Prolate spheroidal wave functions induce Gaussian chip waveforms

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Abstract—Slepian’s prolate spheroidal wave function (PSWF) is optimum in energy concentration within given frequency bandwidth and time-width. However it is not employed in communications mainly because its inter-symbol interference is unfavorable. We show that introducing Markovian spreading codes makes PSWF be a strong candidate for the chip waveform in CDMA systems. For design tractability, Gaussian waveform is recommended.

I. INTRODUCTION

It is not possible to generate completely band-limited signals in finite time; they are physically unrealizable. Defining energy concentration ratios in a time duration \( T \) and a frequency bandwidth \( W \), respectively, as \( \alpha^2 \) and \( \beta^2 \), Slepian and Pollak [1] and Landau and Pollak [2] showed that the maximum attainable \( \alpha^2 \) and \( \beta^2 \) are given by the largest eigenvalue of a Fredholm integral equation with sinc kernel whose eigenfunctions are known as prolate spheroidal wave functions (PSWFs). The completely bandlimited signals are physically unrealizable, and therefore we should assume the channels are approximately bandlimited.

Code division multiple access (CDMA) has become a standard method to realize the multiplex of users. Sum capacity of synchronous and time-discrete CDMA system is maximized by Welch bound equality (WBE) codes [3]. The aim of this paper is to give a rigorous formulation of the approximately band-limited and time-continuous CDMA channels from the Slepian’s standpoint [4], [5]. The most important assumption of this paper is that we cannot make the time delays between different users’ signals infinitesimal and there exists unavoidable time delays. Thus, we consider time delays are continuous-valued random variables.

Spreading codes are currently generated by linear feedback shift registers (LFSRs). In this paper, they are considered to be generated from a stochastic process and are denoted by \( \{X_k\} \).

We assume they have correlations such that \( E_X[X_nX_m]=\lambda^{n-m} \) with \( -1<\lambda<1 \), where \( E_X[\cdot] \) denotes the expectation with respect to codes. As a model of noise process and interferences, such an exponentially decaying autocorrelation has been employed [6], [7].

PSWFs do not satisfy the Nyquist condition; they have inter-symbol-interference (ISI) as well as inter-chip interference (ICI). We therefore analyze the signal-to-interference-plus-noise ratio (SINR) of the receiver output. It is observed SINR performance of the eigenfunction corresponding to the largest eigenvalue is very poor and decreasing \( \beta^2 \) improves the SINR. This implies the spill-over is not a worthless thing but it turns to be advantageous for decreasing interferences. Hence, we should look for a chip waveform whose \( (\alpha^2, \beta^2) \) is in the interior of the Slepian’s attainable region to enhance SINR.

II. TIME-CONTINUOUS BAND-LIMITED CDMA CHANNEL

A time-continuous and bandlimited single-user channel model [8] is depicted in Fig. 1. The channel noise is assumed to be additive and Gaussian. Shannon’s capacity formula \( W \log (1 + P_0/N_0W) \) is based on the ideal low-pass filter

\[
\tilde{\phi}(\omega) = \begin{cases} 
\frac{1}{2W} & \text{if } |\omega| \leq 2\pi W, \\
0 & \text{if } |\omega| > 2\pi W,
\end{cases}
\]

Fig. 1. A model of time-continuous and bandlimited single-user channel.

where \( \tilde{\phi}(\omega) = \int_{-\infty}^{\infty} \phi(t) e^{-\sqrt{-1}\omega t} dt \) and \( P_0 \) is the signal power, \( N_0 \) is the spectral density of Gaussian noise, and \( W \) is the bandwidth. However, the chip waveform \( \phi(t) = \frac{\sin 2\pi W t}{2\pi W t} \) is physically unrealizable. Wyner [8], [9] gave a rigorous proof of the capacity formula, based on physically possible filters and Gaussian noise, where Slepian’s prolate spheroidal wave functions play an important role.

Consider a baseband equivalent model of asynchronous direct sequence/code division multiple access (DS/CDMA) systems with \( K \) active users (Fig.2). The \( i \)-th user’s signature
waveform is defined as
\[ x^{(i)}_{\phi}(t) = \sum_{n=0}^{N-1} x^{(i)}_{n}\phi(t - nT_c), \]  
where \( \{x^{(i)}_{n}\}_{n=0}^{N-1} \in \{+1, -1\}^N \) is a spreading sequence of the \( i \)-th user with spreading factor \( N \) and \( \phi(t) \) is a chip waveform. We assume the chip duration \( T_c = 1 \) and that chip waveform has tails outside this interval. Signature waveform has excess energy outside the data duration \( T_d = NT_c \).

Let \( \{d^{(i)}_{p}\}_{p=-\infty}^{\infty} \) be a data sequence of the \( i \)-th user. Then the received signal is
\[ r(t) = \sum_{i=1}^{K} \sum_{p=-\infty}^{\infty} A_i d^{(i)}_{p} x^{(i)}_{\phi}(t - pT_d - t_i) + n(t), \]  
where \( T_d = NT_c \) is the data duration, \( A_i \) and \( t_i \) are an amplitude and a time delay of the \( i \)-th user’s signal, and \( n(t) \) is an additive white Gaussian noise with two-sided noise spectrum \( \frac{N_0}{2} \). For simplicity, \( A_i = 1 \) is assumed for \( i = 1, \ldots, K \). The matched filter output is given by
\[ Z^{(i)}_{p} = \int_{-\infty}^{\infty} r(t + pT_d + t_i) x^{(i)}_{\phi}(t) \, dt = S^{(i)}_{p} + \eta^{(i)} + \xi^{(i)} + I^{(i)}_{K,p}, \]  
where \( S^{(i)}_{p} \) is the \( i \)-th user’s signal component, \( \eta^{(i)} \) is noise component, \( \xi^{(i)} \) is ISI, and \( I^{(i)}_{K,p} = \sum_{j=1,j\neq i}^{K} I^{(i,j)}_{p} \) denotes MAI from other \( K - 1 \) users.

Assume the noise, ISI and MAI are uncorrelated. The signal-to-noise plus interference ratio (SINR) of the single-user receiver is
\[ \text{SINR}^{(i)} = \frac{\{S^{(i)}_{p}\}^2}{\sum_{j\neq i} \{I^{(i,j)}_{p}\}^2 + \mathbb{E}[\{\eta^{(i)}\}^2] + \{\xi^{(i)}\}^2}, \]  
where \( \mathbb{E}[\{\eta^{(i)}\}^2] = \frac{N_0}{2} \|x^{(i)}_{\phi}(t)\|^2 \) and \( \|\phi\|^2 = \int_{-\infty}^{\infty} |\phi(t)|^2 \, dt \).

For multiuser detection (MUD) receivers, one can select sum energy outside the data duration \( T_d = NT_c \).

III. PROLATE SPHEROIDAL WAVE FUNCTIONS

It is well-known that any waveform has uncertainty trade-off in frequency and time domains and that Gaussian pulse is the optimum one when the variance of the energy distribution function is concerned. On the other hand, Slepian and Pollak [1] and Landau and Pollak [2] discussed this uncertainty problem in a different manner, which is suitable for communication engineering. They introduced linear operators \( B_{T\Omega} \) and \( D_{T} \) for limiting the bandwidth and time-width, respectively, i.e.
\[ B_{T\Omega}\phi(t) = \frac{1}{2\pi} \int_{-\Omega}^{\Omega} \tilde{\phi}(\omega) e^{\omega t} \, d\omega, \]  
\[ D_{T}\phi(t) = \begin{cases} \phi(t) & -\frac{T}{2} \leq t \leq \frac{T}{2}, \\ 0 & |t| > \frac{T}{2}. \end{cases} \]

They investigated the achievable energy concentration ratios \( \alpha^2 \) and \( \beta^2 \), in a given angular frequency bandwidth, \( \Omega = 2\pi W \) and a given time interval \( T \).
\[ \alpha^2 = \frac{\|D_{T}\phi\|^2}{\|\phi\|^2}, \quad \beta^2 = \frac{\|B_{2\pi W}\psi\|^2}{\|\psi\|^2}. \]

There are two fundamental questions: for bandlimited waveforms what is the maximum energy concentration ratio in time slot \( T \) and for time-limited waveforms what is the maximum energy concentration ratio in frequency band \( W \). The answers to these questions are, respectively, given by the maximum eigenvalues of the following integral equations
\[ D_{T}B_{2\pi W}\psi_{\alpha} = \gamma_{\alpha}\psi_{\alpha}, \quad B_{2\pi W}D_{T}\psi_{\beta} = \gamma_{\beta}\psi_{\beta}. \]

Each set of eigenfunctions \( \{\psi_{\alpha}^{(i)}\} \) and \( \{\psi_{\beta}^{(i)}\} \) is doubly orthogonal, i.e.
\[ \int_{-\infty}^{\infty} \tilde{\psi}_{\alpha}^{(i)}(f) \tilde{\psi}_{\beta}^{(j)}(f) \, df = \delta_{i,j}, \]  
\[ \int_{-\infty}^{\infty} \tilde{\psi}_{\beta}^{(i)}(t) \tilde{\psi}_{\beta}^{(j)}(t) \, dt = \delta_{i,j}. \]

Because \( B_{2\pi W} \) and \( D_{T} \) are dual with respect to Fourier transform, they have same eigenvalues \( \gamma_{\alpha}^{(i)} = \gamma_{\beta}^{(i)}(= \gamma^{(i)}) \) \( (i = 0, 1, \ldots) \). The eigenfunctions have a relation \( \psi_{\beta}^{(i)} = \left(\gamma^{(i)}\right)^{-1/2} D_{T} \psi_{\alpha}^{(i)}. \) Therefore it is sufficient to consider one of the two integral equations.
Landau and Pollak [2] showed that the possible combinations of \((\alpha^2, \beta^2)\) is the intersection of the unit square \(0 \leq \alpha^2 \leq 1\) and \(0 \leq \beta^2 \leq 1\) and the elliptical region
\[
\cos^{-1} \alpha + \cos^{-1} \beta \geq \cos^{-1} \sqrt{n(0)}.
\]
Any chip waveform should have its energy concentration ratio in this region. Fig. 3 shows the possible combinations for \(2WT = 1\), as well as rectangular, sinc, and root raised cosine pulses with roll-off factor 0.22. The dashed curve shows \((\alpha^2, \beta^2)\) for Gaussian waveforms \(\phi(t) = a \exp(-c^2t^2)\) where \(\tau\) is a variable parameter and \(a\) is a normalization factor.

The left equation in (10) is expressed as
\[
\gamma \phi(t) = \int_{-T/2}^{T/2} \phi(s) \frac{\sin 2\pi W(t - s)}{\pi(t - s)} ds \quad (|t| \leq \frac{T}{2}). \tag{12}
\]
This equation has isolated positive eigenvalues \(1 > \gamma_0 > \gamma_1, \ldots\). The eigenfunctions, denoted by \(\psi^{(i)}\), are referred to as prolate spheroidal wave functions, which are scaled versions of the solutions of the differential equation \(\frac{d}{dt}(1 - t^2) \frac{d}{dt} + (\chi - c^2t^2)\phi = 0 \quad (t \in [-1, 1])\), where \(c = \pi W T\) and \(\chi = \chi(c)\) is the eigenvalue of this equation [1].

A rigorous proof of Shannon’s capacity formula was given by Wyner [8], [9], where a code word \((x_0, x_1, \ldots, x_{n-1})\) is modulated into \(\sum_{i=0}^{n-1} x_i \psi^{(i)}(t)\) with \(n = \lfloor 2WT \rfloor\). In spite of its rigorousness, such a modulation, however, is not used, probably because \(n \gg 1\) filters are needed in parallel. Instead, we consider simple modulation in the form of \(\sum_{i=0}^{n-1} x_i \phi(t - i)\), where \(\phi(t)\) is called chip waveform.

Let us consider the band-limitedness and time-limitedness of the signature waveforms in CDMA systems in a way similar to Slepian [1], [2]. The signature waveform is \(X_\phi(t) = \sum_{n=0}^{N-1} X_n \phi(t - nT_c + \frac{n\pi}{NT_c})\), where the time shift \(\frac{n\pi}{NT_c}\) is added so that the time slot \(0, NT_c\) is replaced by \([-\frac{NT_c}{2}, \frac{NT_c}{2}]\). Assume the chip duration is \(T_c = 1\) and the symbol duration is \(T_d = N\). The bandwidth of the spreading codes is \(W = 1/(2T_c) = 1/2\). Note that we consider the bandwidth is \(W = 1/2\) for both \(\phi\) and \(X_\phi\) but time slots are \(T_c = 1\) and \(T_d = N\), respectively. Then
\[
\alpha^2 = \frac{\|D_N X_\phi\|^2}{\|X_\phi\|^2}, \quad \beta^2 = \frac{\|B_{T_c} X_\phi\|^2}{\|X_\phi\|^2}. \tag{13}
\]
Obviously, the optimum chip waveform depends on spreading codes \(X\). We can see
\[
\beta^2 = \frac{\int_{-\infty}^{\infty} |\hat{X}(\omega)|^2 |\hat{\phi}(\omega)|^2 d\omega}{\int_{-\infty}^{\infty} |X(\omega)|^2 |\phi(\omega)|^2 d\omega} \tag{14}
\]
where \(\hat{X}(\omega)\) is the Fourier transform of \(\{X_n\}\).

In order to see the effect of codes, let us consider that spreading codes are random variables with exponentially vanishing autocorrelation:
\[
\mathbb{E}_X[X_n X_m] = \lambda^{n-m}, \quad (n, m = 0, 1, \ldots, N - 1) \tag{15}
\]
where \(\mathbb{E}_X[\cdot]\) denotes the expectation with respect to codes and \(-1 < \lambda < 1\) is a parameter. The average power spectrum of the spreading codes is given by
\[
\frac{1}{N} \mathbb{E}_X[|\hat{X}(f)|^2] = \frac{1}{N} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} \mathbb{E}_X[X_n X_m] e^{-2\pi \sqrt{-1}(n-m)f} 
\approx \frac{1}{1 + \lambda^2 - 2\lambda \cos(2\pi f)}, \tag{16}
\]
when \(N \gg 1\), which is a periodic function with period \(2W = 1\) and is illustrated in Fig. 4. This figure shows the spreading codes with positive \(\lambda\) works as a low pass filter, whereas the codes with negative one works as a high pass filter. This function is applied in (14) therefore \(\beta^2\) increases (decreases, respectively) if positive (negative) \(\lambda\) is used. The case \(\lambda = 0\) implies i.i.d. codes which has flat power spectrum. Note that sequences generated by a Markov chain have the above exponential vanishing autocorrelation functions. R.E. Kalman found that such a Markov chain can be embedded into a chaotic dynamic of piecewise-linear Markov map [11]. Mazzini et. al. reported that Markovian spreading codes improves the bit error rate of chip-asynchronous CDMA systems [12].

Landolsi and Stark [13] assumed that chip waveforms are time limited within one chip duration to design inter-chip interference (ICI)-free solitary waves. For symbol-synchronous CDMA systems, two modulation methods using PSWFs are analyzed in [14]. One method is \(\sum_i x_i \psi^{(i)}(t)\), where \(\psi^{(i)}(t)\) are completely timelimited PSWFs in \(0, T_d\] and the other is \(\sum_i x_i \phi(t - i)\), where \(\phi(t)\) is completely timelimited PSWF in \(0, T_c\]. The former method is optimum in terms of fractional out-of-band energy bandwidth and the latter one is suboptimal and found better than rectangular pulse and half sine waves. In [15], a chip waveform design which allows ICI is presented to improve BER performance.

We give new integral equations corresponding to (13):

**Lemma 1:** For an absolutely bandlimited chip waveform, i.e., \(\phi(t) = B_{T_c} \psi(t)\), the maximum energy concentration of \(X_\phi(t)\) in \(t \in [-N/2, N/2]\) is given by the largest eigenvalue of the integral equation:
\[
\gamma_{X, \alpha} \mathbb{E}_X[|\hat{X}(f)|^2] \hat{\psi}_{X, \alpha}(f) = \int_{-1/2}^{1/2} K_\alpha(f, f') \hat{\psi}_{X, \alpha}(f') df'. \tag{17}
\]
where $X(f)$ is a Fourier transform of $\{X_n\}$ and

$$K_\alpha(f, f') = E_X[X(f)X(f')]\sin N\pi(f-f')\pi(f-f').$$

On the other hand, for an absolutely time-limited chip waveform in one chip duration, i.e. $\phi(t) = D_T\phi(t)$, the maximum energy concentration of $X_\phi(t)$ in frequency $f \in [-1/2, 1/2]$ is given by the largest eigenvalue of the integral equation:

$$\gamma_{X,\alpha}\psi_{X,\alpha}(t) = \int_{-1/2}^{1/2} \psi_{X,\alpha}(s)K_\beta(t, s)ds, \quad (|t| < \frac{1}{2})$$

(19)

$$K_\beta(t, s) = \frac{1}{N} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} E_X[X_nX_m]\sin N\pi(t+s+n-m)\pi(t+s+n-m).$$

(20)

Note that if $E_X[X_nX_m] = \delta_{n,m}$, the kernel $K_\beta$ reduces to the Slepian’s sinc kernel.

Let $\gamma_{X,\alpha}^{(0)} > \gamma_{X,\alpha}^{(1)} > \ldots$ and $\gamma_{X,\beta}^{(0)} > \gamma_{X,\beta}^{(1)} > \ldots$ be the eigenvalues of Eqs. (17) and (19) and let $\psi_{X,\alpha}^{(0)}(t), \psi_{X,\alpha}^{(1)}(t), \ldots$ and $\psi_{X,\beta}^{(0)}(t), \psi_{X,\beta}^{(1)}(t), \ldots$ be the corresponding eigenfunctions. These eigenfunctions depend on the spreading codes. The largest eigenvalue changes as shown in Table II. The dominant eigenfunctions $\psi_{\alpha}^{(0)}(f)$ and $\psi_{\beta}^{(1)}(f)$ are depicted in Fig. 5 and 6, respectively, for $\lambda = 0, \pm 0.4, \pm 0.8$. The chip waveform $\psi_{\beta}^{(0)}(t)$ time-limited to $[-0.5, 0.5]$ approaches to a rectangular pulse as $\lambda$ becomes large and to the shape like half sine pulse as $-\lambda$ becomes large.

### IV. Interferences

As far as energy concentrations $(\alpha^2, \beta^2)$ are concerned, spreading code with positive $\lambda$ is better and optimum choice of chip waveforms is $\phi(t) = c_0\psi_{X,\alpha}^{(0)}(t) + d_0\psi_{X,\beta}^{(1)}(t)$ for some constants $c_0$ and $d_0$. However, we do not recommend such spreading codes and chip waveforms for CDMA system, because this pair of codes and chip waveforms has very strong multiple-access interference (MAI), which limits the capacity of CDMA system. Table I shows that $\psi_{X,\alpha}^{(0)}$ and $\psi_{X,\beta}^{(0)}$ for i.i.d. codes case are worse than root raised cosine (rc) with roll-off factor 0.22 and rectangular (rect) pulses, respectively. This table implies we should use the second and third eigenfunctions, too. Hence the chip waveform is expressed by

$$\phi(t) = \sum_{i=0}^{\infty} c_i\psi_{X,\alpha}^{(i)}(t) + \sum_{i=0}^{\infty} d_i\psi_{X,\beta}^{(i)}(t).$$

(21)

The coefficients $c_i$ and $d_i$ should be chosen so as to achieve compromise between the energy concentrations $(\alpha^2, \beta^2)$ and the MAI. PSWFs do not satisfy the Nyquist condition, therefore we should analyze the SIR. Table I suggests that there is a tradeoff between SIR and $\beta^2$. Hence we should use negative $\lambda$, which sacrifices $\beta^2$ but improves SIR performance.
desired chip waveform is chosen so as to be in the interior of the Slepian’s region, but not on the boundary of it.

**Lemma 2:** For chip-asynchronous DS/CDMA systems with \( K \) active users, spreading factor \( N \), and chip waveform \( \phi(t) \), in an additive white Gaussian noise with two-sided spectrum density \( N_0/2 \), the signal to interference ratio of single user receiver output with Markov codes having eigenvalue \( \lambda \) is

\[
\text{SIR}(\lambda, N, K, \Phi) = \frac{1}{N} \left\{ \sum_{k=1}^{N} \left[ (N-k) \lambda^k \Phi(k) \right] \right\}^2 \\
\cdot \left[ (K-1) \sum_{k} \left( |k|^2 + \frac{1 + \lambda^2}{1 - \lambda^2} \lambda^k \Phi(k) \right) \right] + \frac{1}{N} E\{\xi_p^{-2}\}^{-1},
\]

where \( \Phi(t) = \int \phi(t')\phi(t-t')dt' \) and \( \Phi*\Phi(t) = \int \Phi(t')\Phi(t-t')dt' \). Then SIR is given by

\[
\text{SINR}(\lambda, N, K, \frac{N_0}{2}, \Phi) = \frac{1}{\text{SIR}} + \frac{N_0}{2K_0}
\]

where \( E_k = \sum_{k=1}^{N-1} (N-k) \lambda^k \Phi(k) \).

Eq. (22) implies that we need to calculate the fourth order quantity of \( \phi(t) \) to evaluate SIR. Double orthogonality (11) holds for \( \psi_{X,\alpha}^{(i)} \) and \( \psi_{X,\beta}^{(j)} \), but orthogonality does not hold between \( \psi_{X,\alpha}^{(i)} \) and \( \psi_{X,\beta}^{(j)} \). These facts make the choice of \( c_i \) and \( d_i \) bothersome.

In order to avoid choosing \( c_i \) and \( d_i \) and to simplify the problem, a Gaussian chip waveform is examined. A Gaussian pulse is defined by \( \phi_G(t) = a \exp \left( -\frac{t^2}{\sigma^2} \right) \), where \( a \) is a normalization factor and \( \sigma \) expresses the standard deviation of the energy distribution in time domain. Note that \( \sigma^2 \) increases (thus, bandwidth of the signal decreases) as \( \sigma \) increases. In Fig. 7, SIR performance is plotted against a parameter \( \sigma \). Interestingly, SIRs have peaks around \( \sigma = 0.85, 0.95, 1.05 \) for \( \lambda = -0.7, -0.8, -0.9 \), respectively, while SIRs are monotone decreasing for positive \( \lambda \). The curve of SINR is obtained by shifting up the curve of SIR according to (23). Fig. 8 illustrates the SIR performance versus \(-1 < \lambda < 1\). For \( \lambda = 0.9 \), the optimum SIR is achieved by \( \lambda = -0.76 \) with energy concentrations \( (\sigma^2, \sigma^2) = (0.9853, 0.9980) \), SIR performance as well as energy concentrations are better than that of root raised cosine pulses. Optimization of the chip waveform can be performed in terms of \( c_i \) and \( d_i \) in (21), which is left as a future work.

**REFERENCES**


