The consensus models with interval preference opinions and their economic interpretation

Zaiwu Gong\textsuperscript{a,b,*}, Xiaoxia Xu\textsuperscript{b}, Huanhuan Zhang\textsuperscript{b}, U. Aytun Ozturk\textsuperscript{b}, Enrique Herrera-Viedma\textsuperscript{c,d}, Chao Xu\textsuperscript{b}

\textsuperscript{a}Collaborative Innovation Center on Forecast and Evaluation of Meteorological Disasters, Nanjing University of Information Science and Technology, Nanjing 210044, China
\textsuperscript{b}School of Economics and Management, Nanjing University of Information Science and Technology, Nanjing 210044, China
\textsuperscript{c}Department of Computer Science and Artificial Intelligence, University of Granada, Granada 18071, Spain
\textsuperscript{d}Department of Electrical and Computer Engineering, Faculty of Engineering, King Abdulaziz University, Jeddah 21589, Saudi Arabia

\begin{abstract}
This paper aims to explore the case when an individual opinion is interval preference in consensus decision making. And for this purpose, we construct two multi-objective optimization models: one based on the minimum cost from the perspective of the moderator, the other the maximum return from the perspective of the individuals. On the basis of multi-objective programming theories, these multi-objective programming models are then transformed into two single-objective linear programming models, i.e., the primal model and the dual model. The primal model focuses on how to obtain a consensus with the minimum cost, while the dual model is concerned with how to get the maximum return. With the help of dual linear programming theories, we have revealed the following economic significance of the primal-dual consensus models: the primal-dual consensus models can not only help us probe into the relations between the minimum cost paid by the moderator and the maximum return expected by individuals who changed their opinions before, but also help us explore the relations between the unit cost that the moderator pays each individual, unit return that each individual receives, each individual opinion and the consensus opinion. This paper with the aid of theoretical analysis and an illustrative example indicates that once the consensus is obtained, the optimal unit return and optimal consensus opinion value are also solved. This paper also points out that the amount of the total return acquired by all the individuals who have abandoned their original opinions before is equivalent to that of the total cost paid by the moderator to reach the consensus. This paper also argues that there exists compact correlations between the individual’s unit return, the consensus opinion, the individual’s interval opinion, and the moderator’s unit cost.
\end{abstract}

\section{Introduction}

Group decision making (GDM) mainly focuses on unstructured decision making problems that require experts' subjective judgment \cite{1-4}. Generally, experts consist of lots of individual decision makers (DMs) who represent different interests, values and preferences. And therefore, how to arrive at a consensus has become a hot topic in GDM research. The definition of consensus has been varied widely. The American Heritage Dictionary defines consensus as “an opinion or position reached by a group as a whole”. Bezdek et al. \cite{5} and Spillman et al. \cite{6} interpreted consensus as “a full and unanimous agreement”. In fact, these two definitions are concerned with the final outcome, while Ref. \cite{7-9} stress the evolutionary process of reaching consensus. Besides, Ness et al. \cite{10} take consensus to represent the case where most DMs “agree on a clear option”, the few DMs who oppose this option provide rational and essential suggestions, and eventually, all the DMs “agree to support the decision”. In addition, Steve \cite{11} divided consensus into two categories: accidental consensus (i.e., the theory chosen by all individual DMs based on their own independent judgment) and essential consensus (i.e., the viewpoint determined by collective negotiations and discussions). Obviously, how to reach the consensus in the latter category actually involves a multistage process: each individual

\footnotesize
\begin{itemize}
\item This manuscript was processed by Associate Editor Doumpos.
\item * Corresponding author at: Collaborative Innovation Center on Forecast and Evaluation of Meteorological Disasters, Nanjing University of Information Science and Technology, Nanjing 210044, China.
\item E-mail address: zwgong26@163.com (Z. Gong).
\end{itemize}

\url{http://dx.doi.org/10.1016/j.omega.2015.03.003}

\copyright 2015 Elsevier Ltd. All rights reserved.
changes their opinions gradually and all the views tend to be unanimous after many rounds of discussion [12,13]. More importantly, the above evolutionary process usually needs a worthy, effective and efficient moderator (i.e., super-individual defined by [14]) who dominates the whole process of consensus reaching and has strong team leadership and communication skills to convince individuals to change their opinions into an acceptable option [12,14]. Therefore, this research is based on the following hypothesis: a moderator [15,16] introduced in GDM is able to persuade DMs to change their opinions towards a consensus opinion by paying costs (i.e., consuming resources such as time or money), and DMs’ opinions will gradually approach the consensus.

Actually, consensus decision making involves a multi-objective problem-solving process, in which each individual is the minimum decision making unit, and his/her opinion can be regarded as a sub-objective. In the multi-objective optimization theory, a non-inferior (but not globally optimized) solution that meets most goals is a satisfying choice and is good enough for many real-life situations. Generally, multi-objective problems are often transformed into single objective problems with some decision rules [17]. Such aggregation operators in GDM as the weighted averaging (WA) operator [18], the ordered weighted average (OWA) operator [19,20], the power average (PA) operator [21], and the probabilistic weighted average operator [22], representing decision rules in mathematics, are determined by DMs or the moderator under specific GDM background. For example, Ben-Arieh and Chen [23] presented a method which aggregates experts’ judgements into a collective opinion with the fuzzy linguistic OWA operators and calculates the consensus level. Zhang [24] adopted several new hesitant fuzzy aggregation operators to solve multiple attribute GDM problems in hesitant fuzzy environments. His research incorporates both the decision arguments and the relationships between them. Furthermore, Parreiras et al. [25] introduced a flexible consensus scheme for multi-criteria GDM under linguistic assessments based on fuzzy aggregation models. In fact, the optimal solution to a single objective problem is actually a Pareto optimal (non-inferior) solution to the corresponding multi-objective problem [26], meaning that an ideal consensus reached in GDM is merely an optimal solution in the single-objective decision making sense, or merely a Pareto optimal solution in the multi-objective decision making sense.

In 2007, Ben-Arieh and Easton [15] introduced the concept of minimum cost consensus, constructed a multi-criteria consensus model under linear cost opinion elasticity, and presented linear-time algorithms to find the minimum cost consensus. They then generalized their work to derive new algorithms for reaching it [16]. The minimum cost consensus model by Ben-Arieh and Easton [15,16] is to aggregate the deviations between an individual’s opinion and consensus opinion using a weighted arithmetical mean operator and to construct an optimization programming model based on it. In fact, the consensus model, proposed by Ben-Arieh et al., stems from Gonzalez-Pachon and Romero’s distance-based goal programming (GP) models [4,27,28]. The widely used standard GP model, brought forward by Charnes and Cooper [29], aims to minimize the deviations attached to the goal and the aspiration levels determined by DMs. As a result, Gonzalez-Pachon and Romero’s findings have laid a theoretical foundation for the research of Ben-Arieh et al. on the construction of the minimum cost consensus models.

The earlier consensus research based on total cost only takes into account the moderator’s point of view, while in fact, individuals have to continuously modify their opinions to obtain a compromise consensus [28], and therefore it is necessary to take their interests into consideration as they deserve to be compensated. A case in point is the realization of IPCC (Intergovernmental Panel on Climate Change) consensus, which is obtained through multiple rounds of negotiations and constant compromises among political groups as well as experts in various fields. In the process, the consensus will be unilateral, or may be swayed by sectional groups if only the viewpoints of some center groups (i.e., moderator) are considered. And hence, to obtain a more scientific IPCC consensus, we should consider the voice of the developing countries or relatively weak groups (i.e., individual DM) who have abandoned their own interests to arrive at the generally accepted result. As an exchange, they will expect some compensation (e.g., lower taxes or more carbon quotas) and the more the better. Therefore, this paper constructs two linear optimization models based on the minimum cost and maximum return to explore from both the moderator’s and the individual DMs’ perspectives the cost consensus problems in GDM. With the help of the primal-dual linear programming theory, we analyze the relationships between all the variables and present the economic interpretation of the models proposed. This paper is an extension of the minimum cost consensus model presented by Ben-Arieh and Easton [15,16].

The rest of this paper is arranged as follows. Section 2 introduces the distance measure on consensus decision making and then proposes two multi-objective programming models based on minimum cost and maximum return. Section 3 establishes the minimum cost and maximum return models for reaching consensus, which are based on primal-dual theory; and also explores the economic significance of these two models. Section 4 investigates the properties of these two primal-dual linear programming models and further discusses the theoretical meaning and economic significance of the primal-dual models. To further explain the proposed models, we provide an illustrative example in Section 5. Lastly, the conclusion and directions for future research are provided in Section 6.

2. Multi-objective programming models based on minimum cost and maximum return

Suppose that there are m decision makers (DMs) \(D = \{d_1, \ldots, d_m\}\) taking part in a GDM. Let \(o_i, o_e \in R\), represent the opinion of DM \(d_i\), \(i = 1, \ldots, m\), in the GDM. Each opinion of an individual DM represents an individual interest, so we define it to be an individual opinion. In consensus decision making, the ideal state is where there exists an ideal opinion \(o^\ast\) such that \(o_i = \ldots = o_m = o^\ast\). That is, when all opinions are equal to the same ideal opinion \(o^\ast\), the group has arrived at unanimity. Such an ideal opinion in fact represents the collective interest, so we define it as a consensus opinion \((15,16)\) define it when all DMs have the same current group opinion). In reality, it is difficult to obtain such a perfectly identical opinion, even if all individuals have similar values, backgrounds, abilities, knowledge structures, experiences, and so on. On the one hand, to arrive at a consensus, the moderator in GDM believes that he/she can persuade each individual to change his/her opinion to an ideal value by paying a cost (consuming resources such as time or money). On the other hand, all individuals expect to obtain return for changing their opinion to the ideal one (the consensus opinion). In other words, during the process of reaching consensus, the moderator expects to pay a cost to obtain consensus and each individual hopes to receive compensation because he/she has sacrificed his/her interests for the collective interest. In consequence, we construct a deviation function \(f\) to measure the changes between an individual opinion and the consensus opinion. Meanwhile, a unit cost \(w\) is paid by the moderator to persuade each individual to change his/her opinion. The moderator represents the collective interest, so it is natural that he/she hopes to pay as little as possible, while the individual cares for his/her own interest, and so obviously expects to gain return as large as possible.
Let \( f_i(o') \) be the deviation between the opinion \( o_i \) of DM \( d_i \), \( i \in M = \{1, 2, ..., m\} \), and the consensus opinion \( o' \). It is suggested by Gonzalez-Pachon and Romero [27,28] and Cook et al. [30] to denote such deviation with the \( P \)-metric (\( p \in [1, +\infty) \)) distance function. Especially, when \( p = 1 \), we get the simplest expression of \( f_i(o') = |o' - o_i| \) [31], where \( o' \) represents the best consensus opinion under the majority principle. Moreover, when measuring the degree of consensus, this distance metric has two important merits: it intuitively expresses the difference between the individual opinion and the consensus opinion; more importantly, this distance form, which can be easily transformed into a linear form, makes it more convenient to compute the consensus opinion, and much easier to construct the dual models and explore the economic significance of the following models proposed. Clearly, the smaller the \( f_i(o') \) is, the closer the individual opinion and the consensus opinion will be. Let \( F(o') = [f_1(o'), f_2(o'), ..., f_m(o')] \), then \( \min F(o') \) is a multi-objective optimization problem, it denotes the minimum distance between each individual opinion and the consensus opinion, where \( o' \in O \), and \( O = \{o \in R | o' > 0\} \) is the set of all possible opinions of individuals. If there exists an \( o^* \in O \) such that the goal \( f_i(o') \), for any \( i \in M \), attains the minimum value 0, then \( o^* \) is the optimal solution to \( F(o') \). Here, we call \( o^* \) the optimal consensus opinion. In consequence, \( F(o') \) is actually a multi-objective programming function such that every distance \( f_i(o') \) attains the minimum value. \( F = [f_i(o') | o' \in O] \) is the set of distances between each individual opinion and the consensus opinion.

When an individual changes his/her opinion, the moderator pays for his/her loss according to the deviation degree \( f_i(o') \). Let \( w_i \) denote the unit cost that the moderator is willing to pay to individual \( d_i \) to achieve consensus opinion, then \( w_i f_i(o') \) denotes the total cost paid individual \( d_i \). Let \( \min G(o') = [w_1 f_1(o'), w_2 f_2(o'), ..., w_m f_m(o')] \) denote the minimum cost that the moderator pays each individual. From the moderator’s point of view, the smaller the cost paid to each individual, the higher the degree of consensus. Thus a multi-objective programming model based on minimum cost can be constructed as follows:

\[
\begin{align*}
(MOP_1) \quad & \min \ G(o') = [w_1 f_1(o'), w_2 f_2(o'), ..., w_m f_m(o')] \\
& \text{s.t. } o' \in O \tag{1}
\end{align*}
\]

where \( O = \{o' \in R | o' > 0\} \) is the set of all possible opinions. We define the set \( G = \{w_i f_i(o') | o' \in O\} \) to be the total costs paid to each individual.

From the individuals’ viewpoint, as they are required to change their valuable opinions \( o_i \) to conform to the consensus opinion \( o' \), so they deserve to be compensated. For this reason, we suppose that they gain unit return \( y_i \) for opinion deviation between \( o_i \) and \( o' \). Hence, the question arises as to what an appropriate value of unit return \( y_i \) is that contributes to reaching consensus. In the next section, we introduce a useful shadow value [32] that measures whether the unit \( y_i \) is reasonable. In other words, a reasonable unit return \( y_i \) denotes that the individual’s return is rational, the moderator’s cost is justifiable, and it contributes to consensus arrival. If \( y_i, i \in M \), denotes the unit return that \( d_i \) is expecting to obtain, then \( (o_i - o') y_i \) denotes the total return that \( d_i \) is expecting to obtain for changing his/her opinion to consensus opinion. For all individuals, the greater the value \( (o_i - o') y_i \), the higher the total return expected by \( d_i \). Thus a multi-objective programming model based on maximum return can be constructed as follows:

\[
\begin{align*}
(MOP_2) \quad & \max Z(Y) = [(o_1 - o') y_1, (o_2 - o') y_2, ..., (o_m - o') y_m] \\
& \text{s.t. } y_i \in R \tag{2}
\end{align*}
\]

where the feasible set \( Y = \{y_i | y_i \in R\} \) denotes the set of all possible unit return of individuals. We define \( Z = \{(o_i - o') y_i | y_i \in R\} \) to be the set of total return for each individual.

It is natural to discuss the relation between the moderator’s unit cost in Model (1) and the individual’s unit return in Model (2). In the light of multi-objective programming theory, a multi-objective programming model needs to be transformed into a single-objective model by exploring a decision rule. Classical multi-objective programming theory shows that the optimal solution to a single-objective model is a Pareto optimal solution to the corresponding multi-objective programming model [26]. We prove that the single-objective problems of Model (1) and Model (2) are actually dual to each other, and discuss their economic significance using the primal-dual theory of linear programming [32].

3. Construction and interpretation of minimum cost and maximum return consensus models

According to multi-objective programming theory, it is easy to find a set of Pareto efficient solutions by transforming the multi-objective programming model into a single-objective one with some aggregation operators. If we consider the total cost \( \sum_{i=1}^{m} w_i f_i(o') \) of all individuals as a whole, then the above model (1) can be transformed into a single-objective programming model. Generally speaking, we only need to find the optimal solution — also the Pareto optimal solution (non-inferior solution) to the corresponding multi-objective problem — to the single-objective model. In other words, we only need to consider the total cost that the moderator is willing to pay to achieve consensus. The smaller the value \( \sum_{i=1}^{m} w_i f_i(o') \) is, the greater the degree of consensus will be. In fact, \( \sum_{i=1}^{m} w_i f_i(o') \) is a weighted arithmetic mean of \( f_i(o'), i \in M \). The smaller this value is, the closer the distance between an individual opinion and the consensus opinion. The set \( W \) is denoted as \( W = \{w | w_i > 0, 1 \leq i \leq m\} \), where \( W \) has two significant meanings:

- For Model (1), \( w_i > 0, 1 \leq i \leq m \), can be viewed as a unit cost that the moderator paid individual \( d_i \) according to each unit opinion that the individual has changed.
- If we unify \( W \), that is \( w_i > 0, 1 \leq i \leq m \), \( \sum_{i=1}^{m} w_i = 1 \), then \( W \) is the weight vector of the weighted arithmetic mean.

Based on this analysis, \( \sum_{i=1}^{m} w_i f_i(o') \) is regarded as the total cost (resource) paid by the moderator to obtain consensus. Thus, we construct an optimization model \( P(w) \) under the premise that there is consensus opinion such that the total cost to obtain consensus is the minimum:

\[
\begin{align*}
P(w): \quad & \min \ \phi = \sum_{i=1}^{m} w_i f_i(o') \\
& \text{s.t. } \{o' \in O | (3 - 1) \} \tag{3}
\end{align*}
\]

The single-objective programming Model \( P(w) \) is the transformed multi-objective programming Model (1). When \( f_i(o') = |o' - o_i|, i \in M \), \( P(w) \) is equivalent to the linear programming model \( LP(w) \) (see Appendix A):

\[
\begin{align*}
LP(w): \quad & \min \ \phi = \sum_{i=1}^{m} (w_i u_i + w_i v_i) \\
& \text{s.t. } \{o' - u_i + v_i = o_i, i \in M | (4 - 1) \} \quad (4 - 1) \\
& \{o' > 0, u_i \geq 0, v_i \geq 0, i \in M | (4 - 2) \} \tag{4}
\end{align*}
\]

In many practical instances of decision making, linguistic information [33,34] or fuzzy information [35] are widely used to express DM’s preference, which means the crisp value cannot fully denote the real viewpoint of a DM. To represent their opinions more confidently, lots of DDMs prefer to use intervals [36,37], the left/right point of which express the pessimistic/optimistic view respectively. Since the interval opinions can better re...
simulate the characteristics of subjective decision making, we assume \(a_i = [a_{i0}, a_{i1}] (0 < a_{i0} < a_{i1})\), where the left point of this interval denotes the lowest or most pessimistic value and the right point of this interval the highest or most optimistic value. The interval opinion of individual \(d_i\) can be expressed by a series of crisp numbers \(a_{i0}, a_{i1}, \ldots, a_{iM}\), where \(0 \leq a_{ik} \leq 1\) for all \(k\). Once the value of \(a_{ik}\) is determined, the interval opinion \(a_i = [a_{i0}, a_{i1}]\) becomes a crisp value \(a_i = a_{i0} + a_{i1} \alpha_{i}\). A figure of an interval opinion \(a_i = [a_{i0}, a_{i1}]\) and its crisp value is shown as follows (Fig. 1):

In the \(P(w)\) model, we let \(f_i(o') = |o' - o_i| = |o' - (a_{i0} + a_{i1} \alpha_{i})(o_{i0} - o_{i1})|\), \(i \in M\), denote the deviation between the individual opinion \(o_i\) and consensus opinion \(o'\). Obviously, the smaller the deviation is, the closer the \(o_i\) is to \(o'\). Model (3) is equivalent to

\[
P(w): \quad \min \phi = \sum_{i=1}^{m} |o' - [a_{i0} + a_{i1} \alpha_{i}]| w_i
\]

\[
s.t.\begin{align}
& o' \in O \\
& 0 \leq \alpha_{i} \leq 1, \quad i \in M
\end{align}
\]

The nonlinear programming Model (5) can be equivalently transformed into a linear programming model \(LP(w)\) (see Appendix B):

\[
LP(w): \quad \min \phi = \sum_{i=1}^{m} (w_i u_i + w_i v_i)
\]

\[
s.t.\begin{align}
& o' - u_i + v_i = \alpha_{i} \alpha_{i} = a_{i0}, \quad i \in M \\
& 0 \leq \alpha_{i} \leq 1, \quad i \in M
\end{align}
\]

It is easy to prove that the set of feasible solutions \(X^* = (o^*, u_1^*, v_1^*, a_1^*, \ldots, u_m^*, v_m^*, a_m^*)^T\) to Model (6) is nonempty. The optimal solution \(X^* = (o^*, u_1^*, v_1^*, a_1^*, \ldots, u_m^*, v_m^*, a_m^*)^T\) to Model (6) can be solved by the classical Dantzig’s simplex algorithm [32].

In Model (6), the objective function \(\phi = \sum_{i=1}^{m} (w_i u_i + w_i v_i)\) can be considered to be the minimum total cost for obtaining the greatest consensus. The restriction (6-1) denotes the limits of the deviation between the consensus and individual opinion. Obviously, the cost of arriving at the greatest consensus is as small as possible under the restrictions (6-1) and (6-2). In Model (6), we can not only compute the minimum cost paid by the moderator, but also obtain the optimal consensus opinion \(o^*\). Moreover, we can also determine the optimal opinions \(o_i^* = a_{i0} + a_{i1} \alpha_{i}\) from individual 

Next, we explore the specific meaning in economics by discussing the dual model of Model (6). In the primal-dual linear programming theory, the dual model of Model (6) is built as follows:

\[
DLP(w): \quad \max \psi = \sum_{i=1}^{m} (a_{i0} y_i + y_{i+m})
\]

\[
s.t.\begin{align}
& \sum_{i=1}^{m} y_i = 0 \\
& -y_i \leq w_i, \quad i \in M \\
& y_i \leq w_i, \quad i \in M \\
& -a_{i0} y_i + y_{i+m} \leq 0, \quad i \in M \\
& y_{i+m} \leq 0, \quad i \in M
\end{align}
\]

It is easy to prove that the set of feasible solutions \(Y = (y_1, \ldots, y_{2m})^T\) to Models (7) is nonempty. The number of basic feasible solutions is finite, and \(|y_i| \leq w_i, \quad i \in M\), is bounded. This means that the optimal solution to Model (7) exists and can be found by the classical simplex method. We call Model (6) the primary problem (LP (w)), and Model (7) the dual problem (DLP(w)).

### 3.1. Economic interpretation of objective function of the dual problem

In classical primal-dual theory of linear programming, the variables and its optimal values in Model (7) are referred to as marginal values (profits) and shadow value (profits), respectively. In this section, we explore the economic significance of the variable \(y_i\) in our consensus model by discussing the dual objective function: we show that the dual objective function \(\psi = \sum_{i=1}^{m} (a_{i0} y_i + y_{i+m})\) is equivalent to

\[
\psi = \sum_{i=1}^{m} \left[ (a_{i0} + a_{i1} \alpha_{i}) (o_{i0} - o_{i1}) - \alpha_{i}^* \right] y_i = \sum_{i=1}^{m} (o_i - o_i^*) y_i
\]

**Lemma 1.** The dual objective function \(\psi = \sum_{i=1}^{m} (o_i y_i + y_{i+m})\) of Model (7) is equal to \(\psi = \sum_{i=1}^{m} (o_i - o_i^*) y_i\), where \(o_i - o_i^*\) is a feasible solution to Model (6).

**Proof.** See Appendix C.

Let \(\alpha, \alpha' > 0\) be a feasible solution to Model (6). According to Eq. (7-1), we have \(\psi = 0\). This means that \(\sum_{i=1}^{m} (o_i + a_{i1} \alpha_{i} (o_{i0} - o_{i1}) - \alpha_{i}^*) y_i = 0\), where \(\alpha_{i}^*\) holds.

**Theorem 1.** The objective function \(\psi = \sum_{i=1}^{m} (o_i y_i + y_{i+m})\) of Model (7) is equal to \(\psi = \sum_{i=1}^{m} (o_i - o_i^*) y_i\), where \(o_i - o_i^*\) is a feasible solution to Model (6).

The partial derivative of a function \(\psi\) with respect to the variable \(y_i\) is

\[
\frac{\partial \psi}{\partial (o_i - o_i^*)} = y_i
\]

where \(o_i - o_i^*\) is actually the opinion deviation between the individual \(d_i\)’s opinion and the consensus opinion.

The economic interpretation of Eq. (8) shows that \(y_i\) is the marginal profit of individual \(d_i\); in consensus GDM, it is apparent that all individuals would like to preserve their valuable opinions or have their opinions to be considered of great importance. To reach consensus, most of the individuals have to sacrifice their interests to accept the collective interest, also meaning that their interests are damaged when their opinions change. It is natural for individuals to hope to obtain the return from the moderator according to the deviation degree between the individual \(d_i\)’s opinion and the consensus opinion. Therefore, the marginal profit \(y_i\) of individual \(d_i\) can be interpreted as the unit return that individual \(d_i\) expects to obtain for changing his original opinion to the consensus opinion, and thus \(\sum_{i=1}^{m} (o_i - o_i^*) y_i\) as the total return that individual \(d_i\) and all the individuals expect to obtain, respectively.

Now, we turn to interpreting the economic significance of the optimal objective function of dual Model (7). **Corollary 1** is obvious.

**Corollary 1.** Let \(\mathbf{x}^* = (\mathbf{a}^*, \mathbf{u}_1^*, \mathbf{v}_1^*, \ldots, \mathbf{u}_m^*, \mathbf{v}_m^*, \mathbf{a}_1^*, \ldots, \mathbf{a}_m^*)^T\) and \(\mathbf{y}^* = (\mathbf{y}_1^*, \ldots, \mathbf{y}_{2m}^*)^T\) be the optimal solution to Model (6) and Model (7), respectively. The optimal objective function \(\max \psi = \sum_{i=1}^{m} (o_{i0} y_i^* + y_{i+m}^*)\) of dual Model (7) is equal to \(\max \psi = \sum_{i=1}^{m} (o_i - o_i^*) y_i^*\).

In consensus GDM, the economic interpretation \(y_i^*\) is the shadow profit of individual \(d_i\), and \((o_i - o_i^*) y_i^*\) is the optimal return that individual \(d_i\) expects to obtain for changing his original opinion to the optimal consensus opinion. Therefore, the economic significance of the optimal objective function \(\max \psi = \sum_{i=1}^{m} (o_i - o_i^*) y_i^*\) of dual Model (7) is as follows: the maximum total return acquired by all individuals for abandoning their original opinions is determined by the opinion deviation and shadow value, and is equal to the sum of all the products of the opinion deviations from their original opinions to the optimal consensus opinion and the corresponding shadow value.
3.2. Economic interpretation of dual variables and restrictions of the dual problem

Since \( y_i \) is the marginal profit of individual \( i \), this subsection will explore the economic implication of marginal profit when \( y_i \) is positive, negative or zero. And we also need to clarify the economic significance of the dual restrictions of DLP\((w)\) problem.

The economic significance of marginal profit is developed by discussing the relation between the marginal profit \( y_i \) and the individual \( d_i \)'s interval opinion \( \delta_i = o_0 + \alpha_i \#(o_1 - o_0) \) (see Appendix D).

1. The negative marginal profit \( y_i \) implies that the individual \( d_i \)'s interval opinion is smaller than the consensus opinion, and the product of the deviations from his/her original opinion to the consensus opinion and the corresponding unit return, meaning that the total return expected by the set \( \{ i^*_y \ = \{| i; \alpha_i < o' \} \in M \} \) of individuals, is still positive.

2. The positive marginal profit \( y_i \) implies that the individual \( d_i \)'s interval opinion is greater than the consensus opinion, and the product of the deviations from his/her original opinion to the consensus opinion and the corresponding unit return, meaning that the total return expected by the set \( \{ i^*_y \ = \{| i; \alpha_i > o' \} \in M \} \) of individuals, is still positive.

3. The marginal profit \( y_i = 0 \) implies that the individual \( d_i \)'s interval opinion is equal to the consensus opinion, and the product of the deviations from his/her original opinion to the consensus opinion and the corresponding unit return, meaning that the total return expected by the set \( \{ i^*_y = \{| i; \alpha_i = o' \} \in M \} \) of individuals, is zero.

The economic significance of shadow value can be similarly interpreted by discussing the relation between the shadow value \( y^*_o \) and the individual \( d_i \)'s optimal opinion \( \delta_i^* = o_0 + o^*_i \#(o_1 - o_0) \).

In brief, the marginal profit \( y_i \) indicates the unit return that individual \( d_i \) expects to obtain for changing his original opinion to the consensus opinion, and the shadow value \( y^*_o \) indicates the unit return that individual \( d_i \) expects to obtain for changing his original opinion to the optimal consensus opinion.

Consider the sets \( i^*_y \), \( i^*_o \) and \( \delta^*_o \). Obviously, \( M = i^*_y \cup i^*_o \cup \delta^*_o \). The restriction (7-1) can be equivalently rewritten as \( \sum_{i \in i^*_y} y_i = \sum_{i \in i^*_o} |y_i| \), Thus it can be understood as the sum of the non-negative unit return, which is exactly equal to absolute value of the sum of the negative unit return. Meanwhile, the restriction (7-2) and (7-3) can be understood as restricting the unit return \( |y_i| \) paid to \( d_i \) to not exceed the cost \( w_i \) paid by the moderator.

The restriction (7-4) is equivalent to \( 0 \leq y_i \leq y^*_o \). Thus, \( \sum_{i=1}^{m} |(o_0 y_i + y_{i-1,m})| = \sum_{i=1}^{m} (o_i - o')y_i \leq \sum_{i=1}^{m} 0y_i \). So the restriction (7-4) can be interpreted as stating that the total return is no more than the optimistic value of the total return that all individuals expect to acquire.

According to the above discussion, the primal-dual Model (6) and Model (7) have compact relation. Thus Model (7) can be rewritten as a more useful form, and it is more easily and directly to be interpreted:

DLP\((w)\): \[
\begin{align*}
\max \psi & = \sum_{i=1}^{m} o_0 + \alpha_i \#(o_i - o_0) - o'|y_i| \\
\text{s.t.} & \begin{cases}
\sum_{i=1}^{m} y_i = 0 & (9-1) \\
|y_i| \leq w_i, \quad i \in M & (9-2) \\
\sum_{i=1}^{m} (o_0 y_i + y_{i-1,m}) & (9) \\
\sum_{i=1}^{m} (o_i - o')y_i \leq \sum_{i=1}^{m} o_i y_i & (9-3)
\end{cases}
\end{align*}
\]

where \( \alpha_i, i \in M, o' \) are feasible solutions to Model (7). Models (9) are actually the single objective formulation of multi-objective programming Model (2). Hence, the optimal solution to Model (7) is also the Pareto optimal solution to Model (2).

Section 4 further interprets the economic significance of the consensus model by discussing the other relations between the primal problem LP\((w)\) and its dual problem DLP\((w)\).

4. Further interpreting the relation between the primal problem LP\((w)\) and its dual problem DLP\((w)\)

This section further interprets the economic significance of the consensus model by discussing the other relations between the primal problem LP\((w)\) and its dual problem DLP\((w)\). In other words, we will explore the condition which can lead to the consensus of all individuals and the moderator; we discuss the practical significance of the consensus opinion and the unit return; in addition, we will try to figure out the relations between the individual unit return, interval opinion, the moderator’s unit cost and the consensus opinion.

Weak duality [32]: Let \( X \) and \( Y \) be the feasible solution to the primal problem LP\((w)\) and the dual problem DLP\((w)\), respectively, and \( \phi(X) \) and \( \psi(Y) \) be the corresponding value of their objective function, which should be minimized and maximized, respectively. For any feasible solution, the objective value \( \phi(X) \) of the primal problem LP\((w)\) will always be no less than the objective value \( \psi(Y) \) of the dual problem DLP\((w)\).

Economic interpretation of Weak Duality: under rational conditions, the expected return of all the individuals is limited and does not mean the more the better. In fact, their total expected return will be no more than the total cost paid by the moderator to achieve the consensus. Namely, all the resources that the moderator owns will completely satisfy the demands of all the individuals. Obviously, this conclusion is consistent with the facts.

Optimality criterion [32]: If the primal problem LP\((w)\) and the dual problem DLP\((w)\) both have optimal feasible solutions, then their optimal objective values must be equal.

Economic interpretation of optimality criterion: if both the moderator and the individuals achieve the optimal results, that is, the moderator obtains the optimal consensus opinion, and the individuals get the most ideal unit return, then the amount of the minimum total cost paid by the moderator equals that of the maximum return expected by all the individuals who have given up their original opinions. It indicates that once the consensus is obtained, the total return reaches its maximum while the total resource consumption (i.e., the consensus cost) reaches its minimum.

Sufficient optimality criterion [32]: If \( X^* \) and \( Y^* \) are feasible solutions to the primal problem LP\((w)\) and its dual problem DLP\((w)\), respectively, and if the primal objective function \( \sum_{i=1}^{m} w_i (u_i + v_i) \) and the dual objective function \( \sum_{i=1}^{m} (o_0 y_i + y_{i-1,m}) \) satisfy \( \sum_{i=1}^{m} w_i (u_i + v_i) = \sum_{i=1}^{m} (o_0 y_i + y_{i-1,m}) \), then \( X^* \) and \( Y^* \) are the optimal solutions to LP\((w)\) and DLP\((w)\), respectively.

Economic interpretation of sufficient optimality criterion: in the whole process of GDM, as long as the amount of the maximum total return expected by all the individuals who have changed their opinions is equal to that of the minimum total cost to achieve the consensus, the moderator will receive the optimal decision, the individuals will obtain the optimal unit expected return, and the consensus opinion as well as the unit return (shadow profit) will have economic significance in reality (i.e., the optimal solution exists).

Strong duality [32]: If either the primal problem LP\((w)\) or the dual problem DLP\((w)\) have an optimal feasible solution, then the other problem also does and the two optimal objective values are equal. That is, \( \max \psi = \min \phi \).
Economic interpretation of strong duality: once the consensus is reached from the moderator’s perspective, then the unit returns expected by all the individuals have economic significance, indicating that the optimal solution can be obtained from the angle of individual DMs (i.e., the shadow profit exists); if the expected unit returns are solved for all the individuals, then the unit cost and the consensus opinion have practical meaning, that is, the optimal solution to the primal problem exists; taken together, as long as the optimal solution of either the unit return or the consensus opinion exists, the amount of the total return expected by all the individuals who have changed their opinions equals that of the total cost paid by the moderator.

Three important properties can be derived using the same arguments as to the proof of Lemma 1.

**Property 1.** \( y_{i+m} = 0, \alpha_i = 0 \) holds for all \( i \in y_i^+ \subseteq M, y_i > 0 \).

**Property 2.** \( y_{i+m} = (o_i - o_j)y_i, \alpha_i = 1 \) holds for all \( i \in y_i^- \subseteq M, y_i < 0 \).

**Property 3.** \( y_{i+m} = 0 \) holds for all \( i \in y_i^0 \subseteq M, y_i = 0 \).

The physical meaning of Properties 1 and 2 is that the value of \( \alpha_i \) is determined by its corresponding shadow profit or the individual unit return \( y_i \).

The relations between the shadow profit and the individual opinions can be derived using these two properties.

**Theorem 2.** \( \sigma^* \geq o_i \) holds for all \( i \in y_i^+ \subseteq M \) if \( y_i > 0 \) holds; \( \sigma^* \leq o_i \) holds for all \( i \in y_i^- \subseteq M \) if \( y_i > 0 \) holds; and \( \sigma^* \in [o_i, o_i] \) holds for all \( i \in y_i^0 \subseteq M \) if \( y_i = 0 \) holds.

**Proof.** See Appendix E.

Theorem 2 indicates that if an individual’s unit return is positive, then the optimal consensus opinion is no more than the pessimistic value of his/her interval opinion; if the individual’s unit return is negative, then the optimal consensus opinion is no less than the optimistic value of his/her interval opinion; and if the individual unit return is equal to 0, then the optimal consensus opinion is within his/her interval opinion interval. Theorem 3 provides more detailed explanation to the relation between the shadow profit and the individual opinion.

**Theorem 3.** If both the primal problem LP(\( w \)) and the dual problem DLP(\( w \)) have optimal feasible solutions \( y_i \) and \( \sigma^* \), respectively, then as far as each individual \( d_i, i = 1, \ldots, m \) is concerned,

1. if \( y_i = -w_i < 0 \), then \( \sigma^* \geq o_i \) and if \( y_i = w_i > 0 \), then \( \sigma^* \leq o_i \),
2. if \( 0 < y_i < w_i \), then \( \sigma^* = o_i \); if \( -w_i < y_i < 0 \), then \( \sigma^* = o_i \) and if \( y_i = 0 \), then \( 0 \leq \sigma^* \leq o_i \).

**Proof.** See Appendix F.

Theorem 3 implies that, once the consensus opinion is determined (the optimal consensus opinion exists), the unit return \( y_i \) of individual \( d_i \) directly reflects the relation between the consensus opinion and each individual opinion:

1. If the individual’s unit return is exactly equivalent to the opposite value of the moderator’s unit cost, then the optimistic individual opinion is no more than the consensus opinion. If the individual’s unit return is exactly equivalent to the value of the moderator’s unit cost, then the pessimistic individual opinion is no less than the consensus opinion. I.e., if the individual’s unit return is exactly equivalent to the absolute value of the moderator’s unit cost, and if we regard the individual opinion as an open interval, then the individual opinion must not be equal to the consensus opinion.

**Example.** If the individual’s unit return is positive and smaller than the value of the moderator’s unit cost, then the consensus opinion must be equal to the pessimistic individual opinion. If the individual’s unit return is negative and greater than the opposite value of the moderator’s unit cost, then the consensus opinion must be equal to the optimistic individual opinion. If the individual’s unit return is equal to 0, then the consensus opinion must be within the interval of the individual opinion.

Moreover, if the absolute value of the individual’s unit return is smaller than the moderator’s unit cost, that is, if condition \( (-w_i < y_i < w_i) \) holds, then the individual opinion must be equal to the consensus opinion.

Properties 1, 2, 3, Theorem 2 and Theorem 3 show that, once the optimal value of shadow profit \( y_i \) is determined, the individual’s optimistic opinion and the relation between the consensus opinion and the individual’s optimal opinion are also determined, as shown in Table 1 and Fig. 2. Particularly, when the optimal consensus opinion is greater than or equal to the left endpoint of interval opinion \( (\sigma^* \geq o_i) \), the individual \( d_i \)’s optimal opinion is selected as the right endpoint from his/her interval opinion. Obviously, the opinion deviation between the consensus opinion and the individual \( d_i \)’s optimal opinion satisfies \( \sigma^* - o_i \leq \sigma^* - o_i \). Similarly, when the optimal consensus opinion is smaller than or equal to the left endpoint of interval opinion \( (\sigma^* \leq o_i) \), the individual \( d_i \)’s optimal opinion is selected as the left endpoint from his/her interval opinion. Obviously, the opinion deviation between the consensus opinion and the individual \( d_i \)’s optimal opinion satisfies \( o_i - \sigma^* \leq o_i - \sigma^* \). When the optimal consensus opinion is in the interval opinion \( (\sigma^* \in [o_i, o_i]) \), the individual \( d_i \)’s optimal opinion is selected as the optimal consensus opinion. Obviously, the opinion deviation between the consensus opinion and the individual \( d_i \)’s optimal opinion equals 0. It is natural that these rational selections guarantee to minimize the moderator’s cost.

**Theorem 4.** If both the primal problem LP(\( w \)) and the dual problem DLP(\( w \)) have optimal feasible solutions \( y_i \) and \( \sigma^* \), respectively, then as far as each individual \( d_i, i = 1, \ldots, m \) is concerned,

1. if \( \sigma^* > o_i \), then \( y_i = -w_i < 0 \); if \( \sigma^* < o_i \), then \( y_i = w_i > 0 \); and if \( o_i < \sigma^* < o_i \), then \( y_i = 0 \).
2. if \( \sigma^* = o_i \), then \( -w_i \leq y_i \leq 0 \) and if \( \sigma^* = o_i \), then \( 0 \leq y_i \leq w_i \).

**Proof.** See Appendix G.

Theorem 4 indicates that the value of the individual’s unit return \( y_i \) of individual \( d_i \) is determined by the relation between the individual opinion and the optimal consensus opinion:

1. If the individual’s optimistic opinion is smaller than the consensus opinion, then the individual’s unit return is negative and its absolute value is equivalent to the moderator’s unit cost. If the individual’s pessimistic opinion is greater than the consensus opinion, then the individual’s unit return is positive, and its absolute value is equivalent to the moderator’s unit cost. If the consensus opinion is greater than the pessimistic value but
smaller than the optimistic value of the individual opinion, then the individual’s unit return is 0, which denotes that the individual may not expect to acquire any return because his/her opinion is consistent with the consensus opinion.

(2) If the individual’s optimistic opinion is equal to the consensus opinion, then the individual’s return is non-positive; if the individual’s pessimistic opinion is equal to the consensus opinion, then the individual’s return is non-negative. At this time, the unit return of the individual may be lower, greater than, or even equal to the moderator’s unit cost, meaning that an individual’s unit return is fuzzy.

To interpret Theorem 4 more clearly, we show the relationship among the individual’s unit return, the moderator’s unit cost, the individual opinion, and the consensus opinion in Table 2.

5. Numerical example

This example further examines the relationship between the primal problem LP(w) and its dual problem DLP(w). Suppose that there are four DMs $d_1$, $d_2$, $d_3$, and $d_4$ in a GDM and their corresponding interval opinions are $o_1 = [14, 37]$, $o_2 = [22, 30]$, $o_3 = [64, 153]$, and $o_4 = [8, 61]$, respectively. The unit cost paid to the four DMs by the moderator are $w_1 = 1$, $w_2 = 2$, $w_3 = 3$, and $w_4 = 4$, respectively, and the consensus opinion is supposed to be $o'$. The optimistic consensus decision making model based on minimum cost is constructed as follows:

$$
\text{P(w)} : \min \phi = 1 + |o' - (14 + 23\alpha_1)| + 2|o' - (22 + 8\alpha_2)| + 3|o' - (64 + 89\alpha_3)| + 1|o' - (8 + 53\alpha_4)|
$$

subject to

$$
\begin{align}
0 &\leq \alpha_i \leq 1, & i &= 1, 2, 3, 4
\end{align}
$$

The objective function $\phi$ in Model (10) denotes the total cost paid by the moderator. Let $|o' - (o_i + \alpha_i(o_{i-1} - o_i))| = u_i + v_i$. $o' - |o_1 + \alpha_i(o_{i-1} - o_i)| = u_i - v_i$, $i = 1, 2, 3, 4$. Then, the above model can be transformed into a linear programming model:

$$
\text{LP}(w) : \min \phi = u_1 + v_1 + 2u_2 + 2v_2 + 3u_3 + 3v_3 + u_4 + v_4
$$

subject to

$$
\begin{align}
\alpha_1 &\leq 1, & \alpha_2 &\leq 1, & \alpha_3 &\leq 1, & \alpha_4 &\leq 1, & \alpha_5 &\leq 1, & \alpha_6 &\leq 1, & \alpha_7 &\leq 1
\end{align}
$$

The optimal value of the objective function of Model (11) is $\phi = 95$, indicating that the minimum (optimal) total cost paid by the moderator to the four DMs is 95. The optimal consensus opinion calculated from the above model is $o^* = [37, 61]$. When $o^* = 37$, the optimal solution of the model is $X^* = (o^* u_1 v_1 u_2 v_2 u_3 v_3 u_4 v_4)$,

$$
X^* = (37 0 0 7 0 0 2 7 0 0 1 0 0 5 4 7 2)^T
$$

When $o^* = 61$, the optimal solution becomes $X^* = (61 24 0 3 1 0 3 0 0 1 1 0 1 0)^T$. If the reference value $o^*$ is selected as 38, then it is easy to obtain the optimal solution based on the model, which is $X^* = (38 0 1 0 8 0 0 2 6 0 0 1 0 0 5 6)^T$. Then, the moderator’s and the four DMs’ optimal opinions are 38, 37, 30, 64, and 38, respectively (Fig. 3).

To further explore the economic significance of Model (11), the following dual model was constructed according to the primal-dual linear programming theory. We call Model (11) the primal model and Model (12) the dual model:

$$
\text{DLP}(w) : \max \psi = 14y_1 + 22y_2 + 64y_3 + 8y_4 + y_5 + y_6 + y_7 + y_8
$$

subject to

$$
\begin{align}
y_1 + y_2 + y_3 + y_4 &= 0 \\
y_1 \leq 1, & \quad y_2 \leq 2, & \quad y_3 \leq 3, & \quad y_4 \leq 4
\end{align}
$$

subject to

$$
\begin{align}
-23y_1 + y_5 &\leq 0 \\
y_6 + y_7 &\leq 0 \\
y_8 &\leq 0
\end{align}
$$

The optimal solution is $\psi^* = (11 - 23 0 - 23 - 0 0 0)^T$, and the optimal value of the objection function is max $\psi = 95$. That is, the unit returns solved by Model (12) are

---

Table 2

<table>
<thead>
<tr>
<th>The relation between $\sigma^*$ and $d_i$’s opinion</th>
<th>The value of $y_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>If $\sigma^* &gt; o_i$</td>
<td>$y_i = -w_i &lt; 0$</td>
</tr>
<tr>
<td>If $\sigma^* = o_i$</td>
<td>$-w_i \leq y_i \leq 0$</td>
</tr>
<tr>
<td>If $\sigma^* &lt; o_i$</td>
<td>$y_i = w_i &gt; 0$</td>
</tr>
<tr>
<td>Or $\sigma^* = (o_i, o_i)$</td>
<td>$0 \leq y_i \leq w_i$</td>
</tr>
</tbody>
</table>

To interpret Theorem 4 more clearly, we show the relation among the individual’s unit return, the moderator’s unit cost, the individual opinion, and the consensus opinion in Table 2.

Fig. 2. The relation between the consensus opinion and the individual’s optimal opinion based on Theorems 3 and 4.

Fig. 3. Optimal opinions of the moderator and the four DMs.

---
Both the primal model (11) and its dual model (12) have optimal solutions and their optimal objective function are both 95. Economically, the total return of all individuals is equal to the total cost paid by the moderator. Because \( y_1 = -1 \), \( y_2 = -2 \), \( y_3 = 3 \), \( y_4 = 0 \), respectively, while the total return expected by all the individuals is 95.

The optimistic opinion \( o_{d_3} = 30 \) of individual \( d_3 \) is smaller than the optimal consensus opinion \( o_e \in [37, 61] \), so his/her unit return \( y_3 = -2 \) is equal to the opposite value of the moderator’s unit cost \( w_3 = 2 \); individual \( d_1 \)'s pessimistic opinion \( o_{d_1} = 64 \) is greater than the optimal consensus opinion \( o_e \in [37, 61] \), hence his/her unit return \( y_1 = 3 \) exactly equals the moderator’s unit cost \( w_1 = 3 \). If the optimal consensus opinion is \( o_e = 37 \), contained in the fourth DM’s interval opinion \([8, 61]\), then the unit return expected by \( d_4 \) is \( y_4 = 0 \). (The correctness of Theorem 3 is validated.)

The optimistic opinion \( o_{d_3} = 30 \) of individual \( d_3 \) is smaller than the optimal consensus opinion \( o_e \in [37, 61] \), so his/her unit return \( y_3 = -2 \) is equal to the opposite value of the moderator’s unit cost \( w_3 = 2 \); individual \( d_1 \)'s pessimistic opinion \( o_{d_1} = 64 \) is greater than the optimal consensus opinion \( o_e \in [37, 61] \), hence his/her unit return \( y_1 = 3 \) exactly equals the moderator’s unit cost \( w_1 = 3 \). If the optimal consensus opinion is \( o_e = 37 \), contained in the fourth DM’s interval opinion \([8, 61]\), then the unit return expected by \( d_4 \) is \( y_4 = 0 \). (The correctness of Theorem 4(1) is validated.) If the optimal consensus opinion is \( o_e = 61 \), then we have \(-1 \leq y_4 \leq 0 \) (Theorem 4(2)). As \( d_1 \), if the optimistic consensus opinion is \( o_e = 37 \), which equals his/her optimistic value \( o_{d_1} = 37 \), then the unit return of \( d_1 \) satisfies \(-1 \leq y_1 \leq 0 \) (Theorem 4(2)). If the consensus opinion belong to \([37, 61]\), then the unit return satisfies \( y_1 = - w_1 = -1 \) (Theorem 4(1)). As far as \( d_1 \) and \( d_2 \) are concerned, their corresponding expected unit returns \( y_1 = -1 \), \( y_2 = -2 \) are equal to the opposite value of the moderator’s unit cost \( w_1 = 1 \), \( w_2 = 2 \); so the optimal consensus opinion \( o_e \in [37, 61] \) must be no less than their optimistic opinions \( o_{d_1} = 37 \) and \( o_{d_1} = 30 \). For individual \( d_3 \), the expected unit return \( y_3 = 3 \) equals the unit cost \( w_3 = 3 \) paid by the moderator, so the optimal consensus opinion \( o_e \in [37, 61] \) must be smaller than his/her lower limit \( o_{d_3} = 64 \) (Theorem 3(1)); while for \( d_4 \) because the expected unit return is \( y_4 = 0 \), the optimal consensus interval \( o_e \in [37, 61] \) must belong to his/her original opinion \([8, 61]\), that is, \([37, 61] \subseteq [8, 61] \) (Theorem 3(2)).

6. Conclusions and future research

When dealing with inevitably imprecise or not fully reliable opinions, it is difficult for a DM to develop a precise, clear opinion. An interval opinion indicates a range of opinion, in which the left and right points are known but the true value cannot be fully determined. Due to imprecision or not fully reliability, a DM has difficulty developing a precise, clear opinion. Furthermore, the consensus model can only decide the optimistic or pessimistic value of an individual’s unit return is smaller (or greater) than the absolute value of the individual’s unit return, then the individual’s unit return is negative (or positive) and the absolute value of his/her unit return equals the moderator’s unit cost; if the consensus opinion in the open interval of the individual opinion, then his/her expected unit return is equal to 0; if the optimistic (or pessimistic) value of an individual opinion is equal to the value of consensus opinion, then his/her unit return is fuzzy.

Future research will investigate to what extent the unit cost or return can change without changing the optimal consensus opinion. We will also look at how adding some opinions to the GDM changes the optimal consensus opinion. Furthermore, the consensus model of the minimum cost and the maximum return with heterogeneous opinions [38] in GDM problems will also be taken into consideration in the future.

Acknowledgments

The work in this paper was supported by the Excellence Andalusian Project TIC-5991, KAU-project 27-135-35/HiCi, the National Project TIN2013-40658-P, the Innovation Foundation of Postgraduate Science and Technology in Jiangsu Province (KYLX_0858), a Marie Curie International Incoming Fellowship within the 7th European Community Framework Programme (Grant no. FP7-PIIF-GA-2013-629051), the National Natural Science Foundation of China (71171115, 71173116, 70901043), the Natural Science Foundation of Jiangsu, China (Grant no. BK20141481).
Appendix A. The relation Proof of Model (3) and Model (4)

In the $P(w)$ model, if $f_d(o^\ast) = |o^\ast - o_i|$, $i \in M$, then there exist $u_i \geq 0$, $v_i \geq 0$, and $u_i, v_i > 0$, such that $|o^\ast - o_i| = u_i + v_i$, $o^\ast - o_i = u_i - v_i$. In fact, if we let $w_i = |o^\ast - o_i + (o^\ast - o_i) / 2$, $v_i = |o^\ast - o_i - (o^\ast - o_i) / 2|$, then the above conditions hold.

Appendix B. The relation Proof of Model (5) and Model (6)

In Model (5), we let $u_i \geq 0$, $v_i \geq 0$, satisfying $u_i, v_i > 0$, and $|o^\ast - [o_i + a_i(o_i - o_j)]| = u_i + v_i$, $|o^\ast - [o_i + a_i(o_i - o_j)]| = u_i - v_i$. In fact, we only let $u_i = |o^\ast - [o_i + a_i(o_i - o_j)]| + |o^\ast - [o_i + a_i(o_i - o_j)]| / 2$, $v_i = |o^\ast - [o_i + a_i(o_i - o_j)]| - |o^\ast - [o_i + a_i(o_i - o_j)]| / 2$.

Appendix C. Proof of Lemma 1

Considering the fact that $\sum_{i=1}^{m} (o_i y_i + y_i + m) = \sum_{i=1}^{m} (o_i + a_i(o_i - o_j)) + \sum_{i=1}^{m} \alpha_i y_i$, and $\sum_{i=1}^{m} \alpha_i y_i = 0$, we only need to prove that the equality $\sum_{i=1}^{m} (o_i y_i + y_i + m) = \sum_{i=1}^{m} (o_i + a_i(o_i - o_j))$ holds.

Let $I^+_1 = \{j \in J \mid y_j > 0, \exists \alpha_i \in \mathbb{R}, \forall k \in M \}$, $I^+_2 = \{j \in J \mid y_j < 0, \exists \alpha_i \in \mathbb{R}, \forall k \in M \}$ and $I^+_3 = \{k \in M \mid \exists \alpha_i \in \mathbb{R}, \forall j \in J \}$. Then $I^+_1 \cup I^+_2 \cup I^+_3 = M$. We have $\sum_{i=1}^{m} (o_i + a_i(o_i - o_j)) + \sum_{i=1}^{m} \alpha_i y_i = \sum_{i=1}^{m} (o_i + a_i(o_i - o_j)) + \sum_{i=1}^{m} \alpha_i y_i$. For simplicity, we let $A = \sum_{i=1}^{m} (o_i + a_i(o_i - o_j))$, $B = \sum_{i=1}^{m} \alpha_i y_i$, and $C = \sum_{i=1}^{m} (o_i + a_i(o_i - o_j)) + \sum_{i=1}^{m} \alpha_i y_i$. Thus we have $A \geq B$ and $C$. Obviously, $A \geq B$. Then, for all $I^+_1 \cup I^+_2 \cup I^+_3$, we have $\sum_{i=1}^{m} (o_i + a_i(o_i - o_j)) = 0$.

(1) When $t \in I^+_1 \cup I^+_2 \cup I^+_3$, $y_i > 0$.

(a) We first show that for all $t \in I^+_1 \cup I^+_2 \cup I^+_3$, $y_i + m = 0$. Otherwise, if $y_i + m = 0$, then $y_i + m = 0$ holds according to the principle of complementary slackness and then $\alpha_i = 0$. Therefore, $y_i + m = 0$ holds according to the principle of complementary slackness. Thus we have $y_i = -y_i - m$. This shows that when $y_i > 0$, then $y_i + m > 0$, which contradicts the hypothesis $y_i + m < 0$. Thus we have $y_i + m = 0$.

(b) Second, we prove that $\alpha_i = 0$, for all $t \in I^+_1 \cup I^+_2 \cup I^+_3$. Otherwise, if $0 < \alpha_i \leq 1$, then the equality $y_i + m + (\alpha_i o_i - o_j) y_i = 0$ holds according to the principle of complementary slackness. This also shows that when $y_i > 0$, then $y_i + m = 0$. Therefore, $y_i + m = 0$, which contradicts the hypothesis $y_i + m < 0$.

Appendix E. Proof of Theorem 2

1. When $y_i$ is positive, for all $i \in J^+$, we have $\alpha_i = 0$. Then the optimal consensus opinion $\sigma^*$ satisfies $\sigma^* - [o_i + a_i(o_i - o_j)] = u_i - v_i$. Obviously, $\sigma^* = o_i + u_i - v_i$. In the light of the principle of complementary slackness and $y_i > 0$, there must be $u_i = 0$, $v_i > 0$ for all $i \in J^+$. Otherwise, if $i \in J^+$, $v_i < 0$, according to the principle of complementary slackness, we have $y_i = -o_i - m < 0$, which contradicts the hypothesis $y_i > 0$. Thus we have $\sigma^* = o_i - v_i < 0$.

2. When $y_i$ is negative, for all $i \in J^-$, we have $\alpha_i = 1$. Then the optimal consensus opinion $\sigma^*$ satisfies $\sigma^* - [o_i + a_i(o_i - o_j)] = u_i - v_i$. Obviously, $\sigma^* = o_i + u_i - v_i$. According to the principle of complementary slackness and $y_i < 0$, we have $u_i > 0$, $v_i < 0$. Otherwise, if $v_i < 0$, $u_i < 0$, according to the principle of complementary slackness, we have $y_i = o_i + m > 0$, which contradicts the hypothesis $y_i < 0$. Thus we have $\sigma^* = o_i - v_i < 0$.

3. When $y_i = 0$ for all $i \in J^0$. The strict inequalities of restrictions (7-2) and (7-3) in the DLP($w$) problem hold. According to the principle of complementary slackness, we have $u_i = v_i = 0$. The optimal consensus opinion $\sigma^*$ satisfies $\sigma^* - [o_i + a_i(o_i - o_j)] = 0$, that is, $\sigma^* \in [o_i, o_j]$.
(6–1) is equivalent to \( \alpha^* - u_i + v_i = 0 \). Then \( u_i \) must be equal to 0. Otherwise, if \( u_i \neq 0 \), then \( u_i - v_i > 0 \) according to the principle of complementary slackness, which contradicts the hypothesis \( y_i = w_i \). Thus we have \( o^* = o_i \), and \( \alpha^* \leq o_i \).

(2) If \( 0 < y_i < w_i \), then \( u_i = 0, v_i = 0 \) according to the principle of complementary slackness. The restriction (6–1) is equivalent to \( \alpha^* - \alpha_i (o_i - o_j) = 0 \). Then \( \alpha_i = 0 \) according to Property 1, and \( \alpha^* = o_i \). If \( -w_i < y_i < 0 \), then according to the principle of complementary slackness, we have \( u_i = 0, v_i = 0 \). The restriction (6–1) is equivalent to \( \alpha^* = o_i + \alpha_i (o_i - o_j) \), and \( o_j \leq 0 \leq o_i \).

Appendix G. Proof of Theorem 4

(1) If \( \alpha^* > o_i \), then \( \alpha^* - o_i = o_i + \alpha_i (o_i - o_j) - o_i - u_i > 0 \). That is, \( u_i - v_i = (1 - \alpha_i) (o_j - o_i) \geq 0 \). Thus we have \( u_i > v_i > 0 \). For the reason that \( u_i > 0, v_i > 0, u_i w_i = 0 \), then there must be \( v_i = 0, u_i > 0 \). According to the principle of complementary slackness, we have \( y_i = w_i \); similarly, we can prove that if \( \alpha^* < o_i \), then \( y_i = w_i > 0 \), if \( o_j < \alpha^* < o_i \), and suppose that \( y_i \neq 0 \), then \( y_i > 0 \), we have \( o_i \leq \alpha^* \) according to Theorem 3, which contradicts the hypothesis; if \( y_i < 0 \), we have \( \alpha^* \geq o_i \) according to Theorem 3, which contradicts the hypothesis. Therefore \( y_i \) must be equal to 0.

(2) If \( \alpha^* = o_i \), when \( 0 \leq \alpha < 1 \), we have \( o_i - o_j - \alpha_i (o_i - o_j) = u_i - v_i \). That is, \( u_i - v_i = (1 - \alpha_i) (o_i - o_j) \geq 0 \). Thus we have \( u_i > v_i > 0 \). Because \( u_i \geq 0, v_i \geq 0, u_i w_i = 0 \), then there must be \( v_i = 0, u_i > 0 \). According to the principle of complementary slackness, we have \( y_i = -w_i \); when \( \alpha = 1 \), if \( y_i > 0 \), there must be \( \alpha = 0 \) according to Property 1, which contradicts the premise. In consequence, when \( \alpha = 1 \), we have \( y_i \leq 0 \), and according to restriction (7–2), \( -w_i \leq y_i \). Therefore, if \( \alpha^* = o_i \), then we have \( -w_i \leq y_i \leq 0 \). Similarly, we can prove that if \( \alpha^* = o_i \), then \( 0 \leq y_i \leq w_i \).

References