Value at risk methodology under soft conditions approach (fuzzy-stochastic approach)

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Abstract

The paper describes methodology of dealing with financial modelling under uncertainty with risk and vagueness aspects. An approach to modelling risks by the Value at Risk methodology under imprecise and soft conditions is solved. It is supposed that the input data and problem conditions is difficult to determine as real numbers or as some precise distribution function. Thus, vagueness is modelled through the fuzzy numbers of the linear T-number type. The combination of risk and vagueness is solved by fuzzy-stochastic methodology. Illustrative example is introduced.

Keywords: Banking; Decision support systems; Finance; Fuzzy sets; Risk analysis; Uncertainty modelling

1. Introduction

Typical feature of financial environment is uncertainty. This term is understood mostly as risk uncertainty (probability, stochastic) and is modelled by stochastic apparatus. However, the term uncertainty has the second aspect—the vagueness (sometimes called imprecision, non-preciseness, ambiguity) which is often neglected and could be modelled by fuzzy methodology. In this respect it is apparent that the general term uncertainty includes two aspects: risk (stochastic) and vagueness (fuzzy) ones. These terms will be used in the paper.

Distinguishing of risk and vagueness confirms a discussion in financial decision making for several years (see Keynes, 1937; Ellesberg, 1961; Olsen and Throuhton, 2000). What are the impacts of uncertainty on decision making? Interesting characteristics are described in Olsen and Throuhton (2000) here vagueness is called ambiguity. (1) Uncertainty influences selection. (2) Decision makers are ambiguity averse in general. (3) Ambiguity causes more weight to be placed on negative information. (4) Buyers pay lower prices for and insurers require higher premium on objects or hazards subject to greater difficulty in estimation of value or probability of outcome. (5) Risk aversion and ambiguity aversion have not been seen to be highly correlated.

Recently, an application of fuzzy-stochastic methodology in financial modelling is being extensively studied now. We can see four basic application areas. (1) Valuation of financial instruments where basic assumptions of von Neumann and Morgenstern expected utility theory are
not fulfilled, mainly the assumption of subadditivity. Particularly, a probability measure is substituted and generalised by a fuzzy measure (see for instance Simonelli, 2001; Young and Zariphopoulou, 2000). (2) Forecasting financial characteristics are modelled by robust fuzzy methodology or by neuro-fuzzy networks (see for example Riberio et al., 1999; Tseng et al., 2001). (3) Multiple criteria evaluation of corporate financial level or creditworthiness with soft aspects and applying fuzzy aggregate operators in comparing with traditional expected utility criterion, Choquet integral, Sugeno integral or order weighting average operators (OWA) (see Riberio et al., 1999; Zmeskal, 2002). (4) Financial modelling under soft input data and ill-structured decision-making circumstances (see Tanaka et al., 2000; Inuiguchi and Ramík, 2000; Zmeskal, 2001). The last approach will be applied in the paper.

The recent developments of financial models are full of successful applications of stochastic models. We can say that now these models are undoubtedly considered as basic tools for financial modelling. One of useful applications is the value at risk methodology, which is now a financial standard for modelling the market and credit risk (see for instance Longerstaey and Spencer, 1996; Duffie and Pan, 1997; Fong and Vasiczek, 1997; Dowd, 1998; Jorion, 1997; Dempster, 2002). Moreover, J.P. Morgan established a market standard through RiskMetrics system (see Longerstaey and Spencer, 1996; Laubsh, 1999; Kim et al., 1999; Linsmeier and Pearson, 2000).

It is supposed that in traditional value at risk methodology decision-maker’s attitude is exactly stated and quality (precise) data are at disposal. These assumptions are often fulfilled and stochastic value at risk methodology is sufficient and effective approach for estimation the risk. However, complexity and flexibility of financial world show that there are several situations in which described conditions (goals, criteria, and constraints) are not determined precisely (ill-structured problem) and sometimes quality-input data are not at disposal (imprecise input data or distribution functions). Such circumstances exist if financial system is non/stable, after crisis, in transition economies etc. For modelling such situations, the fuzzy-stochastic approach has been developed and could be applied.

The basic intention of the paper is to show a possibility and procedure for dealing with softly formulated financial problem applying fuzzy-stochastic methodology. The problem on value at risk methodology basis will be described, however, under soft conditions, it means that conditions of decision making are not sharply defined and input data are introduced only vaguely. Such problem could be modelled by the fuzzy-stochastic methodology, it means by a combination of VAR methodology (risk aspect) and fuzzy methodology (vagueness aspect). The methodology is explained for the sake of simplicity and stressing problem soft aspects on the analytical delta approximate method. Similarly it could be explained simulation value at risk methodology with soft aspects.

The paper is organised as follows: in Section 2 and 3 analytical delta VaR methodology is described, we explain application reasons for soft parameters in Section 4, Section 5 is devoted to fuzzy-stochastic methodology description and illustrative example is presented in Section 6.

2. Characterisation of value at risk methodology

Value at risk (VAR) is a crucial term, which is defined as the greatest predicted loss under risk level (probability) and determined span (T). It means that the probability of receiving profit (ΔΠ) from an asset portfolio lesser than a determined profit level (VAR) is equal to the stated risk level (γ). Thus VAR characterises a loss or negative profit and it is possible to write this idea as follows:

\[ \Pr(\Delta \Pi \leq -\text{VAR}) = \gamma, \]  

(1)

where the line depicts random variable (the notation is used throughout the paper), VAR is a loss under probability level γ.

It is apparent that it is significant to find a multivariate distribution function of an asset margin portfolio value and then to derive VAR for the stated γ level:

\[ \Delta \Pi = \bar{V} - V_{t-1} = \bar{V}_{t-1} \cdot \bar{r}_{t} = \sum_{i} x_{i} \cdot \bar{r}_{t}, \]  

(2)
where $\Delta \Pi$ is a portfolio asset margin, $\Pi_i$ is a portfolio value, $\bar{r}_p$ is a portfolio return, $x_i$ is money amount of the $i$th financial instrument, $\bar{r}_i$ is the $i$th financial instrument return. Moreover a financial instrument return is defined as a function of basic random factors $\bar{rf}_j$.

The selection of the particular method for VAR calculation depends on (1) financial instruments types if the return instrument functions are of linear or non-linear types, further on (2) random process types of basic random factors and (3) multivariate distribution function types.

We can distinguish two basic groups of methods, analytical and simulated ones. (1) The delta approximate method, for linear instruments, (2) delta–gamma approximate method for non-linear instruments could be assigned to analytical methods. Simulation methods include, (3) historical simulation method, (4) simulation Monte-Carlo method.

Every method means a degree of preciousness and calculation complexity, advantages and disadvantages concerning the implementation aspects as well. Further on, the delta approximation method will be described because it is relatively simple and suitable for dealing with and explaining the soft aspects of VAR method. The purpose of the paper is to demonstrate how to apply soft aspects of VAR methodology and for the sake of simplicity the delta approximate method is explained and applied.

3. Analytical delta normal approximate method description

The basic assumptions concerning the analytical delta normal approximate method are following: (1) The basic random factor returns are normally distributed. (2) Financial instruments returns are modelled by the first order member Taylor (delta) expansion. (3) The financial instrument portfolio return is of a multivariate normal distribution type. (4) The VAR is calculated for inverse normal distribution and a portfolio variance; it is also assumed that the portfolio expected value equal to zero.

The random financial instrument return ($\bar{r}_i$) as a basic random factor function ($\bar{rf}_j$) can be approximated by the first order member of (linear) Taylor expansion, called the delta approximation, as follows:

$$\bar{r}_i = \sum_j c_{ij} \cdot \bar{f}_j \cdot P_i^{-1} \cdot \bar{rf}_j + \bar{u}_i$$

$$= \sum_j a_{ij} \cdot \bar{rf}_j + \bar{u}_i,$$

where $P_i$ is an asset price, $c_{ij} = \frac{\partial P_i}{\partial \bar{rf}_j}$ is an exposure parameter as the first derivation, $\bar{rf}_j$ is a random factor, $a_{ij}$ is a coefficient and $\bar{u}_i$ is a residuum error of particular asset.

The basic random factors are modelled according to the random walk process; covariance and variance are forecasted according to EWMA (exponential weighting moving average) process for expressing heteroskedasticity or GARCH, ARCH, and stochastic volatility models. The EWMA method is usually preferred from the implementation point of view, recurrent formula approach and calculation effectiveness.

We can show the single index model, security market line (SML) for shares, as some simplified examples of the delta financial asset returns

$$\bar{r}_i = \beta_{ij} \cdot \bar{rm}_j + \bar{u}_i,$$

where $\beta_{ij}$ is a beta coefficient exposure of particular asset and market portfolio, $\bar{rm}_j$ is a random market portfolio return (stock exchange index).

A bond return, which is influenced mainly by the term structure (yield) curve, could be modelled as follows:

$$\bar{r}_i = -y_i \cdot \text{MD}_i \cdot \bar{ry} + \bar{u}_i,$$

where $\text{MD}$ is a modified duration, $y$ is a yield and $\bar{ry}$ is a relative yield.

We can introduce the following formula as an example of options instrument:

$$\bar{r}_i = S_i \cdot P_i^{-1} \cdot \text{DELTA}_i \cdot \bar{rS}_i$$

$$+ t_i \cdot P_i^{-1} \cdot \text{THETA}_i \cdot \bar{rt}_i + \bar{u}_i,$$

where DELTA is factor loading to underlying asset ($S$) and THETA is factor loading to time to maturity ($T$).
A portfolio assets type return is of a multivariate normal distribution type under introduced assumptions:

\[ \Delta \Pi = \sum_i \xi_i \cdot \bar{r}_i + \bar{u}_i \]
\[ = \sum_i \xi_i \cdot \sum_j a_{ij} \cdot \bar{r}_f + \bar{u}_i. \]  

(7)

Now, under the above-described assumptions, it is simple to state the mean and variance values of the portfolio derived from Eq. (7). The mean value of asset margin is as follows:

\[ E(\Delta \Pi) = \bar{x}_{k,1} \cdot \bar{A}_{k,n} \cdot E(\bar{F}_{n,1}), \]  

(8)

where \( x \) is an absolute value composition vector of particular assets \( x_i \), \( A \) is coefficient matrix of \( a_{ij} \) coefficients, \( F \) is a vector of basic random factor returns \( \bar{r}_f \). The portfolio variance is formulated like this

\[ \sigma^2(\Delta \Pi) = \bar{x}_{k,1} \cdot \text{COVV}_{k,k} \cdot \bar{x}_{k,1}, \]  

(9)

where \( \text{COVV}_{k,k} = \bar{A}_{k,n} \cdot \bar{C}OVF_{n,n} \cdot \bar{A}_{k,n}^T \), simultaneously COV and COVF are covariance matrix of financial instruments between asset returns and basic random factors returns.

The VAR calculation for the delta approximate method and normal distribution function is calculated according to the well-known formulae:

\[ \text{VAR} = -F^{-1}(\gamma) \cdot \sigma(\Delta \Pi) - E(\Delta \Pi). \]  

(10)

In short-term financial decision making it is supposed that mean value \( E(\Delta \Pi) \) is equal to zero and apparently Eq. (10) is modified because of the symmetry of normal distribution, \( F^{-1}(\gamma) = -F^{-1}(1-\gamma) \), in this way,

\[ \text{VAR} = F^{-1}(1-\gamma) \cdot \sigma(\Delta \Pi). \]  

(11)

This is the well-known simplified formula of VAR methodology, as well.

Now, we can summarise the basic assumptions concerning the analytical delta approximate method as follows. (1) Basic random factor returns have the conditional normal distribution. (2) Variances and covariance are estimated and forecasted in accordance with chosen forecasting methods. (3) Factor loading coefficient \( a_{ij} \) are stated in accordance with Eqs. (3)–(8). (4) The financial instrument returns are expressed by Eq. (3). (5) Under the conditions the portfolio distribution function is multivariate normally distributed with zero mean and conditioned variances and hence the VAR is calculated by the Eq. (11).

4. Risk and vagueness in VAR decision making

Having in mind the basic goal of the paper is to show and apply a fuzzy-stochastic methodology; therefore it is necessary to provide a description of VAR modelling circumstances. We shall introduce assumptions and input parameters that could be difficult to determine precisely either real numbers or distribution functions. This aspect also implies from the decision-making process conditions. The reasons why the VAR methodology parameters could be understood as uncertain imply from the input data uncertainty and the approximate procedures used. There is an excellent description of limitation of value at risk methodology in Jorion (1997). Similar opinion is presented in Pisoult (2000).

The VAR methodology does not provide a measure of worst absolute loss, only some confidence level (risk of exceeding).

The VAR also assumes that the portfolio position is fixed over the horizon (frozen position). This assumption, however, ignores the possibility of changing the trading positions over time in response to changing market conditions (changing position risk).

The VAR models are based on historical data and it is assumed that the recent past is a good projection of future randomness. Some situations where the historical patterns change abruptly will cause havoc with the models. Changing correlation coefficients can lead to drastically different measures of portfolio risk (event and stability risk).

Whenever there is a major change, some possibility of errors exists. This applies for instance to organisational changes, implementation new systems etc. (transitional errors).

All the analytical methods underlying VAR assume that every data are available to measure risk. A meaningful market-clearing process may not exist for some securities (data inadequacy risk).
Some errors in model output can occur due to errors in inputs. Then parameters risk stems from the imprecision in parameters measurement of historical data. Some random errors are bound to happen just because of sampling variation (standard errors) and this problem is often ignored in VAR analysis. Number of observations and number of assets influence this. The bias increases as number of assets increases in relation to number of observations. We face also the problem that higher value of correlation is likely to have been measured with some error probability position. The higher is the correlation coefficient the more likely it will change down (estimation risk).

Users should realise the fundamental trade-off between using more data, which leads to more precise estimates, and focusing on more recent data, which may be safer if the risk changes over the time. The problem is also that the available histories may give a distorted picture of the risk merely due to survival of the time series, because only existing time series are considered (forecasting risk).

A part of the risks are a direct consequence of the model approximation. Predictions will then inevitably produce some errors (implementation risk).

We can summarise the findings in the following way and stress these aspects, which are useful in modelling vague parameters in VAR determination.

1. Probability level because of difficulty to state parameter \( \gamma \) precisely; knowledge of sensitivity of the VAR level \( \gamma \) is useful.
2. Holding period \( T \) should be suitable to consider in some intervals, it means vaguely.
3. A portfolio composition vector \( \vec{x} \) because assumption that a portfolio composition does not remain unchanged throughout the holding period is more realistic; because in a financial planning process a portfolio composition is not often known precisely in the decision-making moment.
4. Instruments are decomposed into simpler instruments, linear approximation.
5. Fixed income instruments are mapped in the chosen vertices and interpolation method is used. Some assets are modelled as a portfolio with identical risk and market value.
6. Multivariate normal distribution approximation is often used.
7. Historical data are used for forecasting and it is assumed that the situation of financial instruments in the future will be similar to the past. Hence, the forecasting is more risky and vague.
8. There are problems with input quality of data, data frequency (expected returns, correlation, volatility) and stochastic validity.

We can assign the first five points to the vaguely structured problem and the last three to the non-precise input data and distribution function (expressed by non-precise distribution parameters).

5. Fuzzy-stochastic model under normal fuzzy sets

Apparently, there are two aspects presented in decision-making process, risk and vagueness. The risk is understood to be a stochastic aspect and we can suppose that a distribution function is known. On the other hand vagueness means that the location and shape of the distribution are opened to question and so it is a function of confidence degree or the weight attached to the probability judgement. Briefly speaking, uncertainty is vague (ambiguous) probability. It is the basic idea and intention how to deal with combination of risk and vagueness. Other parameters are possible to define vaguely, as well. The risk is modelled by random (stochastic) methodology and fuzzy apparatus is used for vagueness modelling. Thus one of the suitable approaches for solving this problem is to apply a fuzzy-stochastic methodology and to create a fuzzy-stochastic model.

Now, there exist several references concerning the combination of fuzzy and stochastic processes (e.g. Dubois and Prade, 1980; Kacprzyk and Fedrizzi, 1988; Luhandjula, 1996; Wang and Qiao, 1993; Viertl, 1996; Zmeskal, 2001). The basic monograph dealing with fuzzy-stochastic aspects modelling with non-precise data is Viertl (1996), here such basic terms as, fuzzy sets, resolution principle, Bayesian inference for non-precise data...
(Section 5) are described etc. In the paper methodology in Wang and Qiao (1993), Viertl (1996), Zmeskal (2001) is primarily followed.

There are several basic fuzzy-stochastic elements, which are very important from an application point of view, (1) fuzzy set, (2) normal fuzzy number, (3) \( \varepsilon \)-cut, (4) extension principle, (5) resolution (decomposition) principle, (6) fuzzy-random variable, (7) fuzzy-probability function.

**Definition 1.** A fuzzy set (depicted with tilde) is commonly defined by a membership function (\( \mu \)) as a mapping from \( E^n \) (Euclidean n-dimensional space, \( n > 1 \)) to a subset of \( E^1 \), particularly the interval [0;1]; we write \( \tilde{\mu} \equiv \mu(x) \), where \( \tilde{\mu} \) is a fuzzy set, \( x \) is a vector and \( x \in X \subset E^n \), \( \mu(x) \) is a membership function.

It is evident that many fuzzy sets could be created. The most common type of the fuzzy set, in \( E^1 \), satisfying conditions of normality, convexity and continuity, is a well-known normal fuzzy number, and also L-R fuzzy number; see Dubois and Prade (1980). These fuzzy sets have quasi-concave membership function shape, in detail; see Ramik and Vlach (2001). If the fuzzy set does not fulfill convexity condition it could be substituted in some situations by convex hull. It was verified in fuzzy applications that this type of the fuzzy set is sufficient and suitable and coincides with human thinking and decision-making conditions. The set of normal fuzzy numbers is denoted by \( F_N(E) \). One of the generalised normal fuzzy numbers type is PW-number.

**Definition 2.** A fuzzy set satisfying conditions of normality, convexity, continuity and closeness denoted by \( \tilde{\mu} = (s^L, s^U, s^a, s^b) \) and defined as follows:

\[
\tilde{\mu}(x) = \begin{cases} 
0 & \text{for } x \leq s^L - s^a; \\
\phi(x) & \text{for } s^L - s^a < x < s^L; \\
1 & \text{for } s^L \leq x \leq s^U; \\
\psi(x) & \text{for } s^U < x < s^U + s^b; \\
0 & \text{for } x \geq s^U + s^b 
\end{cases}
\]

where \( \phi(x) \) are non-decreasing functions and \( \psi(x) \) are non-increasing functions, \( s^L_i + 1 = s^L_i - s^a_i \), \( s^U_i + 1 = s^U_i + s^b_i \), \( i \in \{1; 2; \cdots; n\} \) are limit points of intervals, \( s^a_i, s^b_i \geq 0 \), is called piece-wise fuzzy number, shortly PW-number. Let us denote the set of PW-numbers by \( F_{PW}(E) \), we have \( F_{PW}(E) \subset F_N(E) \). A special case of PW-number is a T-number.

**Definition 3.** Fuzzy set satisfying conditions of normality, convexity, continuity and closeness and being defined as a quadruple \( \tilde{s} = (s^L, s^U, s^a, s^b) \), where \( \phi(x) \) is a non-decreasing function and \( \psi(x) \) is a non-increasing function, as follows:

\[
\tilde{s} \equiv \mu_{s}(x) = \begin{cases} 
0 & \text{for } x \leq s^L - s^a; \\
\phi(x) & \text{for } s^L - s^a < x < s^L; \\
1 & \text{for } s^L \leq x \leq s^U; \\
\psi(x) & \text{for } s^U < x < s^U + s^b; \\
0 & \text{for } x \geq s^U + s^b 
\end{cases}
\]

is called the T-number. Let us denote the set of all T-numbers by \( F_T(E) \), evidently \( F_T(E) \subset F_{PW}(E) \).

An analogy to the convolution principle in probability theory is a very useful and powerful instrument, which might be used for calculating membership function of fuzzy sets is Extension principle (see Zadeh, 1965).

**Definition 4.** The extension principle is derived by the sup min composition between fuzzy sets. Let \( r_1, \ldots, r_n \) be fuzzy sets, \( f : E^n \to E^1 \), then the membership function of a function of fuzzy set \( \tilde{s} = f(r_1, \ldots, r_n) \) is defined by

\[
\mu_s(y) \equiv \tilde{s} = \sup_{x_1, \ldots, x_n} \min[\mu_{r_1}(x_1), \ldots, \mu_{r_n}(x_n)], \\
\text{for } x \in E^n; \mu_s(x) \geq \varepsilon \\
x_1, y \in E^1.
\]

The basic advantage of the Extension principle is its applicability to any type of fuzzy sets. Disadvantages consist in non-possibility to find analytical solution generally in many situations.

**Definition 5.** The \( \varepsilon \)-cut of the fuzzy set \( \tilde{s} \), denoted by \( \tilde{s}^\varepsilon \), is defined as follows. \( \tilde{s}^\varepsilon = \{x \in E^n; \mu_s(x) \geq \varepsilon\} \), \( \tilde{s}^\varepsilon = [\tilde{s}^-, \tilde{s}^+] \), where \( \tilde{s}^- = \inf\{x \in E^n; \mu_s(x) \geq \varepsilon\} \), \( \tilde{s}^+ = \sup\{x \in E^n; \mu_s(x) \geq \varepsilon\} \). Here

\[
\inf\{x \in E^n; \mu_s(x) \geq \varepsilon\} = \{\inf(x_1), \inf(x_2) \ldots \inf(x_n); \mu_s(x) \geq \varepsilon\}, \\
\sup\{x \in E^n; \mu_s(x) \geq \varepsilon\} = \{\sup(x_1), \sup f(x_2) \ldots \sup(x_n); \mu_s(x) \geq \varepsilon\}.
\]
If the fuzzy set is not convex, it could be transformed into the convex hull.

**Definition 6.** The convex hull of fuzzy set $\tilde{s}$ is the smallest convex fuzzy set containing $\tilde{s}$.

The $\varepsilon$-cuts are suitable for creating fuzzy numbers.

**Definition 7.** The resolution (decomposition) principle for the fuzzy number construction by approximate procedure of $\varepsilon$-cuts is defined as follows:

$$\mu_\varepsilon(y) = \sup\{\varepsilon; y \in \tilde{s}\}$$

for any $y \in E^n$ and $\varepsilon \in [0; 1]$, where $\tilde{s} = [-\tilde{s}^-, +\tilde{s}^+]$ is $\varepsilon$-cut. Here

$$(\varepsilon; y \in \tilde{s}) = \begin{cases} 
\varepsilon & \text{if } y \in [-\tilde{s}^-, +\tilde{s}^+] \\
0 & \text{if } y \notin [-\tilde{s}^-, +\tilde{s}^+] 
\end{cases}.$$ 

We could say that there are often some difficulties in solving practical complex problems in fuzzy environment that the analytic solution according to the extension principle is not available. Assuming a fuzzy set is of the fuzzy number type (piecewise number, T-number, L-R number as well), there is a possibility to solve the function of fuzzy numbers $\tilde{s} = f(\tilde{r}_1, \ldots, \tilde{r}_n)$ by the resolution (decomposition) principle as the approximate procedure of $\varepsilon$-cuts.

**Definition 8.** The resolution (decomposition) principle for fuzzy numbers function construction by approximate procedure of $\varepsilon$-cuts, $\mu_\varepsilon(y) \equiv \bar{s} = f(\mu_{\tilde{r}}(x_1), \ldots, \mu_{\tilde{r}}(x_n))$, where $x_i, y \in E^n$, is defined as follows:

$$\bar{s} = \sup\{\varepsilon; y \in \tilde{s}\}$$

for any $y \in E^n$ and $\varepsilon \in [0; 1]$, where $\tilde{s} = [-\tilde{s}^-, +\tilde{s}^+]$ is an $\varepsilon$-cut function.

$$\tilde{s}^-(x) = \min_{x \in E^n} f(x), \quad \tilde{s}^+(x) = \max_{x \in E^n} f(x).$$

In Vierl (1969) this function is named $\varepsilon$-level function. Here

$$(\varepsilon; [-\tilde{s}^-(x), +\tilde{s}^+(x)]) = \begin{cases} 
\varepsilon & \text{if } x \in [-\tilde{s}^-, +\tilde{s}^+] \\
0 & \text{if } x \notin [-\tilde{s}^-, +\tilde{s}^+] 
\end{cases}.$$ 

An interesting feature of Definition 8 is that it is possible to create any fuzzy function from $\varepsilon$-cut functions. It is apparent that applying Definition 8 the function of fuzzy numbers could be constructed by solving several mathematical programming problems for $\varepsilon$ in the following way:

**Problem P1**

$$\max(\min)s = +\tilde{s}^+, (-\tilde{s}^-),$$

s.t. $$s = f(x_1 \ldots x_n),$$

where $x_i \in [-x_i^-, +x_i^+]$ for $i \in \{1; 2; \ldots n\}$ and $\varepsilon \in [0; 1].$

The advantage of the procedure can be seen in generalised application possibility for normal (convex) fuzzy numbers. Some disadvantage consists in a computation laborious and confining only for fuzzy numbers.

The crucial category in fuzzy-stochastic modelling is the fuzzy-random variable.

**Definition 9.** A mapping, $\tilde{s} : \Omega \rightarrow F_N(E)$ is said to be the fuzzy-random variable (denoted by tilde and line), if for every $w \in \Omega$ and $\varepsilon \in [0, 1]$, $\tilde{s}_w = \{x : x \in E^n, \tilde{s}_w \geq \varepsilon\}$ is a random interval (random $\varepsilon$-cut), $-\tilde{s}^-, +\tilde{s}^+$ are random as well. Let us denote the set of all fuzzy-random variables by FR($\Omega, P$), where $P : \Omega \rightarrow [0, 1]$ and thus $\tilde{s} \in FR(\Omega, P)$.

It implies from definition that,

(a) $\tilde{s} = \cup_w \tilde{s}_w$, because $\forall w \in \Omega, \tilde{s}_w = \cup_w \tilde{s}_w$, where $\tilde{s}_w$ is fuzzy set and $\tilde{s}_w$ is random interval,

(b) $\tilde{s}$ is a fuzzy-random variable iff $\tilde{s}_w(w)$ is a random interval, it means $\tilde{s} = \cup_w \tilde{s}_w$ for every $w \in \Omega$ and $\varepsilon \in [0, 1]$.

The following definition is useful for evaluating and computing the functions of fuzzy-random variables and for constructing fuzzy-probabilities and fuzzy-expected values.

**Definition 10.** Let us suppose that $\tilde{s} \in FR(\Omega, P)$,

$$\tilde{s}_w = [-\tilde{s}_w^-, +\tilde{s}_w^+], \varepsilon \in [0; 1], x \in E^n.$$

Let $\tilde{g}_w$ (resp. $\tilde{g}^c_w$) be a function of $-\tilde{s}_w^-$ (resp. $+\tilde{s}_w^+$). Then we call $\mu_\varepsilon(y) = \sup\{\varepsilon; y \in \tilde{g}\}$ the fuzzy probability function in accordance with Definitions 8 and 9, simultaneously $\tilde{g} = [-\tilde{g}^-(x), +\tilde{g}^+(x)]$ is an $\varepsilon$-cut probability function.

6. Illustrative example of fuzzy-stochastic VAR calculation

Since the purpose of the paper is to show an application possibility of the fuzzy-stochastic
approach for stating VAR under soft conditions, the
following example describes a possibility of estimation the delta approximate VAR under
uncertain (risk, vagueness) circumstances. For the
sake of comparison the traditional crisp-stochastic
approach is computed.

We assume to have four assets (home stock, for-
573 eign stock, bond 2 year, bond 3 year) and five
basis random risk factors (home stock exchange
index, foreign stock exchange index, exchange rate, bond 2 year yield, bond 3 year yield). In the
model the delta approximation of stocks due to
Eq. (4) and bonds due to Eq. (5) are used.

The reasons for an application of soft input
data are described in the last paragraph of Section
4. For the sake of simplicity non-precise input data
are introduced as the linear T-numbers. It is not
difficult to solve the problem with piecewise fuzzy
numbers, of course.

The calculation of VAR is based on delta nor-
434 mal approximate method (Section 3) and pre-
sented by Eq. (11), which is supported by Eq. (9).
The fuzzy-stochastic methodology is described in
Section 5 and the resolution principle of function
by procedure of Section 5 and the resolution principle of function
for Eq. (11), which is supported by Eq. (9) and is for-
434 mulated as resolution (decomposition) principle
(11), which is supported by Eq. (9) and is for-
434 mulated as resolution (decomposition) principle

\[ \text{Problem P2} \]

\[
\begin{align*}
\text{max}(\min) \varpsilon & \text{VAR} \equiv \pm \text{VAR}^\varepsilon, (-\text{VAR}^\varepsilon) \\
& \text{for } \varepsilon \in [0, 1], \\
\text{s.t. } \varpsilon & = F^{-1}(1 - \gamma) \cdot \bar{x}_{k,1} \cdot \bar{A}_{k,n} \\
& \cdot \bar{C}OVF_{n,n} \cdot \bar{A}_{k,n}^T \cdot \bar{x}_{k,1},
\end{align*}
\]

where

\[
\begin{align*}
\bar{C}OVF_{n,n} & \in \left[ -\bar{C}OVF^\varepsilon_{n,n}, +\bar{C}OVF^\varepsilon_{n,n} \right], \\
\bar{A}_{k,n} & \in \left[ -\bar{A}^\varepsilon_{k,n}, +\bar{A}^\varepsilon_{k,n} \right], \\
\bar{x}_{k,1} & \in \left[ -\bar{x}^\varepsilon_{k,1}, +\bar{x}^\varepsilon_{k,1} \right], \quad T \in \left[ -\bar{T}^\varepsilon, +\bar{T}^\varepsilon \right], \\
\gamma & \in \left[ -\gamma^\varepsilon, +\gamma^\varepsilon \right].
\end{align*}
\]

6.1. Input data of the fuzzy-stochastic approach

Fuzzy input data are given by the following way. The fuzzy vector portfolio composition \( \bar{x}_{k,1} \), expiry fuzzy period \( \bar{T} \) and probability (risk) fuzzy level \( \bar{\gamma} \) are given subjectively by a financial analyst.

The fuzzy covariance factor matrix \( \bar{C}OVF_{n,n} \) is determined similarly by methodology described in Zmeskal (2001) applying the statistical error due to the simplified formula \( \hat{\sigma}^2 \approx N(\sigma^2; \sigma^4 \frac{\hat{\gamma}}{T^2}) \). Simultaneously quintals of statistical error distribution determine a shape of the linear T-number, \( s^\varepsilon = 40\%, s^U = 60\%, s^L = s^u = 5\%, s^U + s^b = 95\% \).

A similar approach is shown in Rommelfanger (1999). Another possible way of fuzzy number construction of exponential type from a normal probability distribution is described in Inuiguchi and Ramík (2000). The fuzzy exposure matrix \( \bar{A}_{k,n} \) is derived from validity of parameters and an analyst’s judgement.

Another possibility is to create a fuzzy number from a combination of fuzzy number variants as expert’s estimations by the formula,

\[
\mu(x) = \sup \min \left( \mu_1(x), \mu_2(x) \cdots, \mu_n(x) \right).
\]

If the calculated fuzzy set is not a fuzzy number, it is necessary to use the convex hull approach for transformation (Definition 6), then the fuzzy set is normal.

An analyst can introduce some input data by
linguistic terms, as well. Input data matrices of
linear T-numbers have the following form:

\[
\bar{S} = \begin{bmatrix}
S^L & S^U \\
S^L & S^b
\end{bmatrix}.
\]
6.2. Input data of crisp-stochastic approach

We suppose that the covariance matrix is identical with $S_L$ part of fuzzy COVF matrix, the matrix $A$ is identical with $S_L$ part of fuzzy matrix $A$, a portfolio composition coincide with $S_L$ part of fuzzy vector $x$ and $\gamma = 0.05$, $T = 0.3$ are $S_L$ parts of fuzzy parameters.

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>$A$</th>
<th>$B$</th>
<th>$C$</th>
<th>$D$</th>
<th>$E$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Home stock</td>
<td>0.02250</td>
<td>0.02625</td>
<td>0.01200</td>
<td>-0.00540</td>
<td>-0.00525</td>
</tr>
<tr>
<td>Foreign stock</td>
<td>0.02625</td>
<td>0.06250</td>
<td>0.01500</td>
<td>-0.00900</td>
<td>-0.00875</td>
</tr>
<tr>
<td>Exchange rate</td>
<td>0.01200</td>
<td>0.01500</td>
<td>0.01000</td>
<td>-0.00270</td>
<td>-0.00140</td>
</tr>
<tr>
<td>Yield 2 years</td>
<td>-0.00540</td>
<td>-0.00900</td>
<td>-0.00270</td>
<td>0.00810</td>
<td>0.00567</td>
</tr>
<tr>
<td>Yield 3 years</td>
<td>-0.00525</td>
<td>-0.00875</td>
<td>-0.00140</td>
<td>0.00567</td>
<td>0.00490</td>
</tr>
</tbody>
</table>

6.3. Comparison of fuzzy-stochastic and crisp-stochastic solutions

Calculated uncertain VAR for $\varepsilon$-cuts with numbers 0, 0.25, 0.5, 0.75 and 1.0 are clear from Fig. 1 and Table 1. It is obvious that the minimum loss will range from 9.75 to 21.05 for $\varepsilon$-cut 0 and from 13.86 to 15.49 for $\varepsilon$-cut 1.00 under the written
prepositions. This result might be understood as the fuzzy-probability decision-making space and information about sensitivity of the VAR value on input data vagueness too. For the sake of comparison, crisp-stochastic solution is presented too; VAR value is 13.88. It shows that in this example the result is not in the centre of T-number, and potential loss is underestimated.

Applying Definitions 8 and 10, it is suitable to construct fuzzy-probability functions. This interesting information gives fuzzy-probability \( \epsilon \)-cuts functions for VAR and sensitivity of VAR (see Fig. 2).

### 7. Conclusion

Recently, the methodology of Value at Risk is now under deep research and application verification. One of the aspects studied is a possibility to introducing various preciousness of data and applying an approximate and simplified methodology. Unfortunately, in many situations it is very difficult to determine these inputs because the conditions and circumstances of financial decision making are often uncertain (see Section 4). Such a situation is typical for emerging markets and transitional economies. On the other hand it is useful to use the distorted probabilities or the vague distribution functions (ambiguity distributions). The application of the fuzzy-stochastic methodology with the normal fuzzy numbers is one of practical possibilities how to solve this problem. The application of the methodology was presented for the sake of simplicity on the analytical delta approximate method. The described approach might be seen as the generalised sensitivity analysis of calculation the VAR value under imprecise input data and decision-making conditions. The fuzzy-stochastic approach described is suitable for ill structured and non-stable situations with vaguely stated input data and probability distributions. On the other hand the crisp-stochastic approach could be applied if well structured problem with precisely data and probability exists. These two approaches are not in contradiction, but we can consider them to be complement. Further research might concern the simulation technique application under soft (fuzzy-stochastic) conditions.

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References