Investigation of an Optimum Sampling Interval for a Local Clock TIE Model with an Unbiased FIR Filtering Algorithm

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Abstract—An investigation of the optimal time step (sampling interval) is provided for the time interval error (TIE) model of a local crystal clock in GPS-based timekeeping. For the sawtooth noise of a receiver, the local clock states are estimated employing an unbiased finite impulse response (FIR) filtering algorithm. We exploit the local crystal clock imbedded to the Stanford Frequency Counter SR620. The measurements are provided using the SynPaQ III GPS Sensor as a timing receiver and rubidium clock (SR625) as a reference source of time for the crystal clock.

I. INTRODUCTION

The recently designed unbiased finite impulse response (FIR) filter and algorithm [1] allow estimating the time interval error (TIE) model of a local clock in presence of the sawtooth noise induced by the GPS timing receiver with an error less than that obtained by the standard Kalman filter. The case is the Motorola family of the receivers such as the SynPaQ III GPS Sensor. Here noise is uniformly distributed having the sawtooth structure owing to the principle of the one pulse per second (1PPS) signal formation utilized to the receiver. If such a receiver is used without the sawtooth correction, the estimates provided by the standard Kalman filter become biased and noisy that was shown in [1] experimentally.

The problem is efficiently solved with the unbiased FIR filters, which do not claim the noise to be Gaussian. Contrary to the recursive infinite impulse response (IIR) structures of Kalman filters, the FIR filters are bounded input/bounded output (BIBO) stable. The filters are simple for engineering applications, requiring the only optimal value, the optimal number \( N_{\text{opt}} \), of the points in the average (averaging horizon). No knowledge about noise is involved to the unbiased FIR algorithm. Recall that the Kalman filter needs describing the correlation matrix for both the measurement and signal that, for the unknown noise of a local clock, entails difficulties. Finally, a horizon \( N \) in timekeeping is usually large, so one should not expect for a big difference between an unbiased (simple) and optimal (complicated) estimates of all the clock states.

An optimal value \( N_{\text{opt}} \) for the unbiased FIR filter strongly depends on the time step \( \tau \). This step is virtually a sampling interval for the filter and may be obtained by thinning the 1PPS pulses. In this paper, we present the results of the experimental studies of the optimal \( N_{\text{opt}} \) for several states of a local crystal clock. We find \( N_{\text{opt}} \) as a function of \( \tau \), thereby answering on the question: What are optimum values of \( \tau_{\text{opt}} \) and \( N_{\text{opt}} \) in the sense of the minimum mean-square-error (MSE) in the state estimates?

II. AN UNBIASED FIR FILTERING ALGORITHM

Before discussing the results of the experimental studies, we describe in brief the TIE model of a local clock and unbiased FIR filtering algorithm proposed in [1] and investigated in [2].

A. TIE model of a clock

Most commonly, the TIE polynomial model of a clock projects ahead on a horizon of \( N \) points from the start point \( n = 0 \) with the \( K \)-degree Taylor polynomial

\[
x_1(n) = \sum_{p=0}^{K} x_{p+1} \frac{\tau^p n^p}{p!} + w_1(n, \tau)
\]

\[
x_1 = x_1 + x_2 \tau n + \frac{x_3}{2} \tau^2 n^2 + \frac{x_4}{6} \tau^3 n^3 \ldots + w_1(n, \tau),
\]

where \( x_{l+1} \equiv x_{l+1}(0), \ l \in [0, K] \), are initial states of the clock and \( w_1(n, \tau) \) is a clock noise with known properties. By extending the time derivatives of the TIE model to the Taylor series, the signal and observation equations become, respectively,

\[
\lambda(n) = A(n)\lambda(0) + w(n, \tau),
\]

\[
\xi(n) = C\lambda(n) + v(n),
\]

where

\[
\lambda(n) = [x_1(n) x_2(n) \ldots x_{K+1}(n)]^T
\]

is a \((K + 1) \times 1\) vector of the clock states and a \((K + 1) \times (K + 1)\) time-varying system matrix is
For equal number of states and measurements, the observation vector is

\[
\xi(n) = [\xi_1(n)\xi_2(n) \ldots \xi_{K-1}(n)]^T
\]

and a \((K + 1) \times (K + 1)\) measurement matrix \(C\) is typically unit. The \((K + 1) \times 1\) clock noise vector is described by

\[
w(n, \tau) = [w_1(n, \tau)w_2(n, \tau) \ldots w_{K+1}(n, \tau)]^T
\]

with the components caused by the oscillator noises. Finally, the noise vector

\[v(n) = [v_1(n)v_2(n) \ldots v_{K-1}(n)]^T\]

contains correlated or uncorrelated components that are not obligatory Gaussian. The GPS noise \(v(n)\) dominates on a horizon \(N\); that is, typically, \(<w_2^2(n, \tau)>_N \ll v_f^2(n)\). Therefore, \(w(n, \tau)\) is neglected in the FIR procedure.

B. An unbiased FIR filtering algorithm

The unbiased FIR filtering algorithm is illustrated in Fig. 1. The clock first state estimate \(\hat{x}_1(n)\) (TIE) is obtained with \(h_K(i)\) at a horizon of \(N_K\) points. The observation \(\xi_2(n)\) for the second state \(x_2(n)\) (fractional frequency offset) is then formed by increments of \(\hat{x}_1(n)\). Accordingly, \(\hat{x}_2(n)\) is achieved with \(h_{K-1}(i)\) at a horizon of \(N_{K-1}\) points. Inherently, the first accurate value of \(\hat{x}_2(n)\) appears at \((N_K + N_{K-1})\)th point starting from \(n = 0\). Finally, the last state estimate \(\hat{x}_{K+1}(n)\) is calculated with \(h_0(i)\) at a horizon of \(N_0\) points, using \(\xi_{K+1}(n)\) that is formed in the same manner as \(\xi_2(n)\). The first correct value of \(\hat{x}_{K+1}(n)\) appears thus at \((N_K + N_{K-1} + \ldots + N_0)\)th point.

The unique low order FIRs for the algorithm (Fig. 1) are derived in [1] and given below

\[
h_0(i) = \frac{1}{N}, \quad (5)
\]

\[
h_1(i) = \frac{2(2N - 1) - 6i}{N(N + 1)}, \quad (6)
\]

\[
h_2(i) = \frac{3(3N^2 - 3N + 2) - 18(2N - 1)i + 30i^2}{N(N + 1)(N + 2)}, \quad (7)
\]

\[
h_3(i) = \frac{8(2N^3 - 3N^2 + 7N - 3)}{N(N + 1)(N + 2)(N + 3)} \quad \text{and}
\]

\[
h_4(i) = \frac{20(6N^2 - 6N + 5)i - 120(2N - 1)i^2 + 140i^3}{N(N + 1)(N + 2)(N + 3)}, \quad (8)
\]

having the common properties:

\[
\xi(n) = [\xi_1(n)\xi_2(n) \ldots \xi_{K-1}(n)]^T
\]

\[
h(i) = \begin{cases} h(i), 0 \leq i \leq N - 1, \\ 0, \text{ otherwise}, \end{cases}
\]

\[
\sum_{i=0}^{N-1} h(i) = 1.
\]

Depending on the model degree \(K\), we thus have several particular realizations of the algorithm.

If a model is assumed to be constant, \(K = 0\), the only nonzero state is the TIE and the estimate of the TIE is provided by simple averaging,

\[
\hat{x}_1(n) = \frac{1}{N_0} \sum_{i=0}^{N_0 - 1} \xi(n - i). \quad (9)
\]

The linear model, \(K = 1\), is processed for two states by

\[
\hat{x}_1(n) = \sum_{i=0}^{N_1 - 1} h_1(i)\xi_1(n - i), \quad (10)
\]

\[
\hat{x}_2(n) = \frac{1}{\tau N_0} \sum_{j=0}^{N_0 - 1} [\hat{x}_1(n - j) - \hat{x}_1(n - j - 1)]. \quad (11)
\]

For the quadratic model, \(K = 2\), the 3-state unbiased FIR batch algorithm becomes

\[
\hat{x}_1(n) = \sum_{i=0}^{N_2 - 1} h_2(i)\xi_1(n - i), \quad (12)
\]

\[
\hat{x}_2(n) = \frac{1}{\tau} \sum_{j=0}^{N_1 - 1} h_1(j)[\hat{x}_1(n - j) - \hat{x}_1(n - j - 1)], \quad (13)
\]

\[
\hat{x}_3(n) = \frac{1}{\tau N_0} \sum_{r=0}^{N_0 - 1} [\hat{x}_2(n - r) - \hat{x}_2(n - r - 1)]. \quad (14)
\]

Finally, if we assume a cubic model, \(K = 3\), the algorithm is formed with

\[
\hat{x}_1(n) = \sum_{i=0}^{N_3 - 1} h_3(i)\xi_1(n - i), \quad (15)
\]

\[
\hat{x}_2(n) = \frac{1}{\tau} \sum_{j=0}^{N_2 - 1} h_2(j)[\hat{x}_1(n - j) - \hat{x}_1(n - j - 1)], \quad (16)
\]

\[
\hat{x}_3(n) = \frac{1}{\tau} \sum_{p=0}^{N_1 - 1} h_1(p)[\hat{x}_2(n - p) - \hat{x}_2(n - p - 1)], \quad (17)
\]

\[
\hat{x}_4(n) = \frac{1}{\tau N_0} \sum_{r=0}^{N_0 - 1} [\hat{x}_3(n - r) - \hat{x}_3(n - r - 1)]. \quad (18)
\]

As it is seen, the calculus (9)–(18) needs determining two variables for each estimate to be optimal in some sense. Namely, one must take care in evaluating and setting the proper values of \(N\) and \(\tau\). Below, we measure these values experimentally.
III. EXPERIMENTAL EVALUATION OF AN OPTIMAL SAMPLING INTERVAL (TIME STEP $\tau$)

Experimental evaluation of $N_{\text{opt}}$ and $\tau_{\text{opt}}$ for each of the algorithms in (9)–(18) was provided for the crystal clock embedded in the Stanford Frequency Counter SR620. The measurement is made with the GPS timing sensor SynPaQ III and SR620 for $\tau = 1$ s (GPS-measurement). Simultaneously, to get a reference trend, the TIE of the same crystal clock is measured, by SR625, for the rubidium clock (Rb-measurement). Initial time and frequency shifts between two measurements are then eliminated statistically and a transition to $\tau > 1$ s is provided by the data thinning in time. The difference between each estimate and the reference trend (Rb-measurement) or its time derivatives is evaluated in the sense of the MSE and the optimal values of $N$ and $\tau$ are obtained for the minimum MSE.

For signal processing, we exploited measurements obtained during about one day. Therefore, the upper bound of $\tau$ was limited by $10^4$ s.

Two principle limitations of accuracy may be pointed out in evaluating the $N_{\text{opt}}$ and $\tau_{\text{opt}}$:

- Finite data of measurements (about one day) affecting the estimates for $\tau > 10^3$ s.
- Long-term phase drifts in the rubidium reference clock increasing errors in the estimates of the TIE.

Below we discuss the results of evaluating the optimal time step $\tau_{\text{opt}}$ and related $N_{\text{opt}}$ by all four above-listed algorithms for several clock states.

A. The first clock state (TIE)

Fig. 2 illustrates evaluation of $N_{\text{opt}}$ vs. $\tau$ for the first clock state (TIE) with four algorithms. A common conclusion is that the minimum root-mean-square error (RMSE) corresponds to $\tau = 1$ s by all of the algorithms and, hence, estimating the TIE does not require thinning in time the measured data. For TIE, $\tau_{\text{opt}}$ is exactly 1 s.

One may also observe that simple averaging provides for with a minimum number $N_{0(\text{opt})}$ of the points among all other filters. However, the RMSE in simple averaging (5) is larger than that produced by the linear and quadratic kernels, (6) and (7), respectively. The cubic kernel (8) produces even larger error. So, the TIE model of an investigated clock may be identified on a horizon of the filter memory to be either linear or quadratic.
Fig. 3. Optimal values of $N_{opt}$ vs. the time step $\tau$ for the second clock state (fractional frequency offset): (a) optimal $N$ and (b) RMSE.

Fig. 4. Optimal values of $N_{opt}$ vs. the time step $\tau$ for the third clock state (linear fractional frequency drift rate): (a) optimal $N$ and (b) RMSE.

Fig. 5. Optimal values of $N_{opt}$ vs. the time step $\tau$ for the fourth clock state (quadratic fractional frequency drift rate): (a) optimal $N$ and (b) RMSE.
B. The second clock state (fractional frequency offset)

To find optimum values of $N$ and $\tau$ for the second clock state that is the fractional frequency offset, one needs first to estimate optimally the TIE for the optimal $N$ and $\tau$ taken from Fig. 2. Thereafter, the increments of the estimates of TIE are used as measurements of the second state, in accordance with the algorithm (Fig. 1).

Fig. 3 shows the results of measurements of $N_{\text{opt}}$ vs. $\tau$ for the second clock state. Instantly we indicate that the minimum RMSE corresponds to the time step $\tau$ from 10s to 100s for all three filters. Notice that reducing the error in the last decade of $\tau$ in Fig. 3 is owing to the finite data in the processing. Yet, the cubic filter produces a bit lower error than two others. Actually, the differences in errors are not so appreciable here and one may use a linear filter, since its $N_{0(\text{opt})}$ is much smaller that those of the quadratic and cubic filters.

C. The third clock state (linear fractional frequency drift rate)

In a like manner, measurements are provided for the third clock state termed the linear fractional frequency drift rate. The results are shown in Fig. 4. Here we see that the minimum RMSE lies in the region of $\tau$ from 300s to 600s. We notice again that the last decade in the $\tau$ scale (from $10^3$s to $10^4$s in Fig. 4) is not reliable to make conclusions owing to the finite data in the processing. For this minimum, the optimal $N$ is provided by Fig. 4 exactly, even though, on the whole, the $N_{0(\text{opt})}$ was found with large error.

D. The fourth clock state (quadratic fractional frequency drift rate)

To evaluate $N_{\text{opt}}$ and $\tau_{\text{opt}}$ for the fourth clock state that may be called the quadratic fractional frequency drift rate, the only algorithm may be used, that is (15)–(18). Accordingly, Fig. 5 give one curve for $N_{0(\text{opt})}$ and one for the RMSE vs. $\tau$.

The principle observation is that the minimum RMSE lies likely in the last decade of $\tau$ from $10^3$ to $10^4$ or even beyond this decade. It is not unexpected, since the frequency changes very slowly even in crystal oscillators. Therefore, the spectral content associated with the quadratic drift rate inherently lies very closely to zero and much time is necessary to evaluate the $N_{0(\text{opt})}$ and $\tau_{0(\text{opt})}$.

Nevertheless, one may conclude that, even approximately, the quadratic fractional frequency drift rate may be estimated by the unbiased FIR algorithm with $\tau_{\text{opt}}$ about 3hours with only several points in the average.

IV. CONCLUDING REMARKS

In this paper, we presented the results of an experimental evaluation of the optimal time step (sampling interval) $\tau_{\text{opt}}$ for four states of the TIE model of a local crystal clock. The results were obtained with the unbiased FIR filtering algorithms described by (9)–(18). The conclusions are as follows.

For the crystal clock imbedded to the Stanford Frequency Counter SR620, the optimal values of $\tau_{\text{opt}}$ and $N_{0(\text{opt})}$ differ for different clock states, in particular:

- The first state (TIE) must be filtered with $\tau = 1$s and $N_{0(\text{opt})}$ given in Fig. 2 (left).
- The second state (fractional frequency offset) may be estimated with $\tau = 10 \ldots 100$s and $N_{0(\text{opt})}$ provided by Fig. 3 (left).
- The third state (linear fractional frequency drift rate) needs setting $\tau = 300 \ldots 600$s with $N_{0(\text{opt})}$ taken from Fig. 4 (left).
- The fourth state (quadratic fractional frequency drift rate) may be estimated with $\tau$ about $10^4$s and only several points in the average.

REFERENCES
