Pinched hysteretic loops of ideal memristors, memcapacitors and meminductors must be ‘self-crossing’

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Recently, novel findings have been published, according to which some mem-systems excited by harmonic signals can be characterised by the so-called ‘non-crossing-type pinched hysteretic loops’. Presented is a proof that this phenomenon cannot occur in ideal memristors, memcapacitors and meminductors which are defined axiomatically via the corresponding constitutive relations or via other equivalent characteristics, and that the ‘crossing-type hysteretic loop’ is thus one of their typical fingerprints.

Introduction: The nth-order u-controlled memory element (mem-element for short) is defined in [1] by the set of equations

\[ y(t) = g(x, u, t)u(t), \quad x(t) = f(x, u, t) \]  

where y and u are the terminal circuit variables, x is the n-dimensional vector of internal state variables, and \( \dot{x} \) is its derivative with respect to time. The symbols g and f denote the scalar and vector functions of variables x and u, respectively. Time is an additional argument, indicating that the above equations describe a general case of a time-varying mem-system where the given functions of variables x and u are not fixed but they vary with time.

It is shown in [1] that for properly selected variables u and y, (1) represent models of general memristive, memcapacitive, and meminductive systems. The ideal memristor (MR), memcapacitor (MC), and meminductor (ML) are their specific cases. Equations (1) for stationary (time-invariant) elements denoted VCMR (voltage-controlled memristor), CCMR (current-controlled memristor), VCMC (voltage-controlled memcapacitor), QCMC (charge-controlled memcapacitor), CCML (current-controlled meminductor), and FCML (flux-controlled meminductor), which are special first-order mem-systems, are summarised in Table 1. The symbols \( v, i, q, \phi, p, \) and \( \sigma \) denote voltage, current, charge, flux, time-domain integral of charge, and time-domain integral of flux [2, 3]. The symbols \( G_M, R_M, C_M, D_M, L_M, \) and \( A_M \) represent memductance, memristance, memcapacitance, inverse memcapacitance, meminductance and inverse meminductance.

<table>
<thead>
<tr>
<th>( y )</th>
<th>( u )</th>
<th>( x )</th>
<th>( g(x,u,t) )</th>
<th>( f(x,u,t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>VCMR</td>
<td>( v )</td>
<td>( q )</td>
<td>( G_M(q) )</td>
<td>( v )</td>
</tr>
<tr>
<td>CCMR</td>
<td>( v )</td>
<td>( i )</td>
<td>( R_M(q) )</td>
<td>( i )</td>
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<tr>
<td>VCMC</td>
<td>( q )</td>
<td>( v )</td>
<td>( C_M(u) )</td>
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<td>QCMC</td>
<td>( q )</td>
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<td>( D_M(q) )</td>
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<tr>
<td>CCML</td>
<td>( q )</td>
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<td>( L_M(q) )</td>
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<tr>
<td>FCML</td>
<td>( q )</td>
<td>( \sigma )</td>
<td>( A_M(u) )</td>
<td>( \sigma )</td>
</tr>
</tbody>
</table>

It follows from Table 1 that the general equations (1) can be reduced to the above ideal mem-elements to the simple form

\[ y(t) = g(x)u(t), \quad \dot{x}(t) = u \]  

or, after the reduction to a single equation between the quantities u and y,

\[ y(t) = g \left( \int u(\alpha) d\alpha \right) u(t) \]  

Since the general memristive systems are frequently (but not accurately) denoted memristors, and because such trends may also appear for meminductors and memcapacitors, the elements of MR, MC, and ML types from Table 1 are denoted ‘ideal’ in the following text. In this manner, we differentiate between these elements, defined by their specific constitu- tive relations or by equivalent characteristics of the type of g(x), and general mem-systems, which lack some of the typical fingerprints of the elements from Table 1.

Consider the mem-system (1) excited by the variable u in the form of harmonic signal

\[ u(t) = U_{\text{max}} \sin(\Omega t) \]  

where \( U_{\text{max}} \) is the amplitude, \( \Omega = 2\pi/T \) is the repeating angular frequency, and T is the repeating period. For the mem-system in periodical steady state, a typical pinched hysteretic loop appears in the y-u coordinates. According to [1], this loop can be of type I (self-crossing) or type II (not self-crossing), see Figs. 1a and b. Points \( \bullet, \circ, \Delta, \) and \( \Theta \) are placed on the loop and labelled by the time instants at which the operating point, drawing this loop, arrives at these points. The type II loop is a new phenomenon which appears in neither classical [4, 5] nor recent [6] works of Professor Chua. Such loops were observed, for example, in the thermistor or in an elastic memcapacitive system [1]. There are some notes in [1] about possible relations between the loop type and the properties of functions f and g in equations (1) which are numbered (4) and (5) in [1]:

The symmetry of equations (4–5) does not always define the type of crossing. However, as it follows from examples considered in this review, ‘not self-crossing’ loops are very often observed when g(x; u) and f(x; u) are even functions of u. We emphasise that this is not a necessary condition for ‘not self-crossing’.

**Table 1**: Specification of variables in (1) and (2) for ideal memristors (MR), memcapacitors (MC), meminductors (ML)

**Fig. 1** Hysteretic loops of type I and type II

- a) Type I \( b_1 = 1, b_2 = 200 \text{ m}, a_1 = 0, a_2 = 0, a_3 = 0 \)
- b) Type II \( b_1 = 1, b_2 = 0, a_1 = 200 \text{ m}, a_2 = 0, a_3 = 0 \)

The purpose of this Letter is to give a proof that ideal memristors, memcapacitors, and meminductors cannot generate pinched hysteretic loops of type II, and that researching this phenomenon should thus be focused on more general mem-systems.

Note that the hysteretic loops of general mem-systems need not be the way they are shown in Fig. 1, i.e. symmetrical in the first and third quadrants. However, such symmetry is guaranteed for all ideal mem-elements as shown below.

**Pinched hysteretic loops of ideal mem-elements are always odd-symmetric**: It is sufficient to prove that the ideal mem-element which is described by (3) and which is excited by signal u(t) according to (4), always generates the response y(t) as an odd function of time. Substituting (4) into (3) and integrating u(t) yield

\[ y(t) = g \left( C - \frac{U_{\text{max}}}{\Omega} \cos(\Omega t) \right) U_{\text{max}} \sin(\Omega t) \]  

where C is the constant of integration, dependent on the initial value of the state variable x in (2). Then it follows from (5) that \( y(-t) = -y(t) \) for \( t \in R \), and that y(t) is thus an odd function.
Rules for type I and II loops: Let us consider the variable \( y(t) \) with its repeating period \( T \) in the form of the Fourier series

\[
y(t) = a_0 + \sum_{k=1}^{\infty} [a_k \cos(k \Omega t) + b_k \sin(k \Omega t)]
\]

where \( a_k, b_k \) are real coefficients. Let us derive the conditions of the existence of symmetry of the hysteretic loops in Fig. 1. The symmetry condition of the loop in Fig. 1a can be described as follows: for an arbitrary time \( t = t_1 \), it holds (check the \( y \)-axis co-ordinates of the pairs of points \( \ominus, \oplus \) or \( \ominus, \oplus \));

\[
y(t) = -y(T - t) = -y(-t)
\]

The latter equality results from the periodicity of \( y(t) \). If a type I loop is to be symmetrical in the above sense, then the response of the element to the excitation (4) must be an odd function of time. This is, however, a natural feature of the ideal mem-element. Condition (7) also implies that the periodic odd function \( y(t) \), drawing the hysteretic loop of type I together with sine-type excitation \( u(t) \) can contain neither cosine-type spectral terms nor a DC term.

As is obvious from Fig. 1b, another type of symmetry is valid for the hysteretic loop of type II. For an arbitrary time \( t = t_1 \), it holds (check the \( y \)-axis co-ordinates of the pairs of points \( \ominus, \ominus \) or \( \oplus, \oplus \));

\[
y(t) = -y(t + T/2)
\]

Expressing both sides of (8) by the Fourier series (6) and a simple rearrangement yield

\[
\sum_{k=1}^{\infty} a_{2k} \cos \left( k + \frac{1}{2} \right) \Omega t + \sum_{k=1}^{\infty} b_{2k} \sin \left( k + \frac{1}{2} \right) \Omega t = -2a_0
\]

The first left-side sum describes the even function of time whereas the second sum is an odd function. Their sum will be constant for all the time instants if

\[
a_0 = 0, a_{2k} = b_{2k} = 0, k = 1, 2, 3, \ldots
\]

The above formulae mean that for a hysteretic loop to be of type II, there must not be any even harmonic components or a DC component present in the spectrum of the response \( y(t) \).

Let us conclude that the response \( y(t) \):

- Cannot contain the DC component for either type of loop.
- Must contain only sine-type harmonic components for the loop of type I.
- Must contain only odd harmonic components for the loop of type II. The presence of the cosine-type components is necessary for the formation of the hysteretic effect. Otherwise, the response \( y(t) \) would be formed only by odd sine-type components which would result in the well-known effect of degenerating the hysteretic loop into an unambiguous function [6].

The above conclusions are in accordance with the conditions used for the simulations shown in Fig. 1. Loop \( a \) is of type I, and the response \( y(t) \) contains the first two sine-type harmonic components. Loop \( b \) is of type II, and \( y(t) \) contains only odd harmonics. The presence of cosine-type components in the response for the loops of type II confirms that \( y(t) \) cannot be an odd function of time. That provides the proof that ideal mem-elements cannot generate type II pinched hysteretic loops.

Conclusion: A memristor as the fourth fundamental circuit element, axiomatically introduced in the circuit theory by the unambiguous non-linear constitutive relation between charge and flux [4], or by memristance-charge or memductance-flux relation [6], cannot generate pinched hysteretic loops of type II. This means that the loop must have a ‘crossing property’ at the origin of the system of \( i-v \) co-ordinates. This rule can be also extended to a memcapacitor which is defined by the unambiguous relation between the flux and the time-domain integral of charge, and to a meminductor with the constitutive relation between the charge and the time-domain integral of flux. The ‘crossing property’ thus extends the well-known set of specific fingerprints of ideal mem-elements.

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