Hollow fiber membrane adsorber: Mathematical model

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Abstract

Two opposite design strategies for ultrafiltration/microfiltration filters, (1) reduction of concentration polarization and particle deposition to increase permeate velocity and (2) utilization of particle deposition on membrane surface to produce an additional (to permeate) volume of clarified water, are analyzed. It is shown that the first strategy is always associated with additional expenditures in power or other material resources, making it not enough cost-effective to be competitive with non-membrane filtration processes in some water treatment applications. At the same time, the second strategy does not require additional power expenditures and provides high water recovery and cost-effectiveness. The mathematical model describing the performance of hollow fiber membrane adsorber, which represents a second-strategy filter, is studied. A general form of the particle-deposition equation is introduced, and its terms are analyzed. As a result, its linearized form, looking like a linear equation of reversible adsorption, is chosen. A numerical solution to the system of governing equations is obtained and used to assess

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the accuracy of approximate solutions. A new approximate solution allowing one to evaluate the adsorber particle retentions with an acceptable accuracy is suggested.

*Keywords:* Cake deposition; Ultrafiltration; Microfiltration; Depth filtration; Mathematical model

1. Introduction

The conventional theory and practice of ultrafiltration/microfiltration (UF/MF) have been grown from the concept brought forth about 40 years ago in Michaels' paper [1], in which the accumulation of suspended particles (SP) retained at the surface of a semipermeable membrane causes the phenomena of concentration polarization (CP) and cake formation, which account for the efficiency of UF/MF filters. The buildup of a cake layer begins when the "wall" (membrane surface) concentration reaches the value of cake formation (gel polarization). In this concept, the hydraulic resistance of the cake layer mostly dictates the value of permeate velocity and, hence, the efficiency of UF/MF plants. It has since been recognized that the only way to increase the efficiency of UF/MF plants is to reduce CP, cake deposition rate or, which is better, to provide the conditions under which the cake layer is not formed at all.
A dozen of models in which the above concept is a cornerstone were developed to get insight into the UF/MF nature and suggest how to improve the UF/MF efficiency [2, 3].

The convective diffusion models (film, gel-polarization) are able to explain several important phenomena such as limiting flux, critical flux, decline of permeate flux to zero when the bulk concentration is increased up to the gel-polarization value, etc. However, they have several limitations that put their foundation in doubt. For example, the high values of diffusion coefficient needed for the models not to underestimate the experimental data cannot be physically justified by the existing physicochemical theories [2, 3]. The models cannot satisfactorily describe the dependence of the permeate velocity on the feed flow velocity in cross-flow filters in a wide range of Reynolds numbers. The convective diffusion problem assuming the wall concentration being equal to the gel-polarization value leads to the conclusion that the value of permeate velocity in a cross-flow membrane channel under steady state conditions is independent of the specific gel (or cake) resistance [4, 5]. The latter parameter is not involved at all in the problem formulation, which looks much questionable. There are not enough data in the literature confirming that these models correctly describe the dependence of permeate flux on transmembrane pressure (TMP) and feed concentration when the values of empirical constants determined for one TMP and feed concentration are used to calculate the curves for the other values of TMP and feed concentration. It is obvious that the further progress in building a general
convective diffusion theory of membrane filtration depends on the pace at which the theory of concentrated and highly concentrated solutions along with appropriate experimental techniques that are applicable to membrane processes are being developed [6-10]. To eliminate the discrepancies with experiment, a new formulation of the equations governing the mass transfer in UF/MF in deadend and cross-flow filters may be necessary.

To "patch the gaps" that cannot be explained by the convective diffusion theory, such as the experimental data where the permeate flow rate is not a 1/3 exponent function of feed velocity in cross-flow UF/MF, not high enough values of diffusion coefficient, and the like, alternative models were suggested [2, 3]. They include inertial migration (tubular-pinch effect), shear-induced hydrodynamic convection, shear-induced hydrodynamic diffusion, erosion, friction force, particle adhesion, surface renewal, and particle-particle interactions models. With their help, most of the experimental results were interpreted, though more qualitatively than quantitatively because the above models are not able to consider the whole set of factors affecting UF/MF in a wide range of its process parameters.

The resistance-in-series models, in which the permeate velocity is directly proportional to TMP and inversely proportional to the total (concentration layer, cake, membrane) resistance, are mostly used to estimate the value of the specific cake resistance, and determine when one step of the process, for example, pore blocking, ends, and another step, cake formation, begins [2, 11-13]. The reason for this is the
assumptions on which the resistance-in-series model is based: Generally, the concentration of suspended particles is the same at any point beyond the cake layer, the cake growth rate is the same throughout the UF/MF channel, the growth rate of the layer of particles adsorbed on the pore surface is the same along the entire pore, and there is no convective carry-away of particles by the stream tangential to the membrane surface. Clearly, these assumptions are hardly applicable to most experimental data. That is why the model uses empirical parameters which, as a rule, are applicable to the only set of experiments where they were determined.

The main conclusion resulting from the above models is that the UF/MF filter should be designed so that (1) tangential flow velocity and local flow instabilities (eddies, vortices, and the like) in membrane channel be as high as possible to reduce the concentration polarization and cake deposition, (2) the adsorption of particles to the membrane surface be as low as possible, (3) the particle-particle interaction in the boundary layer be unfavorable, and (4) the specific cake resistance be as low as possible.

Therefore, all the efforts undertaken by UF/MF researchers and engineers were directed to fulfill these tasks. High tangential velocities in thin channels, turbulence promoters of various designs, corrugated-plate membrane modules, membrane channels with high curvature, Dean vortices, sonication, vibration, periodic feed flow, air sparging, back pulsing, low TMP, short membrane channels, low-adsorption membranes, etc. are used to reduce the membrane fouling [3, 12-15]. Although the use
of the above techniques for reducing the membrane fouling allowed us to design commercial plants of high productivity, their increased power consumption and more complicated plant design due to these techniques make these plants cost-ineffective, as compared to conventional technologies such as sedimentation, coagulation, depth filtration, and the like, in some areas of water and wastewater treatment [3].

It looks like the fundamental idea that the high efficiency of UF/MF plants could only be achieved by reducing CP and cake growth has its own inherent limitation. Even if we assume that there is no adsorption between the membrane and particles, the particles retained by the membrane would be accumulated at the membrane surface and trying to form a cake. The energy conservation law implies that the work to reduce the cake growth requires an additional amount of power, which would always lead to increased power consumption in such plants. In other words, it means that in this case the production rate can be maintained at the same level only by spending more power or other material resources. It should be noted that traditional cross-flow membrane filters produce permeate, almost completely clean of particles, and retentate, in which the concentration of particles is higher than that in the feed. The retentate is often a problem in water and wastewater treatment because it does not allow to achieve high water recoveries and may be a threat to the environment.

On the other hand, the material balance law implies that accumulation of particles at one point can lead to their lack at another point without any additional power expenditure. For example, in depth filtration the deposition (or adsorption) of particles at
the inlet collectors allows us to obtain filtrate, the liquid with a low concentration of particles. In such a device, which collects the particles, the power is spent only for pumping the liquid across the bed.

Let us look from this point of view at UF/MF outside-in hollow fiber membrane (HFM) filters (Figs. 1, 2), considering how to utilize its natural particle collection ability in order to produce some filtrate in addition to the permeate, rather than "struggling" against the filter particle collection ability. In these filters, which contain several hundreds or thousands of hollow fibers with typical shell diameters of 0.3 to 3 mm and wall thicknesses of 0.1 to 0.75 mm, the feed is supplied to the fiber shells, and the permeate (liquid that passed through the membrane walls) is withdrawn from the HFM lumens [12, 13]. The velocity of liquid flowing around the shells of hollow fibers with diameters of \(~1\) mm, permeate velocities of \(10^{-6}–10^{-5}\) m/s, and membrane packing densities of 0.5, typical for these devices, does not exceed \(1\) cm/s. Owing to the convergent-divergent geometry of the interfiber space and a low equivalent height of the interfiber channels (\(~1\) mm), which are the factors promoting intense flow mixing, the rate of mass transfer in the filters must have much in common with the mass transfer in the narrow channels with turbulence promoters in spiral wound membrane elements. As compared to the turbulence promoters in spiral wound elements, the hollow fibers themselves represent more efficient flow mixing devices because they do not form stagnant zones like the places where the turbulent promoters touch the membrane surface, or zones with low mixing intensity on the membrane surface located in the
centers of the promoter grids [19, 20]. Calculations made with the equations given in Refs. [21, 22] show that the concentration at the HF shell will be at most a couple of dozens of percent higher than the concentration beyond the concentration polarization layer. As it follows from the results of non-invasive observation obtained in Ref. [20], the mixing between the fibers in a real HFM filter due to the wall instabilities and eddies caused by the transverse flow around the cylindrical fibers will be more intense than that calculated with the models studied in Refs. [21, 22]. As a result, the concentration polarization modulus on the HF surface will be even less than the ones estimated theoretically. Thus, the concentration polarization in this type of outside-in HF filters can be ignored. This also implies the perfect mixing beyond the cake layer in the direction perpendicular to the flow across the filter. Consequently, the behavior of particle-liquid separation in these filters may have common features with depth filters.

This analogy of outside-in HFM filters with depth filters was studied in Refs. [16-18]. It was suggested to withdraw from an outside-in HFM filter (Figs. 1, 2) during the initial period of the filtration process not only the permeate but also the clarified liquid at its outlet produced as a result of particle deposition. As in any depth filter, the particles in the feed flow first deposit on the inlet collectors (hollow fibers), and the suspension contains less and less particles as it moves on to deeper layers of collectors. Consequently, at the filter outlet we can obtain a liquid with low concentration of particles, which in depth filtration is conventionally called the filtrate (Fig. 3).
Thus, in the outside-in HFM filter, the decline in permeate flow velocity due to cake deposition can be compensated by the equal increase in flow rate of the filtrate produced due to the same cake deposition. Interestingly, this increase in the filter productivity can be achieved just by installing an additional control valve at the filter outlet to maintain a constant product flow rate at constant TMP.

To estimate the feasibility of this idea, the authors [16] developed a mathematical model in which the separation process is described by a system of equations consisting of differential law of mass conservation, Darcy's law, liquid continuity equation, and deposition rate equation taking the form of a linear reversible adsorption equation. The model takes into account the decrease of liquid velocity across the filter due to permeate withdrawal. The values of the coefficients of particle adsorption (deposition) and peptization (re-entrainment) in the reversible adsorption equation were estimated using the interface force boundary layer (IFBL) approximation [23, 24] modified in Ref. [16] for the case of semipermeable collectors. The system of governing equations was solved by an approximate method that uses the permeate velocity value averaged over time and filter depth together with a Laplace transform solution obtained for the case of constant permeate velocity.

Calculations [16, 17] showed that the outside-in HFM filter with two clarified products flows, permeate and filtrate, which was called the HFM adsorber in Ref. [18], can be operated as a depth filter, which is stopped for flushing when the concentration of particles in the filtrate increases up to a limiting value. As compared with a deadend
membrane filter, it has a considerable advantage in production rate with almost the same power consumption. As compared with crossflow membrane filters, the HFM adsorber has a considerable advantage in power consumption with the same constant production rate at constant TMP. As compared with a depth filter with solid collectors, the adsorber is advantageous in that it uses the permeate withdrawal to slow down the velocity at which the concentration front moves to the adsorber outlet, thus increasing the separation cycle duration and, hence, the volume of produced filtrate. The permeation drag may also enhance the adsorption ability of semipermeable collectors. The calculations also showed that the adsorber productivity will increase with increasing adsorption coefficient, decreasing peptization coefficient, and decreasing TMP. It was demonstrated that the greatest efficiency of an HFM adsorber can be achieved when the feed flow rate remains constant and equal to the initial permeate flow rate at a constant TMP.

However, the accuracy of the approximate solution used in Refs. [16, 17] was estimated only for two asymptotic regions (unsteady operation at adsorber inlet and the steady state operation of the adsorber after particle breakthrough), which are not of practical interest. The estimates of the adsorption and peptization coefficients based on the IFBL approximation theory may considerably differ from their experimental values [25, 26], thus putting the results obtained in Refs. [16, 17] in doubt. For example, the values of these coefficients determined by best fitting the results of numerical solution to the experimental data obtained in deadend HFM filters [27] were several times different.
from those calculated theoretically in [16]. Also, the choice of the linear reversible adsorption equation for this problem requires a more detailed discussion in view of the modern theories that consider the peptization of colloids unlikely [25, 26].

In the present paper, a numerical solution to the system of governing equations developed in [16] will be obtained, and the accuracy of the approximate method [16] will be evaluated using the empirical values of adsorption and peptization coefficients determined in [27]. A general form of the particle deposition equation and its linearized form will be considered. The physical meaning of the term accounting for the decrease of deposition rate will be discussed.

2. Mathematical model

As in Ref. [16], consider a process in which a dilute suspension with constant density and viscosity is clarified in an HFM adsorber (Fig. 1). The vacuum providing a constant transmembrane pressure, which is the driving force of membrane filtration, is produced in the HF lumens. Assume that the porous HF membranes completely reject the suspended particles. The feed concentration and temperature remain constant. The diffusion of particles beyond the layer of surface forces is ignored, and the suspension is perfectly mixed in the plane perpendicular to the flow of liquid across the filter due to local flow instabilities and interfiber vortices and eddies [20]. Particle dispersion and
effects of filter housing walls are assumed to be negligible. The differential law of mass conservation can then be written as

\[\frac{\partial c}{\partial t} + \frac{\partial (cw)}{\partial z} = -s \frac{\partial \Gamma}{\partial t},\]  

(1)

where \(c\) is the concentration of suspended particles; \(t\), the time; \(z\), the filter depth coordinate; the liquid velocity \(w\) is the velocity averaged over the cross section of all interfiber channels at \(z\); \(s = S_m/(Sd)\), the ratio of HFM shell surface area to the suspension volume inside the filter; \(\Gamma\), specific cake deposit (cake mass per square meter of HFM shell surface area), \(S_m\), the total HFM shell surface area; \(S\), the total cross section area of the interfiber space; \(d\), the overall filter depth.

The permeate velocity of a hollow fiber membrane is governed by Darcy’s law:

\[V_p = \frac{P}{\mu (R_m + r_c \Gamma)},\]  

(2)

where \(V_p\) is the permeate velocity; \(P\), the transmembrane pressure (TMP); \(\mu\), the liquid viscosity; \(R_m = P/(\mu V_0)\) is the clean membrane resistance; and \(r_c\) is the specific cake resistance.

Liquid continuity equation takes the following form:

\[\frac{\partial w}{\partial z} = -sV_p.\]  

(3)

Integration of equation (3) over \(z\) leads to
\[ w = w_0 - \int_0^z s V_p \, dz, \quad (4) \]

where \( w_0 \) is the constant feed velocity.

By contrast to the problem studied in [27], in which the liquid flow rate across the filter is only determined by the rate of permeate withdrawal, the liquid flow rate in the HFM adsorber is defined by expression (4) implying that the decline in permeate flow rate is compensated by the equal increase in filtrate flow rate. The latter accounts for a constant adsorber production rate at constant TMP.

It is also assumed that the head loss across the fiber array (feed side) and axial pressure drop in fiber lumens are negligible, and that the value of pressure is the same throughout the interfiber channels. These assumptions can be justified by simple calculations with Yuan [29], or modified Hagen–Poiseuille, and Bergelin formulas as well as by the results of direct non-invasive observation of cake growth (see Appendix A).

A clean filter initial condition will be used. The suspension concentration at filter inlet is assumed to be constant:

\[ c = c_0 \quad \text{when} \quad z = 0, \, t > 0; \quad (5) \]

\[ c = 0, \, \Gamma = 0 \quad \text{when} \quad t = 0, \, z > 0. \quad (6) \]

To close the system of equations, a specific expression for the deposition rate in equation (1) has to be selected. As Tien indicated in his book [26], "this selection is, to a large degree, arbitrary. Generally speaking, the presence of more constants, i.e.
phenomenological parameters, in the rate expression allows a better fit of experimental results." For example, the list of such expressions in Ref. [26], which were used to describe the performance of depth filters, contains as many as 9 formulas.

In general, the deposition rate equation for an outside-in HFM filter, that is, for a depth filter with semipermeable particle collectors, may be written as

\[
\frac{d\Gamma}{dt} = k_1 (\psi, \Gamma) c - k_2 \Gamma + k_3 V_p c, \tag{7}
\]

where \( k_1 \) is the deposition coefficient; \( \psi \), the parameter vector characteristic of the filtration process; \( k_2 \), the re-entrainment coefficient; \( k_3 \) is a constant.

The first term on the right side of equation (7) is adopted from depth filtration theories [25, 26] and describes the deposition flux of particles to the membrane surface (or to the cake layer after it is formed). For Brownian (submicron) particles, the deposition is usually caused by Brownian diffusion and surface (colloidal) interaction forces, such as electrical double layer and van der Waals forces. For micron particles, the deposition is typically attributed to inertial impaction, interception, sedimentation, electrostatic forces, and surface interaction forces. In general, the deposition coefficient \( k_1 \) varies with specific cake deposit \( \Gamma \), which takes into account the effect of deposited particles on the deposition flux [25, 26].

The second term on the right side of equation (7) is adopted from depth filtration [26, 30] and reversible adsorption [23, 24, 31] theories and describes the particle detachment flux from the membrane surface (or cake layer). For Brownian particles, the
re-entrainment is usually caused by desorption of particles from the collector surface. Though a number of experimental studies show that deposition of colloidal particles is sometimes reversible [32-35], the physical mechanisms of colloid release are still not well understood [36]. For micron particles, the re-entrainment is commonly attributed to unfavorable hydrodynamic conditions (flow instabilities) [37, 38]. It was shown in Ref. [39] that unfavorable surface interactions may lead to reversible deposition of microbial cells in crossflow microfiltration. Generally, the re-entrainment coefficient $k_2$ may be variable.

The last term in equation (7) accounts for the deposition rate increase due to the permeation drag. The linear dependence of the deposition rate on the product of permeate velocity and particle concentration is selected because the classical cake filtration equation, which corresponds to $k_1 = 0, k_2 = 0, k_3 = 1$, can describe the permeate flux decline in some surface microfiltration experiments without axial flow [13, 39].

At the same time, approximation of the experimental data by the classical cake filtration method for deadend outside-in HF filters shows that equation (7) with $k_1 = 0, k_2 = 0, k_3 = 1$ cannot adequately describe the variation of kinetic curves with time and transmembrane pressure (Fig. 4) [27]. It was shown that the classical cake filtration model leads to a uniform concentration profile throughout most of the filter after the initial clean water is almost completely forced out of the filter by the suspension front, which usually takes no longer than several seconds. This explains why the classical cake filtration mechanism, which assumes the square-root dependence of $\Gamma$ on time, is unable
to describe the variation of the particle concentration profile with depth coordinate that takes place in deadend outside-in HFM filters [19, 20, 27]. Obviously, the classical cake filtration model cannot describe the operation of outside-in HFM filters, which by its nature are a type of depth filter.

As there is no experimental data on particle deposition behavior for HFM adsorbers yet, the selection of specific expressions for $k_1$ and $k_2$ would be somewhat arbitrary. Therefore, in this study it is suggested to use a linearized approximation of equation (7), which captures the essential features of particle deposition,

$$\frac{\partial \Gamma}{\partial t} = \beta c - \alpha \Gamma, \quad (8)$$

where $\beta$ and $\alpha$ are phenomenological (averaged) constants. Here the first term on the right side of equation (8) represents the deposition flux. The second term accounts for the effects of deposition rate decrease and possible particle re-entrainment that can be caused by the cake growth on the membrane surface. Equation (8) was already used to describe the performance of deadend outside-in HFM filters, where it was shown to be in good agreement with the experimental data (Fig. 4) [27]. Mathematically, equation (8) looks like a linear equation of reversible adsorption in which $\beta$ accounts for the membrane ability to adsorb or collect suspended particles. Therefore, using the analogy between the adsorption and depth filtration theories, $\beta$ was called the coefficient of adsorption (deposition), and $\alpha$ the coefficient of peptization (re-entrainment) [27].
The system of equations (1), (2), (4)-(6), (8) can describe the performance of both rectangular cartridge-type (Fig. 1) and radial filters (Fig. 2). In the rectangular case, $z$ is the actual filter depth coordinate. In the radial case, it can easily be shown using equations (1) and (4) that the radius is related to the effective depth coordinate $z$ by the formula (see Appendix B):

$$z = \frac{r_0^2 - r^2}{2r_0},$$  \hspace{1cm} (9)

where $r_0$ is the external radius of the hollow-fiber bundle.

In contrast to the conventional model of depth filtration or chromatographic separation, in which the liquid velocity across the filter is constant, in the process under study the liquid velocity in the filter decreases with depth coordinate because of the pure water suction through hollow fiber membranes distributed over the filter volume. This feature of mass transfer in an outside-in HFM filter leads to a nonlinear integrodifferential equation. Physically, this equation describes two individual separation processes combined in one filter: (1) suction of the pure liquid through the semipermeable porous surface and (2) retention of the particles by their collection on the same surface. The processes described by this equation may take place in biological and chemical membrane reactors in which the volume-distributed suction of one of the mixture components is used to maintain the rate of the reaction.

The problem (1), (2), (4)-(6), (8) with the help of simple mathematical transformations takes the following form for the function $\Gamma$:
\[
\frac{\partial^2 \Gamma}{\partial t^2} + (\alpha + s\beta) \frac{\partial \Gamma}{\partial t} + sV_0 \frac{\partial}{\partial z} \left( \frac{\partial \Gamma}{\partial t} + \alpha \Gamma \right) \times \left[ \frac{w_0 - \int_0^{z} \frac{dz}{1 + \chi_i \Gamma}}{sV_0} \right] = 0
\]  
(10)

\[
\Gamma = \beta c_0 \left(1 - \exp[-\alpha t]\right)/\alpha, \quad \text{when } z = 0, t > 0;
\]  
(11)

\[
\Gamma = 0, \frac{\partial \Gamma}{\partial t} = 0 \quad \text{when } t = 0, z > 0.
\]  
(12)

Here \( \chi_i = r_c / R_m \).

Expression (11) was obtained by solving equation (8) for \( c = c_0 \), which corresponds to the filter inlet boundary condition given by (5) [16].

The other functions to be found are the permeate velocity averaged over the adsorber depth:

\[
\frac{V}{V_0} = \frac{1}{d} \int_0^{d} \frac{dz}{1 + \chi_i \Gamma},
\]  
(13)

the concentration of suspended particles \( c \) defined by the equation

\[
c = \frac{1}{\beta} \left( \frac{\partial \Gamma}{\partial t} + \alpha \Gamma \right),
\]  
(14)

the concentration of suspended particles \( c_f \) at the filter outlet (filtrate concentration)

\[
c_f = \frac{1}{\beta} \left( \frac{\partial \Gamma(t,d)}{\partial t} + \alpha \Gamma(t,d) \right),
\]  
(15)

the product (permeate plus filtrate) concentration of suspended particles \( c_{pf} \) for a continuous flow operation:

\[
c_{pf} = \frac{c_f}{w_0} \left( w_0 - \int_0^{d} sV_p dz \right),
\]  
(16)
the adsorber retention $R$ for a continuous flow operation:

$$R = 1 - \frac{c_{pf}}{c_0}, \quad (17)$$

the product (permeate plus filtrate) concentration of suspended particles $c'_{pf}$ for a batch operation:

$$c'_{pf} = \frac{1}{t} \int_0^t c_{pf} \, dt, \quad (18)$$

and the adsorber retention $R'$ for a batch operation:

$$R' = 1 - \frac{c'_{pf}}{c_0}. \quad (19)$$

Here the adsorber depth is expressed as

$$d = \frac{r_0^2 - r_{in}^2}{2r_0} \quad (20)$$

for the radial case, where $r_{in}$ and $r_0$ are the internal and external radii of the hollow-fiber bundle, respectively.

Converting the problem (10)–(12) into dimensionless form gives:

$$\frac{\partial^2 \gamma}{\partial \tau^2} + (N_\alpha + 1) \frac{\partial \gamma}{\partial \tau} + \frac{1}{N_\beta} \frac{\partial}{\partial Z} \left( \left( \frac{\partial \gamma}{\partial \tau} + N_\alpha \gamma \right) \left[ 1 - \xi \int_0^Z \frac{dZ}{1 + N_\alpha \gamma} \right] \right) = 0, \quad (21)$$

$$\gamma = \frac{1 - \exp \left[ -N_\alpha \tau \right]}{N_\alpha} \quad \text{when } Z = 0, \tau > 0; \quad (22)$$

$$\gamma = 0, \frac{\partial \gamma}{\partial \tau} = 0, \quad \text{when } \tau = 0, Z > 0, \quad (23)$$

where $\tau = s \beta t, Z = \frac{Z}{d}, \gamma = \frac{\gamma}{c_0 \Gamma}, N_\alpha = \frac{c_0}{s}, N_\beta = \frac{s \beta d}{w_0}, N_\alpha = \frac{\alpha}{s \beta}, \xi = \frac{s V_0 d}{w_0}$. 
An exact analytical or immediate numerical solution to the above nonlinear
integrodifferential problem is hardly possible. Therefore, as in [27], for numerical
solution the above integrodifferential equation will be transformed into a partial
differential equation for a function of two independent variables.

3. Numerical and approximate solutions to problem (21)–(23)

3.1 Numerical Solution

Introducing a new function

\[ v = \int_{0}^{Z} \frac{dZ}{1 + N_{x} \gamma}, \quad (24) \]

which represents the permeate flow rate for the filter region found between the adsorber
inlet and the plane with coordinate \( Z \), reduces the problem to solving the equation:

\[
-2 N_{\beta} \left( \frac{\partial^{2}v}{\partial Z \partial \tau} \right)^{2} + N_{\beta} \frac{\partial^{3}v}{\partial Z \partial \tau^{2} \partial Z} + N_{\beta} (N_{\alpha} + 1) \frac{\partial^{2}v}{\partial Z \partial \tau} \frac{\partial v}{\partial Z} - \]

\[
- 2(1 - \xi v) \frac{\partial^{2}v}{\partial Z \partial \tau} \frac{\partial^{2}v}{\partial Z^{2} \partial \tau} + (1 - \xi v) \frac{\partial^{3}v}{\partial Z \partial \tau^{2} \partial \tau} \frac{\partial v}{\partial Z} + N_{\alpha} (1 - \xi v) \frac{\partial^{2}v}{\partial Z^{2} \partial \tau} \frac{\partial v}{\partial Z} - \]

\[
- \xi \left( \frac{\partial v}{\partial Z} \right)^{2} \frac{\partial^{2}v}{\partial Z \partial \tau} + \xi N_{\alpha} \left( \frac{\partial v}{\partial Z} \right)^{3} - \xi N_{\alpha} \left( \frac{\partial v}{\partial Z} \right)^{4} = 0 \quad (25)
\]

with initial and boundary conditions
\( \nu(0, Z) = Z, \quad \frac{\partial \nu(0, Z)}{\partial \tau} = 0, \quad \frac{\partial \nu(\tau, 0)}{\partial Z} = \frac{1}{1 + N_x \left(1 - \exp[-N_\alpha \tau]\right)/N_\alpha} \), \( \nu(\tau, 0) = 0. \) \hspace{1cm} (26)

Transition to a system of equations with a first-order time derivative gives

\( u = \frac{\partial \nu}{\partial \tau}, \) \hspace{1cm} (27)

\[
-2 N_\beta \left( \frac{\partial u}{\partial Z} \right)^2 + N_\beta \frac{\partial^2 u}{\partial Z \partial \tau} \frac{\partial \nu}{\partial Z} + N_\beta \left( N_\alpha + 1 \right) \frac{\partial u}{\partial Z} \frac{\partial \nu}{\partial Z} - \]

\[-2 \left( 1 - \xi \nu \right) \frac{\partial u}{\partial Z} \frac{\partial^2 \nu}{\partial Z^2} + \left( 1 - \xi \nu \right) \frac{\partial^2 u}{\partial Z^2} \frac{\partial \nu}{\partial Z} + N_\alpha \left( 1 - \xi \nu \right) \frac{\partial^2 \nu}{\partial Z^2} \frac{\partial \nu}{\partial Z} - \]

\[-\xi \left( \frac{\partial \nu}{\partial Z} \right)^2 \frac{\partial u}{\partial Z} + \xi N_\alpha \left( \frac{\partial \nu}{\partial Z} \right)^3 - \xi N_\alpha \left( \frac{\partial \nu}{\partial Z} \right)^4 = 0 \] \hspace{1cm} (28)

\( \nu(0, Z) = Z, \quad u(0, Z) = 0, \quad \frac{\partial \nu(\tau, 0)}{\partial Z} = \frac{1}{1 + N_x \left(1 - \exp[-N_\alpha \tau]\right)/N_\alpha}, \quad \nu(\tau, 0) = 0. \) \hspace{1cm} (29)

Problem (27)–(29) can be solved by the generalized implicit Crank–Nicholson scheme with a second-order approximation in coordinate and time [27], the algorithm of which is built in the mathematical software package Maple 9.5. For the function \( f = (u, \nu) \), the finite-difference approximations with constant steps in time \( k \) and coordinate \( h \) have the form:

\[
\hat{f} = \frac{f_i^{n+1} + f_i^n}{2}, \quad \frac{\partial \hat{f}}{\partial Z} = \frac{1}{2} \left( \frac{f_i^{n+1} - f_{i-1}^{n+1}}{2h} + \frac{f_{i+1}^n - f_i^n}{2h} \right),
\]

\[
\frac{\partial^2 \hat{f}}{\partial Z^2} = \frac{1}{2} \left( \frac{f_{i+1}^{n+1} - 2f_i^{n+1} + f_{i-1}^{n+1}}{h^2} + \frac{f_{i+1}^n - 2f_i^n + f_{i-1}^n}{h^2} \right),
\]

\[
\frac{\partial^2 \hat{f}}{\partial \tau \partial Z} = \frac{1}{k} \left( \frac{f_{i+1}^{n+1} - f_{i-1}^{n+1}}{2h} - \frac{f_{i+1}^n - f_{i-1}^n}{2h} \right),
\]
where \( i = 1..H \) is the space index and \( n \) is the time index.

To determine the function \( v \) at time moment \((n + 1)k\), we will use the following system of nonlinear equations, which can be solved by the Newton method:

\[
v^{n+1}_i = 0, \\
\frac{-v^{n+1}_i + 4v^{n+1}_{i+1} - 3v^{n+1}_{i-1}}{2h} = \frac{1}{1 + \frac{N_z \left(1 - \exp\left[-N_\alpha (n+1)\right]\right)}{N_\alpha}},
\]

\[
-\frac{u^{n+1}_i}{2} + \frac{v^{n+1}_i}{k} - \frac{u^n_i}{2} - \frac{v^n_i}{k} = 0, \quad i = 1..H,
\]

\[
-2N_\beta \left(\frac{\partial \hat{u}}{\partial Z}\right)^2 + N_\beta \frac{\partial^2 \hat{u}}{\partial Z \partial \tau} \frac{\partial \hat{v}}{\partial Z} + N_\beta (N_\alpha + 1) \frac{\partial \hat{u}}{\partial Z} \frac{\partial \hat{v}}{\partial Z} - \\
-2(1 - \xi \hat{v}) \frac{\partial \hat{u}}{\partial Z} \frac{\partial^2 \hat{v}}{\partial Z^2} + (1 - \xi \hat{v}) \frac{\partial^2 \hat{u}}{\partial Z^2} \frac{\partial \hat{v}}{\partial Z} + N_\alpha (1 - \xi \hat{v}) \frac{\partial^2 \hat{v}}{\partial Z^2} \frac{\partial \hat{v}}{\partial Z} - \\
-\xi \left(\frac{\partial \hat{v}}{\partial Z}\right)^2 \frac{\partial \hat{u}}{\partial Z} + \xi N_\alpha \left(\frac{\partial \hat{v}}{\partial Z}\right)^3 - \xi N_\alpha \left(\frac{\partial \hat{v}}{\partial Z}\right)^4 = 0
\], \quad i = 2..H - 1.

The error, stability, and convergence of this type of finite-difference approximations are discussed elsewhere [27].

The desired dimensionless depth-averaged permeate velocity \( V/V_0 \) is equal to the value of function \( v \) at point \( Z = 1 \).

Dimensionless specific cake deposit \( \gamma \) can be derived from formula (24)

\[
\gamma = \frac{1}{N_z} \left[\left(\frac{\partial v}{\partial Z}\right)^{-1} - 1\right].
\]  

(30)

Dimensionless concentration can be calculated using expression
Expressions to determine the key performance characteristics of the adsorber are

\[ C_f = \frac{\partial \gamma(\tau,1)}{\partial \tau} + N_\alpha \gamma(\tau,1), \]  \hspace{1cm} (32)

\[ C_{pf} = C_f (1 - \xi v(\tau,1)), \]  \hspace{1cm} (33)

\[ C'_{pf} = \frac{1}{\tau_0} \int_{\tau_0}^{\xi} C_{pf} \, d\tau. \]  \hspace{1cm} (34)

As the performance characteristics of the adsorber are functions of partial derivatives of \( v \) in time and coordinate, the latter should be calculated with a very high accuracy. To calculate the coordinate partial derivative of \( v \) at \( Z = 1 \), the left finite difference approximation can be used:

\[ \left. \frac{\partial v}{\partial z} \right|_{Z=1} = \frac{\hat{v}(nk,H-2) - 4\hat{v}(nk,H-1) + 3\hat{v}(nk,H)}{2h}. \]

3.2 Approximate solution

An iterative method based on averaging the permeate velocity over the filter depth and time for HFM adsorbers was suggested in Ref. [16]. The same approximation technique will be used in this study.
In this method, the value of the time- and depth-averaged dimensionless permeate velocity

\[ v_{av} = \frac{1}{\tau} \int_{0}^{\tau} \int_{0}^{1} \frac{1}{1 + N_{x} \gamma(Z, \tau)} dZ d\tau \]  

(35)

is used to find the profiles of the SP concentration and specific cake deposit.

In this case, the error of approximation in calculating the performance characteristics of the adsorber can be reduced if the characteristics are determined from integral mass conservation relations [16]. In dimensionless form, they are written as

\[ C_f = \frac{1 - N_{p} \int_{0}^{1} \left( \frac{\partial \gamma}{\partial \tau} + \frac{\partial C}{\partial \tau} \right) dZ}{1 - \xi \int_{0}^{1} dZ / 1 + N_{x} \gamma} , \]  

(36)

\[ C_{pf} = 1 - N_{p} \int_{0}^{1} \left( \frac{\partial \gamma}{\partial \tau} + \frac{\partial C}{\partial \tau} \right) dZ , \]  

(37)

\[ C'_{pf} = \frac{1}{\tau} \left( \tau - N_{p} \int_{0}^{1} (\gamma + C) dZ \right) . \]  

(38)

Equations (36)-(38) are valid after the concentration front reaches the filter outlet. Initially (up to the time equal to the adsorber depth \( d \) divided by the velocity \( w \) at the filter outlet), the assumption of clean filter (6) requires that all three concentrations are set to zero.
The basis of the method is the analytical solution to problem (21)–(23) obtained in [16] for the case in which the dimensionless permeate velocity is constant in time and depth and equal to the value of \( v_{av} \).

In this case, the problem (21)–(23) in view of (14) for \( C = c/c_0 \) can be transformed to

\[
\frac{\partial^2 C}{\partial \tau^2} + \frac{1}{N_\beta} \frac{\partial^2 \left(C[1 - \xi_{av} Z]\right)}{\partial \tau \partial Z} + \left(N_\alpha + 1\right) \frac{\partial C}{\partial \tau} + \frac{N_\alpha}{N_\beta} \frac{\partial \left(C[1 - \xi_{av} Z]\right)}{\partial Z} = 0, \tag{39}
\]

\[C = 1 \quad \text{when } Z = 0, \tau > 0; \tag{40}\]

\[C = 0, \frac{\partial C}{\partial \tau} = 0, \quad \text{when } \tau = 0, Z > 0. \tag{41}\]

where \( \xi_{av} = \xi v_{av} \).

With \( X = -\ln \left(1 - \frac{\xi_{av} Z}{\xi_{av}}\right) \), the solution of the above problem by applying the Laplace transform in time leads to the following expressions [16]:

\[C = 0 \quad \text{when } \tau < N_\beta X, \tag{42}\]

\[C = \exp \left[\xi_{av} X - N_\alpha \tau - N_\beta X + N_\alpha N_\beta X\right] \times \\
\sum_{m=0}^{\infty} \frac{I_m}{m!} \left[2 \sqrt{N_\alpha N_\beta X \left(\tau - N_\beta X\right)}\right] \left(\frac{N_\alpha \left(\tau - N_\beta X\right)}{N_\beta X}\right)^m \quad \text{when } \tau > N_\beta X ; \tag{43}\]

\[\gamma = 0 \quad \text{when } \tau < N_\beta X, \tag{44}\]
\[
\gamma = \exp \left[ \xi_{av} X - N_\alpha \tau - N_\beta X + N_\alpha N_\beta X \right] \times \frac{1}{N_\alpha} \times \sum_{m=1}^{\infty} I_m \left[ 2 \sqrt{N_\alpha N_\beta X \left( \tau - N_\beta X \right)} \right] \left( \frac{N_\alpha \left( \tau - N_\beta X \right)}{N_\beta X} \right)^{\frac{m}{2}} \\
\text{when } \tau > N_\beta X ; 
\] (45)

\[
\frac{\partial C}{\partial \tau} = 0 \quad \text{when } \tau < N_\beta X , 
\] (46)

\[
\frac{\partial C}{\partial \tau} = N_\alpha \exp \left[ \xi_{av} X - N_\alpha \tau - N_\beta X + N_\alpha N_\beta X \right] \times \\
\sqrt{\frac{N_\beta X}{N_\alpha \left( \tau - N_\beta X \right)}} \times I_1 \left[ 2 \sqrt{N_\alpha N_\beta X \left( \tau - N_\beta X \right)} \right] \\
\text{when } \tau > N_\beta X ; 
\] (47)

\[
\frac{\partial \gamma}{\partial \tau} = 0 \quad \text{when } \tau < N_\beta X , 
\] (48)

\[
\frac{\partial \gamma}{\partial \tau} = \exp \left[ \xi_{av} X - N_\alpha \tau - N_\beta X + N_\alpha N_\beta X \right] \times \\
I_0 \left[ 2 \sqrt{N_\alpha N_\beta X \left( \tau - N_\beta X \right)} \right] \\
\text{when } \tau > N_\beta X . 
\] (49)

The approximate method uses the following iterative algorithm along with the averaging procedure:

LET \( T_w \) – given filter operation time, \( v_{av}^* \) – averaged value of the dimensionless permeate velocity \( v_{av} \) at \( i \)-th iteration, \( \varepsilon_v \) – given maximum relative difference between two consecutive values of \( v_{av} \).

STEP 1. \( t := T_w, v_{av}^* := 1 \) (specify time \( t \) and initial value for \( v_{av} \)).
STEP 2. IF $i > 0$ AND \( \left( v^*_i - v^*_i \right) / v^*_i < \varepsilon_v \), THEN STOP (execute the further steps until the relative difference between two consecutive values of $v_{av}$ does not become less than $\varepsilon_v$).

STEP 3. EVALUATE $v^*_i$ by (35) using the value of $\gamma$ determined at $v^*_i$.

STEP 4. IF $i > 1$, THEN $v^*_i \coloneqq \left( v^*_{i+1} + v^*_i \right) / 2$ (convergence acceleration).

STEP 5. GO TO STEP 2.

4. Results and discussion

For calculations we will use the empirical values of $\beta$, $\alpha$, and $r_c$ determined for a deadend HFM filter in Ref. [27].

Figure 5 demonstrates the dependence of particle concentration on time in the product (permeate and filtrate) calculated by the numerical and approximate methods. The approximate results are represented as a series of dashed curves calculated for various time intervals of averaging. It is seen that the use of the approximate curve calculated for a time interval of averaging equal to the entire interval of operation time can bring a noticeable error as compared with the numerical solution. The error is higher for the initial period of the process, where the averaged permeate velocity highly
underestimates the true value, and becomes smaller as the curve comes closer to the end of the interval, where the averaged permeate velocity overestimates the true value. As the averaged value of permeate velocity is closer to its true value at the end of operation, the error of the approximate curve for the end of operation is much lower than that for the initial period. To obtain a product-concentration-versus-time curve with an acceptable accuracy calculated with the approximate method, it is suggested to use the end points of curves calculated for a series of averaging intervals smaller than the operation time. It is easy to see that in this case the approximate curve, though a little bit lower than the numerical one, runs almost parallel to the numerical solution. Keeping in mind that some of the parameters in the model, such as the adsorption and peptization coefficients, should be determined empirically by best fitting the experimental data, the end-point approximate technique may adequately describe the performance of the adsorber.

To build the end-point approximate curve, it is suggested to use the method of asymptotic interpolation, which is based on asymptotic solutions and interpolation (switching) functions with coefficients determined from the known values of the function in the intermediate region [40]. For the adsorber, the values of $v_{av}$ are computed for three different time intervals of averaging using the above iterative algorithm. Then they are used to determine the coefficients in the interpolation function:

$$v_{av} = \frac{1}{\left(1 + a_i t^{a_i}\right)^{a_j}},$$

(50)
where $t$ is the time interval of averaging; $a_1$, $a_2$, $a_3$ are coefficients determined using the three known values of $v_{av}$ and corresponding time intervals.

The desired performance characteristics $c_f$, $c_{pf}$, $c^\prime_{pf}$, and $V$ are computed using the particle concentration and specific cake deposit functions and their time derivatives given by (42) - (49) with the values of $\xi_{av}$ calculated using the dependence of $v_{av}$ on time determined from Eq. (50).

Figure 6 presents the decline of permeate flow rate in the adsorber calculated by the numerical and end-point approximate methods. The values of permeate flow rate calculated by the end-point approximate method are very close (within 4%) to those calculated by the finite-difference procedure. The good accuracy of this end-point approximate method is mostly attributed to the fact that, as our estimates show, the actual profile of $v$ in $z$ is close to a linear one. For example, Eq. (26) implies that at the initial moment the profile of $v$ is linear in $z$.

The curve representing the time dependence of the filtrate particle concentration is shown in Fig. 7. As expected for the particle deposition mechanism based on the linear reversible adsorption equation, the curve shape looks like the typical breakthrough curve for adsorption or depth filtration beds. The end-point approximate method is in good agreement with the numerical solution.

Figures 8 and 9 show the time dependence of the adsorber particle retentions for continuous-flow and batch operations, respectively. The data imply that for a given
adsorber particle retention of 0.9 the continuous-flow operation time between flushings, during which the constant production rate at constant TMP is maintained, is higher than 20 min, whereas the batch operation time is about 40 min. The end-point approximate method slightly overestimates the values of the particle retention for a continuous flow operation (Fig. 8). In Fig. 9, the solid circles are calculated from Eq. (38) by the end-point approximate method, and the solid triangles are calculated from Eq. (18) with \( c_{pf} \) determined by the end-point approximate method. It is seen that the circles slightly underestimate the batch particle retention while the triangles slightly overestimate it.

Interestingly, the time dependences of both continuous-flow and batch adsorber particle retentions are close to the linear ones. If this fact would be supported by experimental studies, their results could be used to formulate simple linear relations involving the key dimensionless numbers.

It is worth noting that the iterative algorithm with averaged permeation velocity usually requires only 4-5 iterations to converge when the relative error \( \varepsilon_v \) is set to 0.1%.

5. Conclusions

The design strategy for ultrafiltration/microfiltration filters based on reduction of the concentration polarization and particle deposition to increase the permeate velocity is inherently associated with additional expenditures in power or other material
resources, which can make it not enough cost-effective to be competitive with non-membrane filtration processes in water and wastewater treatment applications. At the same time, the design strategy based on utilization of particle deposition on membrane surface to produce an additional (to permeate) volume of clarified water does not require additional power expenditures. It can be cost-effective and provides high values of water recovery. The calculation using the empirical values of deposition and re-entrainment coefficients shows that the HFM adsorber, which produces two flows of clarified water, permeate and filtrate (the latter due to particle deposition on membrane surface), can provide a constant permeate flow rate equal to its initial value at constant TMP and a particle retention of 0.9 for 20 and 40 min for continuous flow and batch operations, respectively. These operation times between flushings are close to the typical ones for commercial dead-end hollow fiber filters. The concept of HFM adsorber gives us a chance to transit from the feed-permeate-retentate flow diagram, which results from the strategy of reducing concentration polarization and cake growth, to the feed-permeate-filtrate flow diagram, which results from the cake-utilization strategy.

The cake formation mechanism based on the linear particle deposition-re-entrainment equation involving the term representing the deposition flux and the term accounting for the decline of deposition rate due to the growth of cake layer and particle re-entrainment can be justified by its good agreement with the experimental data obtained for an outside-in HFM filter, as well as by the common practice of linearizing a complicated relation when its nature is not understood enough. For the same reasons, the
particle deposition and re-entrainment coefficients can be taken to be phenomenological constants.

The system of equations governing the performance of a hollow fiber membrane adsorber can be solved by the implicit Crank–Nicholson finite-difference scheme, as well as by the approximate method based on the value of permeate velocity averaged over time and depth coordinate along with the Laplace transform solution for the case of averaged permeate velocity. The accuracy of the approximate method in determining the dependence of the particle concentration in the filtrate and permeate flow rate on time can considerably be improved by using the end points of the curves calculated by the approximate method for a series of averaging intervals smaller than the operation time. These end points with the help of an interpolation formula can be used to determine the desired time functions of adsorber performance characteristics. The approximate method modified in this manner, which is called the end-point approximate method, can provide an accuracy acceptable for adsorber design calculations.

Nomenclature

\[ C \quad \text{dimensionless concentration of suspended particles} \]

\[ C_f \quad \text{dimensionless filtrate concentrate} \]

\[ C_{pf} \quad \text{dimensionless product (permeate plus filtrate) concentration for continuous-flow} \]
operation

$C'_{pf}$ dimensionless product (permeate plus filtrate) concentration for batch operation (kg/m$^3$)

c concentration of particles in suspension (kg/m$^3$)

c$_0$ concentration of suspended particles in feed suspension (kg/m$^3$)

c$_f$ filtrate concentration (kg/m$^3$)

c$_{pf}$ product (permeate plus filtrate) concentration for continuous-flow operation (kg/m$^3$)

c$_{pf}'$ product (permeate plus filtrate) concentration for batch operation (kg/m$^3$)

d filter depth (m)

$I_m$ modified Bessel function of order $m$

$k_1, k_2, k_3$ coefficients in Eq. (7)

$N_{a} = \alpha / s \beta$ dimensionless number given by the ratio of desorption and adsorption rate constants

$N_{\beta} = s \beta \omega_0 / d$ dimensionless number accounting for the adsorption ability of membrane surface with respect to suspended particles

$N_{\chi} = \chi_1 c_0 / s$ dimensionless number accounting for cake and membrane resistance

$P$ transmembrane pressure (Pa)

$R$ filter retention for continuous-flow operation

$R_m$ clean membrane resistance (m$^{-1}$)

$R'$ filter retention for batch operation
\( r \)  \hspace{1em} \text{radial coordinate (m)}

\( r_0 \)  \hspace{1em} \text{external radius of hollow-fiber bundle in radial filter (m)}

\( r_{in} \)  \hspace{1em} \text{internal radius of hollow fiber bundle in radial filter (m)}

\( r_c \)  \hspace{1em} \text{specific cake resistance (m/kg)}

\( S \)  \hspace{1em} \text{cross section area of filter interfiber space (m}\^2\)

\( S_m \)  \hspace{1em} \text{membrane surface area (m}\^2\)

\( s \)  \hspace{1em} \text{specific membrane surface area in filter (m}\^{-1}\)

\( t \)  \hspace{1em} \text{time (s)}

\( V \)  \hspace{1em} \text{permeate velocity averaged over filter depth (m/s)}

\( V_0 \)  \hspace{1em} \text{initial permeate velocity (m/s)}

\( V_p \)  \hspace{1em} \text{permeate velocity (m/s)}

\( v \)  \hspace{1em} \text{permeate flow rate for the filter region found between adsorber inlet and plane with coordinate } Z

\( v_{av} \)  \hspace{1em} \text{dimensionless permeate velocity averaged over filter depth and time}

\( w \)  \hspace{1em} \text{filtration velocity (m/s)}

\( w_0 \)  \hspace{1em} \text{feed velocity (m/s)}

\( X \)  \hspace{1em} \text{dimensionless corrected coordinate}

\( Z \)  \hspace{1em} \text{dimensionless effective filter depth coordinate}

\( z \)  \hspace{1em} \text{effective filter depth coordinate (m)}
Greek letters

\[ \alpha \] peptization (re-entrainment) coefficient (s\(^{-1}\))
\[ \beta \] adsorption (deposition) coefficient (m/s)
\[ \Gamma \] specific cake deposit (kg/m\(^2\))
\[ \gamma \] dimensionless specific cake deposit
\[ \mu \] fluid viscosity (Pa s)
\[ \xi \] ratio of initial permeate flow rate to feed flow rate
\[ \xi_{av} \] average ratio of permeate flow rate to feed flow rate
\[ \tau \] dimensionless time
\[ \chi_i = r_i / R_m \] ratio of specific cake resistance to membrane resistance (m\(^2\)/kg)
\[ \psi \] parameter vector

Acknowledgments

The author is much grateful to Dr. Serge Timashev, Dr. Glenn Lipscomb, Dr. Kamalesh Sirkar, and Dr. George Belfort for their valuable remarks and comments, which were taken into account in preparing this paper.
Appendix A

The pressure loss $\Delta p_p$ due to the fluid friction inside a hollow fiber (permeate side) can be estimated using the Hagen–Poisseuille and continuity equations [41–43].

In this case, the Hagen–Poisseuille equation can be rewritten as

$$\Delta p_p = \frac{128\mu L}{\pi D_{in}^4} \int_0^1 Q_p(x) \, dx,$$  \hspace{1cm} (A.1)

where volumetric permeate flow rate $Q_p$ is given by

$$Q_p(x) = k P D_{in}^2 \sqrt{\frac{\pi}{128 \mu}} \sinh \left[ \sqrt{\frac{128 \mu}{\pi}} \left( k L / D_{in}^2 \right) \right] \cosh \left[ \sqrt{\frac{128 \mu}{\pi}} \left( k L / D_{in}^2 \right) \right],$$  \hspace{1cm} (A.2)

$L$ is the hollow fiber length, $D_{in}$ the internal diameter of a hollow fiber, $k$ the membrane permeability, $x$ the dimensionless axial coordinate.

Let $\mu = 0.0012 \, Pas$, $P = 0.5 \times 10^5 \, Pa$, $L = 0.3 \, m$, $D_{in} = 0.8 \, mm$, $D_{ext} = 1.2 \, mm$, $V_0 = 7 \times 10^{-5} \, m/s$ . The approximate value of $k$ can be evaluated by solving Eq. (A.2) when the value of $Q_p$ at $x=1$ is known. Assuming that $Q_p(1) \approx \pi D_{ext} L V_0$, where $D_{ext}$ is the external diameter of a hollow fiber, which implies the highest (initial) value of pressure losses inside the hollow fiber, we obtain $k = 9.4 \times 10^{-7} \, m/\sqrt{Pas}$ . Then evaluating (A.1) in view of (A.2) yields $\Delta p_p = 236 \, Pa$, which can be ignored as compared with the operational pressure of 50 kPa.
The pressure loss across the fiber array (feed side) when the liquid flow runs normal to the fiber axes can be estimated using Bergelin's formula for tube banks [44].

In this case, the Reynolds number $Re_v$ is defined as

$$Re_v = \frac{D_v w_{\text{max}} \rho}{\mu},$$

where $D_v$ is the volumetric hydraulic diameter, $w_{\text{max}}$ the maximum liquid velocity in the interfiber channels, $\rho$ the liquid density.

$$D_v = 4 \times \frac{1 - \epsilon_h}{\epsilon_h} \times \frac{N \pi D_{\text{ext}}^2 L}{4} \times \frac{1}{N \pi D_{\text{ext}} L} = \frac{D_{\text{ext}} (1 - \epsilon_h)}{\epsilon_h},$$

where $\epsilon_h$ is the filter packing density, $N$ the number of hollow fibers.

With $\rho = 10^3 \text{kg/m}^3$, $\mu = 10^{-3} \text{Pas}$, $D_{\text{ext}} \approx 1 \text{mm}$, $\epsilon_h \approx 0.5$, one obtains

$$Re_v = 10^6 \times \frac{D_{\text{ext}} (1 - \epsilon_h)}{\epsilon_h} \times w_{\text{max}} \approx 10^3 \times w_{\text{max}}.$$

With $w_{\text{max}} \approx 10^{-2} \text{ m/s}$, the Reynolds number $Re_v < 100$, which allows us to use Bergelin's formula:

$$\Delta p_f = \frac{280 N_r}{Re_v} \left( \frac{D_{\text{ext}}}{p} \right)^{1.6} \left( \frac{\rho w_{\text{max}}^2}{2} \right),$$

where $\Delta p_f$ is the pressure drop across the hollow fiber array, $N_r$ the number of major restrictions encountered in the flow, $p$ the pitch factor.

We select a staggered arrangement of hollow fibers (Fig. 1) with an array depth $d = 0.05 \text{ m}$ and an angle of diagonal lines of $60^\circ$. Then $N_r = 2d/(\sqrt{3} g_1)$, where $g_1$ is
the diagonal fiber-to-fiber distance. The shortest distance between two nearest fibers \( p \) is equal to \( g_1 \). For an array packing density of 0.5, it is reasonable to take \( g_1 = 1.5 D_{\text{ext}} \).

Then

\[
\Delta p_f = \frac{280 \times 2 \times d}{\sqrt{3} \times 1.5 \times D_{\text{ext}} \times \text{Re}_v} \left( \frac{1}{1.5} \right)^{1.6} \left( \frac{10^3 w_{\text{max}}^2}{2} \right) \approx 5.63 \times 10^4 \frac{d_{\text{max}}^2}{D_{\text{ext}} \times \text{Re}_v} \approx 5.63 \times 10^{-2} \frac{\varepsilon_h}{1 - \varepsilon_h} \frac{d_{w_{\text{max}}}}{D_{\text{ext}}^2} \approx 2.8 \times 10^3 \times w_{\text{max}}
\]

With \( w_{\text{max}} \approx 10^{-2} \text{ m/s} \), the pressure loss is less than 100 Pa, which is much lower than the operational pressure.

The extension to which the narrowing of interfiber channels due to cake formation can affect the operational pressure (head) can be estimated from the following consideration. The equivalent diameter of the interfiber channel is about 1 mm. The size of suspended particles is about 0.001 mm. The adsorber filtration cycle is supposed to get terminated when the product concentration is about 0.1 of feed concentration [18]; that is, at about 0.33 hr (Fig. 5). The permeate flow velocity at this moment is about 0.3 of the initial value (Fig. 6). This is close to the value at which the industrial submerged dead-end hollow fiber filters are halted for flushing. We assume that during the filtration cycle only several layers of deposited particles can form on the membrane surface. Accordingly, the decrease in interfiber channel cross-section area will not exceed several percent. This assumption is supported by the results of direct observation of cake formation [20]. It is easily seen from the photographic images reported in Ref. [20] that
the average thickness of the cake layer is many times (>10) less than the fiber diameter after 4 hr of operation. It is obvious that after 0.5 hr it will be far less.

Appendix B

The mass conservation equation for the radial filter (Fig. 2) can be written as

\[ \frac{\partial c}{\partial t} - \frac{1}{r} \frac{\partial (rwc)}{\partial r} = -s \frac{\partial \Gamma}{\partial t}. \] (B.1)

The continuity equation in this case takes the form:

\[ \frac{dw}{dr} + \frac{w}{r} = sV_p. \] (B.2)

Integration of equation (B.2) over \( r \) leads to

\[ w = \frac{1}{r} \left[ w_0 r_0 - \int r V_p r dr \right]. \] (B.3)

Hence

\[ - \frac{1}{r} \frac{\partial (rwc)}{\partial r} = - \frac{1}{r} \frac{\partial}{\partial r} \left[ c \left( w_0 r_0 - \int r V_p r dr \right) \right]. \] (B.4)

Introducing a new variable \( z = \frac{r_0^2 - r^2}{2r_0} \) and substituting it into the right side of equation (B.4) transforms the latter to

\[ - \frac{1}{r} \frac{\partial (rwc)}{\partial r} = \frac{\partial}{\partial z} \left[ c \left( w_0 - \int z V_p dz \right) \right]. \]

Thus, equation (B.1) with the new variable \( z \) transforms to equation (1).
References


Figure captions

Fig. 1. Rectangular cartridge-type outside-in hollow fiber membrane filter: (a) cartridge of HFM wafers (1, top plate; d, filter depth), (b) HFM wafer (2, perforated frame; 3, hollow fiber membrane), and (c) flow diagram (gray solid rings – cake layers; porous rings – hollow fibers) [28].

Fig. 2. Radial outside-in hollow fiber membrane filter: (a) flow diagram in vertical cross section, (b) flow diagram in horizontal cross section.
Fig. 3. Profiles of (a) dimensionless SP concentration $C$ and (b) dimensionless specific deposit $\gamma$ in an outside-in HFM adsorber ($\beta = 1.81 \times 10^{-4}$ m/s, $\alpha = 4.20 \times 10^{-4}$ 1/s, $s = 3.88 \times 10^3$ 1/m, $V_0 = 1.39 \times 10^{-5}$ m/s, $\xi = 0.99$, $d = 0.05$ m, $N_\chi = 0.0072$) [18].

Fig. 4. Comparison of theoretical curves with experimental data obtained for deadend outside-in HFM filter at constant transmembrane pressure (phenomenological constants for both models were determined from the experiment with 20 kPa) [27].

Fig. 5. Variation of product concentration with time for continuous-flow operation ($\beta = 1.81 \times 10^{-4}$ m/s, $\alpha = 4.20 \times 10^{-4}$ 1/s, $s = 3.88 \times 10^3$ 1/m, $V_0 = 6.94 \times 10^{-5}$ m/s, $\xi = 1$, $d = 0.05$ m, $N_\chi = 0.0072$).

Fig. 6. Decline of permeate flow rate with time ($\beta = 1.81 \times 10^{-4}$ m/s, $\alpha = 4.20 \times 10^{-4}$ 1/s, $s = 3.88 \times 10^3$ 1/m, $V_0 = 6.94 \times 10^{-5}$ m/s, $\xi = 1$, $d = 0.05$ m, $N_\chi = 0.0072$).

Fig. 7. Variation of filtrate concentration with time ($\beta = 1.81 \times 10^{-4}$ m/s, $\alpha = 4.20 \times 10^{-4}$ 1/s, $s = 3.88 \times 10^3$ 1/m, $V_0 = 6.94 \times 10^{-5}$ m/s, $\xi = 1$, $d = 0.05$ m, $N_\chi = 0.0072$).

Fig. 8. Variation of particle retention with time for continuous-flow operation ($\beta = 1.81 \times 10^{-4}$ m/s, $\alpha = 4.20 \times 10^{-4}$ 1/s, $s = 3.88 \times 10^3$ 1/m, $V_0 = 6.94 \times 10^{-5}$ m/s, $\xi = 1$, $d = 0.05$ m, $N_\chi = 0.0072$).

Fig. 9. Variation of particle retention with time for batch-flow operation ($\beta = 1.81 \times 10^{-4}$ m/s, $\alpha = 4.20 \times 10^{-4}$ 1/s, $s = 3.88 \times 10^3$ 1/m, $V_0 = 6.94 \times 10^{-5}$ m/s, $\xi = 1$, $d = 0.05$ m, $N_\chi = 0.0072$).
Figure 4

Reciprocal dimensionless permeate flow rate $V_0/V$ vs. filtration time (hr)

- Classical cake filtration
- Depth filtration Eq. (8)
- 20 kPa
- 40 kPa
- 60 kPa
Figure 6

- Line: numerical
- Circles: end-point approximate

Graph showing the dimensionless permeate flow rate, $V/V_0$, as a function of filtration time (hr). The flow rate decreases exponentially with time.
Figure 7

The graph shows the relationship between filtration concentration, $C_f/C_0$, and filtration time (sec). The curve is labeled as 'numerical' and the points are marked as 'end-point approximate'. The x-axis represents filtration time, ranging from $10^1$ to $10^4$ seconds, while the y-axis represents the filtration concentration, ranging from 0.00 to 1.00.
Figure 8

Particle retention (continuous flow), R

Filtration time (hr)

Numerical

End-point approximate