A MODEL FOR PAIRS OF BEATTY SEQUENCES

Y. GINOSAR AND I. YONA

ABSTRACT. Beatty sequences \([n\alpha_1 + \beta_1]_{n\in\mathbb{Z}}\) and \([m\alpha_2 + \beta_2]_{m\in\mathbb{Z}}\) are recorded by two athletes running in opposite directions in a round stadium. This approach suggests a nice interpretation for well known partitioning criteria: such sequences (eventually) partition the integers essentially when the athletes have the same starting point.

A remarkable observation due to S. Beatty says that if \(w\) is any positive irrational number, then the sequences \(1 + w, \quad 2(1 + w), \quad 3(1 + w), \ldots\) contain one and only one number between each pair of consecutive positive integers.

Denoting \(\alpha_1 := 1 + w, \quad \alpha_2 := 1 + \frac{1}{w}\),

the corresponding sequences of (floor) integer parts

\[S(\alpha_1) := [n\alpha_1]_{n\in\mathbb{N}}, \quad S(\alpha_2) := [m\alpha_2]_{m\in\mathbb{N}}\]

are called Beatty sequences with moduli \(\alpha_1, \alpha_2\) respectively.

Note that the sum of \(\alpha_1\) and \(\alpha_2\) is equal to their product, i.e. they satisfy

\(\frac{1}{\alpha_1} + \frac{1}{\alpha_2} = 1.\)

Beatty’s result can thus be reformulated as follows:

**Theorem 1.** (Beatty, see [1]) Let \(\alpha_1, \alpha_2\) be two positive irrational numbers satisfying \(\frac{1}{\alpha_1} + \frac{1}{\alpha_2} = 1\), then the Beatty sequences \(S(\alpha_1), S(\alpha_2)\) partition \(\mathbb{N}\).

The converse of Theorem 1 is also valid: since the density of a Beatty sequence \([n\alpha]_{n\in\mathbb{N}}\) in \(\mathbb{N}\) is equal to \(\frac{1}{\alpha}\), then \(S(\alpha_1), S(\alpha_2)\) partition \(\mathbb{N}\) only if \(\alpha_1\) and \(\alpha_2\) satisfy (0.1) (and hence there exists a positive number \(w\) such that \(\alpha_1 = 1 + w\) and \(\alpha_2 = 1 + \frac{1}{w}\)).

In 1957 Th. Skolem generalized the above theorem to non-homogeneous Beatty sequences, i.e. double infinite sequences of the form

\[S(\alpha, \beta) := [n\alpha + \beta]_{n\in\mathbb{Z}}, \quad \alpha \in \mathbb{R}^+, \quad \beta \in \mathbb{R}.\]

When \(\alpha_1\) and \(\alpha_2\) are irrational, the question is when \(S(\alpha_1, \beta_1), S(\alpha_2, \beta_2)\) eventually partition \(\mathbb{Z}\), that is any sufficiently large (and any sufficiently small) integer belongs exactly to one of the sequences.

**Theorem 2.** (Skolem [11], see also [3, 5]) Let \(\alpha_1, \alpha_2\) be two positive irrational numbers satisfying \(\frac{1}{\alpha_1} + \frac{1}{\alpha_2} = 1\), and let \(\beta_1, \beta_2\) be real numbers. Then \(S(\alpha_1, \beta_1), S(\alpha_2, \beta_2)\)
eventually partition \( \mathbb{Z} \) if and only if
\[
\frac{\beta_1}{\alpha_1} + \frac{\beta_2}{\alpha_2} \in \mathbb{Z}.
\]
Moreover, if (0.2) holds, then \( S(\alpha_1, \beta_1), S(\alpha_2, \beta_2) \) partition \( \mathbb{Z} \) with an exception of, perhaps, one repeated integer \( n_0 \) and one missing integer \( n_0 - 1 \).

Theorem 1 clearly follows from Theorem 2, since 0 belongs to both sequences \( S(\alpha_1, 0) = S(\alpha_1, \beta_1) \) and \( S(\alpha_2, 0) = S(\alpha_2, \beta_2) \) (whereas \(-1\) is in neither one of them).

Hence \( \{1, 2, 3, \ldots\} \) (as well as any subset of \( \mathbb{Z} \setminus \{-1, 0\} \)) are disjointly covered by these sequences.

A rational analogue of Theorem 2 was established in 1969 by A.S. Fraenkel. Obviously, if the moduli are rational, then the notions of a partition and an eventual partition of the integers are the same. The criterion in [3] is given here in a slightly different formulation:

**Theorem 3.** (Fraenkel [3]) Let \( r > s \) be two coprime positive integers. Then the Beatty sequences \( S(\frac{r}{s}, \beta_1) \) and \( S(\frac{r-s}{s}, \beta_2) \) partition \( \mathbb{Z} \) if and only if
\[
s\beta_1 + [(r-s)\beta_2] \equiv r - 1 \mod r.
\]

This note suggests a perceptible approach to Beatty sequences by interpreting \( \omega \) as a certain ratio of speeds of two athletes running in opposite directions in a round stadium. This interpretation yields new proofs of Theorems 1 and 2 (in sections 1 and 3 respectively herein). Other proofs of these theorems can be found in [2, 4, 6, 9].

In §5, we show that in both the rational and the irrational cases, the partition condition says that the athletes essentially have the same starting point.

A natural question then arises:

**Question 4.** Given two real numbers \( \alpha_1 \) and \( \alpha_2 \), do there exist \( \beta_1, \beta_2 \) such that the corresponding Beatty sequences \( S(\alpha_1, \beta_1) \) and \( S(\alpha_2, \beta_2) \) are disjoint?

A complete answer to Question 4 was given by R. Morikawa. It suggests an interesting notion of “coprimeness” of pairs of real numbers.

In §6 we review the answer to Question 4 and apply again the running model to prove the following case where \( \alpha_1 \) and \( \alpha_2 \) are irrational and of rational ratio.

**Theorem 5.** (Morikawa [8]) Let \( \alpha_1 = r\gamma, \alpha_2 = s\gamma \), where \( \gamma \) is irrational and \( r, s \) are coprime positive integers. Then there exist \( \beta_1, \beta_2 \) such that \( S(\alpha_1, \beta_1) \) and \( S(\alpha_2, \beta_2) \) are disjoint if and only if \( \gamma > 2 \).

An extensive bibliography on Beatty sequences and their relations to various topics such as Sturmian words and Wythoff’s game can be found in [7, 12, 14].

**Acknowledgement.** We thank Mota Frances, the third runner.

1. **Proof of Beatty’s Theorem**

Assume that two athletes \( X \) and \( Y \) run in opposite directions in a round stadium of length 1. Their common starting point is denoted by \( O \) and their speeds are \( \frac{1}{\alpha_1} \) and \( \frac{1}{\alpha_2} \) respectively. In other words, \( X \) and \( Y \) complete a full round of the stadium in \( \alpha_1 \) and \( \alpha_2 \) time units respectively. Since their speeds sum up to 1, \( X \) and \( Y \) meet every time unit. Each time one of them passes \( O \), the number of times \( X \) and
Y have met so far is recorded. It is easily verified that when $X$ passes $O$ for the $n$-th time, the number \([n\alpha_1]\) is recorded, and when $Y$ passes $O$ for the $m$-th time the number \([m\alpha_2]\) is recorded. The recorded sequences are therefore the Beatty sequences $S(\alpha_1)$ and $S(\alpha_2)$.

When $\alpha_1$ and $\alpha_2$ are irrational, then so is $w = \alpha_1 - 1$, which, by (0.1), is just the ratio of these speeds. Hence $X$ and $Y$ never meet exactly at $O$ (except at $t = 0$). Therefore, between two meetings of $X$ and $Y$, exactly one of them passes $O$. It follows that any natural number can be uniquely expressed either as $[n\alpha_1]_{n\in\mathbb{N}}$ or as $[m\alpha_2]_{m\in\mathbb{N}}$, proving that $S(\alpha_1), S(\alpha_2)$ partition $\mathbb{N}$.

\[\square\]

2. The Non-Homogeneous Model

The above model can be fitted for non-homogeneous Beatty sequences $S(\alpha_1, \beta_1)$ and $S(\alpha_2, \beta_2)$ with any positive real numbers $\alpha_1, \alpha_2$ as follows:

Firstly, since for any integer $m$, the set of values of the double infinite Beatty sequences $S(\alpha, \beta)$ and $S(\alpha, \beta + m\alpha)$ are equal, then by adding or subtracting an appropriate multiple of $\alpha_i$ to $\beta_i$, we may assume

\begin{equation}
0 \leq \beta_i < \alpha_i
\end{equation}

for $i = 1, 2$.

As before, let two athletes $X$ and $Y$ run in opposite directions in a round stadium of length 1 with speeds $\frac{1}{\alpha_1}$ and $\frac{1}{\alpha_2}$ respectively. At time $t = 0$, place $X$ and $Y$ at the points whose distances from $O$ equal $\frac{\beta_1}{\alpha_1}$ and $\frac{\beta_2}{\alpha_2}$ respectively, opposite their running directions.

Whenever one of the athletes passes $O$, at time $t$ say, the number $[t]$ is recorded. It is easily verified that when $X$ passes $O$ for the $n$-th time, the number $[(n - 1)\alpha_1 + \beta_1]$ is recorded, and when $Y$ passes $O$ for the $m$-th time the number $[(m - 1)\alpha_2 + \beta_2]$ is recorded.

Assume that the athletes have been recording the integers $[t]$ also while running at negative times $t < 0$. Then the running model produces the non-homogeneous Beatty sequences $S(\alpha_1, \beta_1)$ and $S(\alpha_2, \beta_2)$.

Evidently, these sequences are disjoint (alternatively, cover the integers) if and only if at most (alternatively, at least) one of these athletes passes $O$ between any two consecutive integer time units. The sequences are eventually disjoint (eventually cover the integers) if and only if the above properties are respectively satisfied as from some $k \in \mathbb{Z}$.

Remark 2.1. A set of $n$ non-homogeneous Beatty sequences $S(\alpha_1, \beta_1), \ldots, S(\alpha_n, \beta_n)$ can similarly be modelled as follows: firstly, we may once again assume (2.1) for any $1 \leq i \leq n$ without changing the values of the Beatty sequences.

Next, let $X_0, X_1, \ldots, X_n$ be $n + 1$ athletes running in a round stadium of length 1 in the same direction with speeds $x_0, x_1, \ldots, x_n$ respectively. Their speeds relative to the slowest athlete $X_0$ are given by

\[x_j - x_0 := \sum_{i=1}^{j} \frac{1}{\alpha_i}, \quad 1 \leq j \leq n.\]
At time $t = 0$, place the $j$-th athlete at the point whose distance from $X_0$ opposite the running direction equals

$$d_j := \text{frac}(\sum_{i=1}^{j} \frac{\beta_i}{\alpha_i}), \quad 1 \leq j \leq n,$$

where $\text{frac}(u) := u - \lfloor u \rfloor$ denotes the fractional part of a real number $u$.

If at time $t$, $X_i \ (1 \leq i \leq n)$ passes $X_{i-1}$ (which is indeed slower), then the number $\lfloor t \rfloor$ is recorded. It is easily verified that $X_i$ records the Beatty sequence $S(\alpha_i, \beta_i)$ for any $1 \leq i \leq n$.

It follows that the sequences $S(\alpha_1, \beta_1), \ldots, S(\alpha_n, \beta_n)$ are disjoint (alternatively, cover the integers) exactly if at most (alternatively, at least) one of the athletes $X_i$ passes $X_{i-1}$ between two consecutive integer time units. The two athletes’ model in §2 can be regarded as a special instance for $n = 2$, where $X_0$ is $X$, $X_2$ is $Y$ and where we “accompany” the athlete $X_1$ as a steady point $O$.

3. Proof of Skolem’s Theorem

Define two domains in the stadium: let $A$ be the set of all points whose distance from $O$ are less than $\frac{1}{\alpha_1}$ opposite the running direction of $X$ and let $B$ be the set of all points whose distance from $O$ are less than $\frac{1}{\alpha_2}$ opposite the running direction of $Y$. Since (0.1) is assumed, then the half closed domains $A$ and $B$ almost partition the stadium. Their intersection is the point $O$, while their union misses only the other edge point which we denote by $E$ (see figure 1).

At time $t = 0$, $X$ and $Y$ see each other at the distance

$$d_0 := \text{frac}(\frac{\beta_1}{\alpha_1} + \frac{\beta_2}{\alpha_2}),$$

Figure 1.
Lemma 4.1. With the above notation, the following are equivalent
nor condition (0.3) in the hypothesis.

Now, assume that the Beatty sequences satisfy (0.2). By (3.1), X and Y are at
the same point at time t = 0 and hence they actually meet every t ∈ Z. Each one of
these meeting points is exactly in one of the domains A or B, unless it is on the two
edges of these domains, either O or E. However, as before, when α₁ and α₂, and
hence also w, are irrational, X and Y may meet exactly at O at most once. If this
happens, denote the number of this specific meeting by n₀. Then n₀ is recorded
twice, both by X and by Y. Clearly, in this case, the n₀ − 1-th meeting was in the
other edge E (which is neither in A nor in B) and hence the integer n₀ − 1 is not
recorded. Therefore, with this possible exception, between two meetings of X and
Y, exactly one of them passes O. It follows that any integer other than n₀ − 1 and
n₀ can be uniquely expressed either as |nα₁ + β₁| or as |mα₂ + β₂|, proving that
S(α₁, β₁), S(α₂, β₂) is a partition of Z \ {n₀ − 1, n₀}.

Conversely, suppose then that condition (0.2) does not hold. Then by a standard
argument about the density of irrational rotations on the unit circle (Jacobi, see
Theorem 2.1), there are infinitely many tᵢ ∈ Z, in both time directions, such
that at time tᵢ the athlete X is in A and the athlete Y is in B. Similarly, there are
infinitely many sᵢ ∈ Z, in both time directions, such that at time sᵢ X is outside A
and Y is outside B. By the above interpretation of the domains A and B, both sets
S(α₁, β₁) ∩ S(α₂, β₂) and S(α₁, β₁) ∩ S(α₂, β₂) are doubly infinite and therefore
do not admit an eventual partition of Z.

4. Proof of Fraenkel’s Theorem

Firstly, by adding or subtracting a multiple of αᵢ to βᵢ (i = 1, 2), we keep as-
suming (2.1) changing neither the values of our non-homogeneous Beatty sequences
nor condition (1.3) in the hypothesis.

We apply once again the running model (2.2) and make the following notation: let
xₖ and yₖ denote the distances of X and Y from O, opposite the running direction
of X, at the integer times k ∈ Z. We need the following

Lemma 4.1. With the above notation, the following are equivalent

1. S(α, β) and S(β, α) partition Z.
2. for every k ∈ Z, xₖ ∈ [0, 1/₇) if and only if yₖ ∈ (0, 1/₇].
3. for every k ∈ Z, there exists an integer 0 ≤ jₖ ≤ r − 1 such that xₖ ∈
   [jₖ/₇, jₖ/₇ + 1] and yₖ ∈ (jₖ/₇ + 1/₇, jₖ/₇ + 1/₇ + 1]
4. there exists an integer 0 ≤ j₀ ≤ r − 1 such that x₀ ∈ [j₀/₇, j₀/₇ + 1/₇] and y₀ ∈
   (j₀/₇ + 1/₇, j₀/₇ + 1/₇ + 1].

Proof of Lemma 4.1. Note that the same argument as in the proof of Theorem
2 yields a necessary and sufficient condition for the (eventual) partition of Z by
S(α₁, β₁), and S(α₂, β₂) as follows: for every integer time t ∈ Z, one of the athletes
is in A if and only if the other athlete is not in B. It is easily checked that the half
open segment (0, 1/₇] is the domain A, whereas the half open segment (0, 1/₇] is the
complement of the domain $B$, proving $(1)\iff(2)$. Next, $X$ and $Y$ return to the same relative position every integer time units. More precisely, for every $k \in \mathbb{Z}$,
\[
x_k - y_k \equiv d_0 \mod \mathbb{Z}.
\]
Now, since $s$ is prime to $r$, and since $X$ and $Y$ pass the distances $\frac{s}{r}$ and $\frac{r-s}{r}$ respectively every integer time units, then each of the sets $\{x_k\}_{k \in \mathbb{Z}}$ and $\{y_k\}_{k \in \mathbb{Z}}$ divide the stadium into $r$ arcs of equal length $\frac{1}{r}$. In particular, both sets admit exactly one member in every segment $\left[j \frac{r}{r}, j + 1 \frac{r}{r}\right)$, for every integer $0 \leq j \leq r - 1$. This observation implies that $(2),(3)$ and $(4)$ are equivalent, and completes the proof of the lemma.

Back to the proof of Theorem 3, we make use of condition $(4)$ in Lemma 4.1. This condition was shown to be necessary and sufficient for partitioning the integers by $S(\frac{s}{r}, \beta_1)$ and $S(\frac{r-s}{r}, \beta_2)$. Recall the initial location of $X$ and $Y$ given in §2:
\[
x_0 = \beta_1 \alpha_1, \quad y_0 = 1 - \beta_2 \alpha_2.
\]
Let $0 \leq j_0 \leq r - 1$ be an integer such that $\frac{s\beta_1}{\alpha_1}$ belongs to the half open segment $\left[j_0 \frac{r}{r}, j_0 + 1 \frac{r}{r}\right)$. Then condition $(4)$, which is equivalent to the partitioning condition, says that $1 - \frac{\beta_2}{\alpha_2}$ belongs to the half open segment $\left(\frac{s\beta_1}{r}, \frac{s\beta_1 + 1}{r}\right]$. Equivalently,
\[
j_0 \leq s\beta_1 < j_0 + 1 \quad \text{and} \quad j_0 < r - (r-s)\beta_2 \leq j_0 + 1.
\]
Consequently,
\[
[s\beta_1] + [(r-s)\beta_2] = r - 1
\]
is a necessary and sufficient condition for $S(\frac{s}{r}, \beta_1)$ and $S(\frac{r-s}{r}, \beta_2)$ to partition the integers under the assumption $(2.1)$. Relaxing this assumption, we obtain that $(0.3)$ is a necessary and sufficient partitioning condition. Theorem 3 is proven. \hfill \Box

5. Comparison between the cases

When the moduli $\alpha_1, \alpha_2$ are irrational and satisfy $(0.1)$, then Theorem 2 says that $S(\alpha_1, \beta_1)$ and $S(\alpha_2, \beta_2)$ eventually partition $\mathbb{Z}$ precisely when the athletes $X$ and $Y$ are at the very same point at $t = 0$. The rational case can be interpreted similarly:

Let $\alpha_1 = \frac{s}{r}, \alpha_2 = \frac{r-s}{r} \in \mathbb{Q}$. Condition $(0.2)$, which says that the two athletes have a common starting point, is equivalent in this case to the condition
\[
s\beta_1 + (r-s)\beta_2 \in r\mathbb{Z}.
\]
Next, note that the Beatty sequences $S(\frac{s}{r}, \beta_1)$ and $S(\frac{s}{r}, \beta_2')$ (with the assumption $(2.1)$ on both) are equal if and only if
\[
\frac{[s\beta_1]}{s} \leq \beta_1' < \frac{[s\beta_1] + 1}{s}.
\]
Similarly, under the assumption $(2.1)$, $S(\frac{r-s}{r}, \beta_2)$ and $S(\frac{r-s}{r}, \beta_2')$ are equal if and only if
\[
\frac{[(r-s)\beta_2]}{r-s} \leq \beta_2' < \frac{[(r-s)\beta_2] + 1}{r-s}.
\]
Now, suppose the partitioning condition $(0.3)$ of Theorem 3 holds. Let
\[
\beta_1' := \frac{[s\beta_1] + \nu}{s}
\]
and
\[ \beta'_2 := \frac{[(r - s)\beta_2] + 1 - \nu}{r - s}, \]
for some \(0 < \nu < 1\).

We have relocated the athletes such that
\[ s\beta'_1 + (r - s)\beta'_2 \in r\mathbb{Z}, \]
that is to a common starting point.

Since (5.2) and (5.3) are satisfied, then
\[ S(r s, \beta_1) = S(r s, \beta'_1) \]
and
\[ S(r r - s, \beta_2) = S(r r - s, \beta'_2), \]
and hence these Beatty sequences still partition the integers.

Conversely, if the common starting point condition (5.1) is satisfied, then the partitioning condition (0.3) holds precisely when
\[ s\beta_1 \in \mathbb{Z} \]
(equivalently, \( (r - s)\beta_2 \in \mathbb{Z} \)). This happens exactly when this common starting point is not one of the lattice points \( \{\frac{j}{r}\}_{j=0}^{r-1} \). We summarize the above discussion:

**Corollary 5.1.** Let \( r > s \) be coprime positive integers. Suppose that the Beatty sequences \( S(r s, \beta_1) \) and \( S(r r - s, \beta_2) \) partition \( \mathbb{Z} \). Then \( \beta_1 \) and \( \beta_2 \) can be chosen (leaving these sequences unchanged) such that with the interpretation of §2, the athletes \( X \) and \( Y \) are at the same point at \( t = 0 \). Conversely, suppose that the athletes are at the same point at \( t = 0 \), with the additional demand that this point is not one of the lattice points \( \{\frac{j}{r}\}_{j=0}^{r-1} \). Then the corresponding Beatty sequences \( S(r s, \beta_1) \) and \( S(r r - s, \beta_2) \) partition \( \mathbb{Z} \).

### 6. Disjoint Sequences

By viewing arithmetical progressions as Beatty sequences with integral moduli, one can formulate the well known Chinese remainder theorem as follows:

**Theorem 6.** (the Chinese remainder theorem) Two integers \( n, m \) are coprime if and only if \( S(n, \beta_1) \cap S(m, \beta_2) \neq \emptyset \) for any \( \beta_1, \beta_2 \in \mathbb{R} \).

Two real numbers \( \alpha_1, \alpha_2 \) satisfying the property that any two Beatty sequences \( S(\alpha_1, \beta_1) \) and \( S(\alpha_2, \beta_2) \) intersect, may therefore be called *coprime* for short.

Th. Skolem gave the following necessary condition for two Beatty sets to be disjoint:

**Theorem 7.** (Skolem, see [3]) Suppose \( S(\alpha_1, \beta_1) \) and \( S(\alpha_2, \beta_2) \) are disjoint. Then either

1. \( \frac{\alpha_1}{\alpha_2} \) is rational, or
2. there exist positive integers \( m, n \) such that
\[ \frac{m}{\alpha_1} + \frac{n}{\alpha_2} = 1, \quad \frac{m\beta_1}{\alpha_1} + \frac{n\beta_2}{\alpha_2} \in \mathbb{Z}. \]

Condition (2) in Theorem 7 is also sufficient for the two sequences to be disjoint. It takes care of integral multiples of complementary Beatty sequences (see Theorem 2) and is therefore well understood.

The case \( \alpha_1, \alpha_2 \in \mathbb{Q} \) is captured by the following result known as the *Japanese remainder theorem*:

**Theorem 8.** (Morikawa [8], see also [10]) Let \( \alpha_1 := \frac{p_1}{q_1}, \alpha_2 := \frac{p_2}{q_2} \) two rational numbers in reduced forms, let \( p := (p_1, p_2), q := (q_1, q_2), u_1 := \frac{p_1}{q}, u_2 := \frac{p_2}{q} \). Then
\( \alpha_1, \alpha_2 \) are coprime if and only if there do not exist positive integers \( k \) and \( l \) such that

\[
ku_1 + lu_2 = p - 2u_1u_2(q - 1).
\]

Note that with the notation of the Japanese remainder theorem, when \( q = 1 \) the existence of \( k, l \in \mathbb{Z}^+ \) that satisfy (6.1) is equivalent to the existence of \( m, n \in \mathbb{Z}^+ \) in condition (2) of Theorem 7 by the equations

\[
kp_1 = mp, \quad lp_2 = np.
\]

In particular, Theorem 7(2) covers all the cases of \( \alpha_1, \alpha_2 \in \mathbb{Z} \) (i.e., where \( q_1 = q_2 = 1 \)).

Theorem 5 completes the picture given in Theorems 7 and 8. It deals with the remaining case, which falls under condition (1) in Theorem 7, namely, the moduli \( \alpha_1, \alpha_2 \) are irrational numbers of rational ratio.

We describe the disjointness condition using the running model in §2 as follows: suppose that at time \( t \) the athlete \( X_1 \) passes the point \( O \) and hence records the integer \( [t] \). At the same time \( t \), let the athlete \( X_2 \) be at a point whose distance from \( O \) opposite the running direction is \( \rho(t) \). The disjointness condition implies that \( X_2 \) does not record the integer \( [t] \). This means that \( X_2 \) does not pass the point \( O \) within the time interval \( [t], [t] + 1 \). Hence, the distance \( \rho(t) \) satisfies

\[
1 - \frac{t}{\alpha_2} < \rho(t) < 1 - \frac{\frac{t}{\alpha_2}}{\alpha_2}.
\]

Suppose now that \( \frac{\alpha_1}{\alpha_2} = \frac{r}{s} \in \mathbb{Q} \), where \( r \) and \( s \) are coprime positive integers and let \( \gamma := \frac{\alpha_1}{\alpha_2} = \frac{r}{s} \). Then \( X_1 \) reaches \( O \) every \( \alpha_1 \) time units. By this time, \( X_2 \) passes \( \frac{r}{s} \) of the stadium. From the condition \( (r, s) = 1 \), these steps determine a lattice \( \Gamma \) of length \( \frac{1}{s} \).

We are ready to prove Theorem 5. This is done by applying once more the argument about density of irrational rotations on the unit circle. If the modulus \( \alpha_1 \) is irrational, then for any small \( \epsilon > 0 \), \( X_1 \) passes the point \( O \) at times whose fractional parts are less than \( \epsilon \) as well as at times whose fractional parts are larger than \( 1 - \epsilon \). Hence, (6.2) is satisfied in this case if and only if \( O \) is located exactly between two points of \( \Gamma \), and the distance \( O \) to the closest lattice points is larger than the speed of \( X_2 \). That is to say \( \frac{1}{2s} > \frac{1}{\alpha_2} \). This completes the proof of Theorem 5.

Remark 6.1. Let \( r, s \) be coprime positive integers. Then with the terminology of the Chinese Remainder Theorem, Theorem 5 says that if \( \gamma \) is irrational, then \( r\gamma \) and \( s\gamma \) are coprime if and only if \( \gamma < 2 \). When \( \gamma \in \mathbb{Q} \), it is not hard to see that the condition \( \gamma < 2 \) implies that (6.1) is never satisfied. Hence, \( \gamma < 2 \) is sufficient for \( r\gamma \) and \( s\gamma \) to be coprime . In fact, when \( \gamma \) is an integer, the condition \( \gamma < 2 \) is clearly necessary and sufficient for \( \alpha_1 = r\gamma \) and \( \alpha_2 = s\gamma \) to be coprime.

References


A MODEL FOR PAIRS OF BEATTY SEQUENCES


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HAIFA, HAIFA 31905, ISRAEL

E-mail address: ginosar@math.haifa.ac.il