \( e \) tends to zero if
a) \( F_w \) divides \( P \) for 1DOF
b) \( F_w \) divides (1-BR) for 2DOF which gives the second Diophantine equation in the form:
\[ F_w S + BR = 1 \]  
(12)
Another control problem of practical importance is disturbance rejection and disturbance attenuation. In both cases, the effect of disturbances \( v \) and \( n \) should be asymptotically eliminated from the plant output. Since the both disturbances are external inputs into the feedback part of the system, the effect must be processed by a feedback controller. It means that
\[ y = \frac{AP}{AP + BQ} \frac{G_v}{F_v} \]  
(13)
\[ y = \frac{BP}{AP + BQ} \frac{G_n}{F_n} \]  
(14)
must belong to \( \mathbb{R}_{\mathbb{P}}(s) \), i.e. all \( AP+BQ, F_v, F_n \) should cancel. In other words, a multiple \( F_v, F_n \) must divide \( P \). More precisely \( F_v, \) must divide the multiple \( AP \) and \( F_n \) the multiple \( BP \). When define relatively prime elements \( A_0, F_{v0} \) and \( B_0, F_{n0} \) in \( \mathbb{R}_{\mathbb{P}}(s) \)
\[ A = \frac{A_0}{F_{v0}}, \quad B = \frac{B_0}{F_{n0}} \]  
(15)
then the problem of disturbance rejection and attenuation is solvable if and only if the pairs \( F_v, B \) and \( F_n, B \) are relatively prime and the feedback controller is given by
\[ C_s = \frac{Q}{P} = \frac{Q}{P_0 F_{v0} F_{n0}} \]  
(16)
where \( P_0, Q \) are the solution of the equation
\[ AF_{v0} F_{n0} P_0 + BQ = 1 \]  
(17)
The final feedback controller is then given by the ratio \( \frac{Q}{F_0F_{z_0}P} \).

### 4 Robust analysis and tuning

The fractional approach developed by Vidyasagar in [14] enables a deeper insight into control tuning and robustness. Let \( R_P \) be a set of proper and Hurwitz stable rational functions. This set is a ring and the norm \( H_\infty \) can be easily defined through the frequency response:

\[
\|G\| = \sup_{\omega \geq 0} |G(j\omega)|
\]

\[
\|G_1G_2\| = \sup_{\omega \geq 0} \left( |G_1(j\omega)|^2 + |G_2(j\omega)|^2 \right)^{1/2}
\]

Almost all models differ from a physical system. Let \( G(s) = \frac{B(s)}{A(s)} \) be a nominal plant and consider a family of perturbed systems \( G'(s) = \frac{B'(s)}{A'(s)} \) where

\[
A-A' \leq \varepsilon_1, \quad B-B' \leq \varepsilon_2
\]

Robust control tuning method requires to choose a part of stabilizing controllers \( P, Q \) which stabilize also perturbed plants. For perturbed plants choose such \( P, Q \) in (10) which fulfil the sufficient conditions

\[
\varepsilon_1\|P_0 + BT\| + \varepsilon_2\|Q_0 - AT\| < 1
\]

or the necessary and sufficient condition in the form

\[
\varepsilon\|P_0 + BT\| \leq \|Q_0 - AT\| < 1
\]

In the case where the perturbations are not known, the notion of the sensitivity function

\[
\varepsilon = \frac{y}{v} = \frac{e}{w} = A(P_0 + BT)
\]

can be used in the sense as in Doyle, Francis and Tannenbaum [5]. Generally, sensitivity function \( \varepsilon \) is a non-linear function of \( m > 0 \) and the norm of the \( \varepsilon \) can be minimized by a simple scalar optimization method. In this way the "most robust" controller of given structure can be obtained, where \( \varepsilon_1, \varepsilon_2, \varepsilon \) are positive constants.

Another tool for robust stability analysis can facilitated via interval polynomials and Kharitonov’s theorem. The class of transfer functions (2), (3) can be confined by a two dimensional convex polygon. The vertices of this polygon are characterized by the following values of parameters \( a_0, b_0 \):

\[
\begin{align*}
V_1: & \quad b_0^+ = \beta_0 + \lambda_1, \quad a_0^+ = a_0 + \lambda_2 \\
V_2: & \quad b_0^- = \beta_0 - \lambda_1, \quad a_0^+ = a_0 + \lambda_2 \\
V_3: & \quad b_0^+ = \beta_0 + \lambda_1, \quad a_0^- = a_0 - \lambda_2 \\
V_4: & \quad b_0^- = \beta_0 - \lambda_1, \quad a_0^- = a_0 - \lambda_2
\end{align*}
\]

Then for a designed feedback controller \( Q / P \) the open loop Nyquist plots \( \frac{BQ}{AP} \) for all vertices \( V_1...V_4 \) should be verified. A natural way for the stability test is the Nyquist criterion. If all four vertices exhibit the stable result then the designed controller \( Q / P \) achieves the robust stability for (2), (3).

### 5 Illustrative example

A nominal transfer function (1)-(2) is a first order system with the relative degree 1. Further, stepwise reference with \( F_s = \frac{s}{s+m} \) and no disturbances are assumed. The Diophantine equation (10) takes the form

\[
\frac{s + a_0}{s + m}p_0 + \frac{b_0}{s + m}q_0 = 1
\]

Multiplying by \( (s+m) \) and comparing coefficients give the general stabilizing solution in the form

\[
P(s) = p_0 + \frac{b_0}{s + m}T, Q(s) = q_0 - \frac{s + a_0}{s + m}T
\]

where \( q_0 = \frac{m-a_0}{b_0} \) and \( P = 1 \) and \( T \) is free in \( R_P(s) \).

The asymptotic tracking for a stepwise reference \( w \) will be given by divisibility \( F = \frac{s}{s+m} \) and \( P \). It is achieved for \( T = t_0 = -m/b_0 \) so that \( P(s) \) has zero absolute coefficient in the numerator. Then inserting \( t_0 \) into (25) gives
\[ P(s) = \frac{s}{s + m} \quad (26) \]

and
\[ Q(s) = \frac{\tilde{q}_1 s + \tilde{q}_0}{s + m} \quad (27) \]

where \( \tilde{q}_1 = \frac{2m - a_0}{b_0} \), \( \tilde{q}_0 = \frac{m^2}{b_0} \).

The 1DOF controller has the transfer function:
\[ \frac{Q}{P} = \frac{\tilde{q}_1 s + \tilde{q}_0}{s} \quad (28) \]

which is a traditional PI control law governed by:
\[ u(t) = \tilde{q}_1 (y(t) - w(t)) + \tilde{q}_0 \int (w(\tau) - y(\tau)) d\tau \quad (29) \]

For 2DOF control it is necessary to solve (12) in the form
\[ \frac{s}{s + m} z_0 + \frac{b_0}{s + m} r_0 = 1 \quad (30) \]

with the general solution
\[ R(s) = r_0 + \frac{s}{s + \tilde{T}}, \quad r_0 = \frac{m}{b_0} \quad (31) \]

The choice \( \tilde{T} = t_0 = 0 \) gives the feedforward part as
\[ \frac{R}{P} = \frac{r_0 (s + m)}{s} \quad \text{and the choice } t_0 = -r_0 \text{ represents} \]
\[ \frac{R}{P} = \frac{r_0 m}{s}. \]

6 Simulation and analysis

A nominal transfer function (1)-(2) is a first order system
\[ G(s) = \frac{1}{s + 2} \quad (32) \]

and the controlled system is given by the transfer function:
\[ G(s) = \frac{1 + 0.1 \sin 0.6t}{s + 2(1 + 0.2 \sin t)} \quad (33) \]

The step response of system (33) is depicted in Fig.5. The 1DOF and 2DOF controllers were developed according to (24) - (31). The vertices of parameter envelope is given by the following transfer functions:

\[ G_1(s) = \frac{1.1}{s + 1.6} \]
\[ G_2(s) = \frac{0.9}{s + 1.6} \]
\[ G_3(s) = \frac{0.9}{s + 2.4} \]
\[ G_4(s) = \frac{1.1}{s + 2.4} \quad (34) \]

The open loop Nyquist plots of all vertices with the feedback controller for \( m=5 \) are shown in Fig.6. The choice \( m=5 \) gives controller parameters in (270 as \( q_1 = 4, q_0 = 9 \). It is clear that all four plots represent stable systems. The minimal distance of curves from the critical point (-1, 0) represents the robustness of the designed controller. It is simple to derive that the open loop plot will be identical with the imaginary axis for \( a_0 = p_1/p_0 = 2.25 \) for this case. It is also the explanation of the fact that asymptotic lines for \( G_1, G_2 \) are in the left half plane and \( G_3, G_4 \) are in the right half plane in Fig. 5. The 1DOF control behaviour of (33) for this controller is depicted in Fig.7. Fig.8 shows the control behaviour for the same value \( m=5 \), but the structure is 2DOF. The influence of the control law is shown in Fig.9. There a similar plant with
\[ G'(s) = \frac{1 + 0.2 \sin t}{s + 2(1 + 0.4 \sin t)} \]

is controlled by PI controller (28), (29) for three value of the tuning parameter \( m=1, 2, 4 \).
7 Conclusion

A class of first order time-varying systems is investigated. Both parameters of the system description is periodically varying. The task of PI generalized controllers was considered for control of the systems. A controller design methodology based on fractional representation was developed for nominal stable or unstable systems. The proposed methodology enables to tune and influence the robustness and control behaviour by a single scalar parameter \( m > 0 \). The tuning parameter can be chosen arbitrarily or it is a result of the robust and sensitivity optimization. The stability analysis is based on interval polynomials and the Nyquist criterion. The developed design is supported by a Matlab + Simulink program system for automatic design and simulation.

Fig. 6: Open loop Nyquist plots for vertices (34)

- a - \( G_1 \);
- b - \( G_2 \);
- c - \( G_3 \);
- d - \( G_4 \).

Fig. 7: Control behaviour of (33) by 1DOF controller for \( m=5 \).

Fig. 8: Control behaviour of (33) by 2DOF controller for \( m=5 \).

Fig. 9: 1DOF control of \( G'(s) = \frac{1 + 0.2\sin t}{s + 2(1 + 0.4\sin t)} \)
for \( m=1, 2, 4 \) (a - 1, b - 2, c - 3).

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References


