Laguerre entire functions and the Lee–Yang property

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Abstract

Laguerre entire functions are the polynomials of a single complex variable possessing real nonpositive zeros only or their limits on compact subsets of \( \mathbb{C} \). These functions are employed to establish a property of isotropic (i.e., \( O(N) \)-invariant) probability measures on \( \mathbb{R}^N \), \( N \in \mathbb{N} \). It is called the Lee–Yang property since, in the case \( N = 1 \), it corresponds to the property of the partition function of certain models of statistical physics, first established by T.D. Lee and C.N. Yang. A class of measures possessing this property is described. Certain connections of the Lee–Yang property with other aspects of the analytic theory of probability measures are discussed.

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1. Introduction

Discovered in the fifties by the famous physicists Lee and Yang [11,23], a property of the partition functions of certain models of statistical physics later was called the Lee–Yang property. Since that time methods based on this property appeared in statistical physics and quantum field theory (see e.g., [4,19]). The original variant of the Lee–Yang theorem was refined and extended in a number of papers (see e.g., [1,6,9,13,17,21,22] and the references therein).
As examples of the application of methods based on this theory we mention [6–8,16].

A simplest version of the Lee–Yang theorem may be described as follows. Obviously,

\[ U(z) = \sum_{s \in \Xi} \exp \left( z \sum_{j=1}^{n} s_j + \frac{1}{2} \sum_{i,j=1}^{n} J_{ij} s_i s_j \right), \quad z \in \mathbb{C} \]  

(1.1)

is an exponential type entire function. Summation in (1.1) is taken over \( 2^n \) configurations \( s = (s_1, \ldots, s_n) \in \Xi = \{-1, +1\}^n \). The Lee–Yang theorem states that all zeros of \( \Phi \) are purely imaginary provided all \( J_{ij} \) are nonnegative. In statistical physics this function is known as a partition function of a ferromagnetic Ising model in an external magnetic field \( z \), which describes a system of spins \( \{s_1, \ldots, s_n\} \) interacting via ferromagnetic coupling \( J_{ij} \geq 0 \).

Thus, for a symmetric probability measure \( \mu \) on \( \mathbb{R} \), the Lee–Yang property may be said to occur if

\[ \Phi_{\mu}(z) = \int_{\mathbb{R}} \exp(\lambda x) \, d\mu(x), \quad z \in \mathbb{C} \]  

(1.2)

is an entire function, which possesses imaginary zeros only or does not vanish at all. Hence the measure \( \mu \) which describes the probability distribution of the total magnetic moment \( m(s) = \sum_{j=1}^{n} s_j \) of the ferromagnetic Ising model, defined in the following way:

\[ \mu(B) = \frac{1}{Z} \sum_{s:m(s) \in B} \exp \left( \frac{1}{2} \sum_{i,j=1}^{n} J_{ij} s_i s_j \right), \]

where \( B \) is a Borel subset of \( \mathbb{R} \) and

\[ Z = \sum_{s \in \Xi} \exp \left( \frac{1}{2} \sum_{i,j=1}^{n} J_{ij} s_i s_j \right) \]

possesses the Lee–Yang property. By means of the so-called Ising approximation, in [20] it was proved that the above fact implies the Lee–Yang property for the probability measure

\[ d\nu(x) = C \exp(-ax^2 - x^4) \, dx \]

with arbitrary \( a \in \mathbb{R} \). Later, it was shown [4,19], that there exist even polynomials \( \phi \), \( \lim_{x \to +\infty} \phi(x) = +\infty \), \( \deg \phi = 6 \), such that the probability measure

\[ d\nu_{\phi}(x) = C \exp(-\phi(x)) \, dx \]  

(1.3)

does not possess the Lee–Yang property. The search of the polynomials \( \phi \) for which the measures \( \nu_{\phi} \) possess the Lee–Yang property was stimulated by the attempts to construct polynomial models of quantum field theory [19], which
took place in the seventies and eighties. In a more general setting this problem may be formulated as follows.

**Problem 1.1.** To find necessary and sufficient conditions for an even function $\phi : \mathbb{R} \to \mathbb{R}$, under which the probability measure $v_\phi$ (1.3) possesses the Lee–Yang property, i.e., its function (1.2) is entire and has imaginary zeros only or has them none.

Obviously, this is a particular case of a much more older problem of description of probability measures $m$ on $\mathbb{R}$, for which the characteristic functions $\varphi_\nu(z) = \Phi_\nu(iz)$ are entire functions possessing real zeros only. A partial answer to this question was given by Pólya in his article [18] on the problem of zeros of Riemannian entire functions (for a detailed discussion of this problem, see [2]). Namely, he showed that the characteristic function of the measure (1.3) with $\phi(x) = a \cosh x$, $a > 0$ is an entire function possessing real zeros only. In their brilliant paper, Gol'dberg and Ostrovski [5] have proved that a ridge entire function $u$, which is of finite order, (ridge means $|\varphi(x+iy)| \leq |\varphi(iy)|$, all characteristic functions are ridge) possesses real zeros only if and only if it has the following infinite-product representation:

$$
\varphi(z) = C \exp(-\kappa z^2 + i\delta z) \prod_{k=1}^{\infty} (1 - \gamma_k z^2), \quad \kappa \geq 0, \quad \delta \in \mathbb{R},
$$

which means, in particular, that its order is at most two. This establishes the form of the function $\Phi_\mu$ (1.2) for $\mu$ possessing the Lee–Yang property. But the functions obeying the representations of this type are known as Laguerre entire functions [2,10]. Putting all these facts together one concludes that the Laguerre entire functions could constitute a proper setting for developing the notion of the Lee–Yang property.

Another problem of this kind is how to generalize the notion of the Lee–Yang property to the measures on $\mathbb{R}^N$, $N > 1$. Of course, one can choose a direction in $\mathbb{R}^N$ (i.e., a unit vector $e \in \mathbb{R}^N$) and set

$$
\Phi_\mu(z) = \int_{\mathbb{R}^N} \exp(\langle z, e, x \rangle) \, d\mu(x), \quad z \in \mathbb{C},
$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^N$. Then, the Lee–Yang property of $\mu$ could be said to occur if $\Phi_\mu$ would be an even entire function possessing imaginary zeros only or not vanishing at all. This approach was used in [3] where it was proved that for the probability measure

$$
dv(x) = C \exp(-a(x,x) - (x,x)^2) \, dx,
$$
the function $\Phi_e$ (1.5) is entire and has real zeros only, that occurs for all $a \in \mathbb{R}$ and $N = 2, 3$. Let $\sigma$ stand for the measure which is concentrated on the unit sphere $S^{N-1} \subset \mathbb{R}^N$ and, being restricted to it, coincides with the uniform measure. For this measure, the function $\Phi_\sigma$ may be obtained explicitly (see below). It appears, this entire function has imaginary zeros only, hence $\sigma$ possesses the Lee–Yang property as well. One observes that all these measures are isotropic, that is they are invariant with respect to the orthogonal group $O(N)$. For isotropic measures, the above property is independent of the choice of the unit vector $e$ in (1.5). For nonisotropic measures, it would be not the case, therefore, the notion of this kind may be really useful first of all for isotropic measures. In this article, we describe a construction based on the use of the Laguerre entire function, which establishes the Lee–Yang property for isotropic measures and gives some description of such measures. Partially, we review our results already published in [6,9,10], but such statements as Theorems 3.1, 3.4 are new.

One more problem connected to the analytic properties of characteristic functions of probability measures was communicated by Lukacs in [14]. Let $\mathcal{D}$ be the set of characteristic functions $\varphi$, such that the functions $\psi(z) = 1/\varphi(iz)$ are again characteristic functions. Thus, according to [14] the van Danzig problem consists in the description of $\mathcal{D}$. Let $\mathcal{D}_E$ stand for the subset of $\mathcal{D}$ consisting of entire functions. Lukacs gave some examples of the elements of $\mathcal{D}_E$, namely, the functions

$$\cos z, \quad \frac{\sin z}{z}, \quad \exp(-z^2/2)$$

belong to this class. One observes that all the three corresponding measures possess the Lee–Yang property. Further, in [14] it was shown that the operations

$$T_1 : \varphi(z) \mapsto \frac{\varphi'(z)}{z\varphi''(0)}, \quad T_2 : \varphi(z) \mapsto \frac{\varphi''(z)}{\varphi''(0)}$$

preserve $\mathcal{D}_E$. In what follows, we have one more problem.

**Problem 1.2.** What would be another examples of operations which preserve $\mathcal{D}_E$?

## 2. Definitions and preliminaria

In the sequel, $\mathcal{F}$, $\mathcal{P}$, $\mathcal{P}^+$ will stand for the set of all entire functions of a single complex variable, for the set of all such polynomials, and for the set of all polynomials which have real nonpositive zeros only, respectively.
Definition 2.1. The closure of the set $\mathcal{P}^+$ in the topology of uniform convergence on compact subsets of the complex plane $\mathbb{C}$ is called the set of Laguerre entire functions $\mathcal{L}$.

Due to Laguerre and Pólya we have the following characterization of $\mathcal{L}$.

Proposition 2.1. Every element of $\mathcal{L}$ may be written as

$$f(z) = C \exp(\kappa z) z^m \prod_{k=1}^{\infty} (1 + \gamma_k z), \quad C \in \mathbb{C},$$

$$\kappa \geq 0, \quad m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \quad \gamma_k \geq 0, \quad \sum_{k=1}^{\infty} \gamma_k < \infty. \quad (2.1)$$

More information about such functions may be found in [2,10]. In particular, for every $f \in \mathcal{L}$, the sequence of its derivatives at zero $(f^{(n)}(0))_{n \in \mathbb{N}_0}$ is a sequence of Pólya numbers [2,12]. The sequence of complex numbers $(\alpha_k)_{k \in \mathbb{N}_0}$ is called a sequence of Pólya numbers if, for every polynomial $p(z) = \pi_0 + \pi_1 z + \cdots + \pi_m z^m$ possessing real zeros only, the polynomial $(\alpha p)(z) = \pi_0 \alpha_0 + \pi_1 \alpha_1 z + \cdots + \pi_m \alpha_m z^m$ possesses real zeros only as well.

Given $N \in \mathbb{N}$, let $\mathcal{A}_N$ stand for the set of all isotropic (i.e., $O(N)$-invariant) probability measures on $\mathbb{R}^N$, such that

$$\int_{\mathbb{R}^N} \exp(a(x,x)) \, d\mu(x) < \infty,$$

for some $a > 0$. A function $F : \mathbb{R}^N \to \mathbb{C}$ is called isotropic if, for every $U \in O(N)$, $F_U = F$. Here

$$F_U(x) \overset{\text{def}}{=} F(Ux), \quad x \in \mathbb{R}^N.$$  

An infinitely differentiable function $F : \mathbb{R}^N \to \mathbb{C}$ is called real analytic if its Taylor series expansion, centred at a given $x \in \mathbb{R}^N$, converges absolutely at any point and for any such $x$. Clearly, each a polynomial $P : \mathbb{R}^N \to \mathbb{C}$ is a real analytic function. Let $\mathcal{P}_N$ stand for the set of all isotropic polynomials. There exists a classical statement characterizing this set.

Proposition 2.2 (Study–Weyl Theorem). There exists a bijection between $\mathcal{P}$ and $\mathcal{P}_N$, which has the following form

$$\mathcal{P} \ni p((x,x)) = P(x) \in \mathcal{P}_N.$$  

Generalizations of this theorem to real analytic functions may be found in [15]. Let $\mathcal{F}_N$ stand for the set of all isotropic real analytic functions. Obviously, for every $f \in \mathcal{F}$, the function $F$ defined by $f$ as follows:
\[ F(x) = f((x,x)), \quad \forall x \in \mathbb{R}^N \]  
(2.3)
begins to \( \mathcal{F}_N \). We set
\[ \mathcal{F}_N(\mathcal{L}) = \{ F \in \mathcal{F}_N | (\exists f \in \mathcal{L}) F(x) = f((x,x)) \} . \]  
(2.4)
It is not hard to show that, for every \( \mu \in \mathcal{M}_N \), the function
\[ F_{\mu}(x) = \int_{\mathbb{R}^N} \exp((x,y)) \, d\mu(y) \]  
(2.5)
begins to \( \mathcal{F}_N \).

**Definition 2.2.** A measure \( \mu \in \mathcal{M}_N \) is said to possess the Lee–Yang property if \( F_{\mu} \in \mathcal{F}_N(\mathcal{L}) \). The set of all such measures will be denoted by \( \mathcal{M}_N(\mathcal{L}) \).

3. The results

In the sequel for a measure \( \mu \in \mathcal{M}_N(\mathcal{L}) \), \( F_{\mu} \) will stand for the function (2.5), \( f_{\mu} \) will stand for the functions connected with \( F_{\mu} \) by (2.3). By Definition 2.2, each such \( f_{\mu} \) possesses the representation
\[ f_{\mu}(z) = \exp(\kappa z) \prod_{k=1}^{\infty} (1 + \gamma_k z), \quad \kappa \geq 0, \quad \gamma_k \geq 0, \quad \sum_{k=1}^{\infty} \gamma_k < \infty . \]  
(3.1)
In view of the Gol’dberg–Ostrovs’kii theorem, for the parameters \( \kappa \) and \( \gamma_k \), one has the following possibilities: (a) \( \kappa = \gamma_1 = \cdots = \gamma_k \cdots = 0 \), i.e., \( f_{\mu}(z) = 1 \) or, equivalently, \( \mu \) is the Dirac measure concentrated at zero; (b) \( \kappa > 0, \gamma_1 = \cdots = \gamma_k \cdots = 0 \), which corresponds to a nondegenerate isotropic Gaussian measure; (c) \( \kappa = 0 \), all \( \gamma_k \) are positive, in this case \( \mu \) is a sub-Gaussian isotropic measure, certain examples of such measures are given below; (d) \( \kappa > 0 \) and all \( \gamma_k \) are positive, which may correspond to the convolution of the measures of the above kind. The parameters \( \kappa, \gamma_k, k \in \mathbb{N} \) determine the following numerical characteristics of \( \mu \):
\[ m_1(\mu) = \kappa + \sum_{k=1}^{\infty} \gamma_k, \quad m_s(\mu) = \sum_{k=1}^{\infty} \gamma_k^s, \quad s \in \mathbb{N} \setminus \{ 1 \} \]  
(3.2)
which in turn determine the moments and semi-invariants of \( \mu \). From the representation (3.1) one concludes that, for \( \mu \in \mathcal{M}_N(\mathcal{L}) \), the function
\[ g_{\mu}(z) \overset{\text{def}}{=} \log f_{\mu}(z) \]  
(3.3)
is holomorphic in a neighborhood of the origin, where it can be expanded
\[ g_\mu(z) = \sum_{s=1}^{\infty} \frac{1}{s!} g_\mu^{(s)}(0) z^s, \quad g_\mu^{(s)}(0) = (-1)^{s-1} (s-1)! m_s(\mu) \] (3.4)

which, amongst others, establishes a sign rule for its derivatives.

Now let us give some examples of the measures possessing the Lee–Yang property. First of all, it is anisotropic Gaussian measure, for which one has \[ g_\mu(z) = jz. \] Another example is the above mentioned measure \( \sigma \), concentrated on the unit sphere, for which

\[ f_\sigma(z) = w_{N/2}(z), \quad \Phi_\sigma(z) = w_{N/2}(z^2), \quad w_\theta(z) = \frac{\Gamma(\theta)}{\Gamma(\theta+n)} z^n. \]

The function \( w_\theta(z) \) is a Wright function, it may be written also with the help of the Bessel functions,

\[ w_\theta(z) = \Gamma(\theta) \exp \left( \frac{i\pi}{2} (1-\theta) \right) z^{(1-\theta)/2} J_{\theta-1}(2i\sqrt{z}), \quad (3.5) \]

its order is one half (hence its \( \kappa = 0 \)), and all its zeros are negative. The latter fact may be established either by direct methods given in [2] or on the base of (3.5). For the Gaussian measure, \( g_\mu^{(1)}(0) = \kappa, g_\mu^{(s)}(0) = 0 \) for all \( s \geq 2 \). On the other hand, for the measure \( \sigma \), one has

\[ \frac{|g_\sigma^{(2)}(0)|}{|g_\sigma^{(1)}(0)|^2} = \frac{2}{N+2}. \]

These two examples are, in a sense, extreme, which is stated in our first theorem.

**Theorem 3.1.** For every \( \mu \in \mathcal{M}_N(\mathcal{L}) \) the following bounds hold:

\[ 0 \leq \frac{|g_\mu^{(2)}(0)|}{|g_\mu^{(1)}(0)|^2} \leq \frac{2}{N+2}. \quad (3.6) \]

These bounds are achieved only for the Gaussian measures and the measure \( \sigma \), which is discussed above.

As it follows from [20] and [3], the measure (1.6) possesses the Lee–Yang property for all \( a \in \mathbb{R} \) and for \( N = 1, 2, 3 \). Our next theorem essentially extends this result.

**Theorem 3.2.** Given a function \( \phi : \mathbb{C} \rightarrow \mathbb{C} \), which maps \( \mathbb{R}_+ = [0, +\infty) \) into itself, let there exist \( b \in \mathbb{R}_+ \), such that the derivative \( \phi' \) obeys the condition \( \phi'(z) + b \in \mathcal{L} \). Then, for all \( N \in \mathbb{N} \) and for arbitrary \( h \in \mathcal{L} \), which maps \( \mathbb{R}_+ \) into itself, the probability measure
\[ d\mu(x) = Ch((x,x)) \exp(-\phi((x,x))) \, dx, \quad (3.7) \]

possesses the Lee–Yang property.

This theorem includes all the results of this kind mentioned above. Thus, for 
\( N = 1, \ h \equiv 1, \) and \( \phi(z) = a \cosh(\sqrt{z}), \ a > 0, \) it gives Pólya's result [18]. For 
\( N = 1, 2, 3, \ h \equiv 1, \) and \( \phi(z) = az + z^2, \ a \in \mathbb{R}, \) it gives the results established in 
[20] \((N = 1)\) and in [3] \((N = 2, 3)\).

For the measures \((3.7)\) with \( h \equiv 1, \) we have the following statement.

**Theorem 3.3.** Let \( \phi : \mathbb{C} \to \mathbb{C} \) satisfy the conditions of the previous theorem. Then, for the probability measure

\[ d\mu(x) = C \exp(-\phi((x,x))) \, dx, \quad (3.8) \]

the function \( f_\mu \) \((2.3)\) obeys the following differential equation

\[ (4D\phi(\Lambda_N)f_\mu)(z) = f_\mu(z). \quad (3.9) \]

Here

\[ \Lambda_N = 2ND + 4zD^2, \quad D = \frac{d}{dz} \quad (3.10) \]

and the differential operator \( \phi(\Lambda_N) \) (possibly of infinite order) acts on an entire
function \( f \) as follows

\[ (\phi(\Lambda_N)f)(z) = \sum_{n=0}^{\infty} \frac{1}{n!} \phi^{(n)}(0)(\Lambda_N^n f)(z), \]

where the series converges uniformly on compact subsets of \( \mathbb{C}. \)

For the measures \((3.8)\) with \( \phi(z) = az + z^2, \) we have one more result of the type of \((3.6)\).

**Theorem 3.4.** Let the measure \( \mu \) be given by \((3.8)\) with \( \phi(z) = az + z^2. \) Then, for all \( N \in \mathbb{N}, \) one has the following estimate for the derivatives defined in \((3.3)\),

\[ \frac{|g^{(2)}_\mu(0)|}{|g^{(1)}_\mu(0)|^2} \leq \frac{2}{a^2}. \]

Finally, let us say some words about the proof of these results. Given an entire function \( f \) and a positive \( b, \) we set

\[ \|f\|_b = \sup_{z \in \mathbb{C}} \{|f(z)| \exp(-b|z|)|. \quad (3.11) \]

\[ ^{1} \text{A detailed consideration of infinite order differential operators of this kind is given in [10].} \]
Set also
\[ \mathcal{L}_a \overset{\text{def}}{=} \{ \phi \in \mathcal{L} \ | \ \| \phi \|_b < \infty, \forall b > a \}, \quad a \geq 0 \] (3.12)
and, for \( \theta \geq 0 \),
\[ \Delta_\theta = \theta D + zD^2, \quad D = \frac{d}{dz}. \] (3.13)
Given \( a \geq 0 \), the set \( \mathcal{L}_a \) defined in (3.12) may be equipped in the topology \( \mathcal{T}_a \) generated by the family of norms \( \{ \| \cdot \|_b, \ b > a \} \). For \( \phi, f \in \mathcal{F} \), we set
\[ (\phi(\Delta_\theta)f)(z) = \sum_{n=0}^{\infty} \frac{1}{n!} \phi^{(n)}(0)(\Delta^n_\theta f)(z). \] (3.14)

In [10], we proved the following statement.

**Proposition 3.1.** For every \( \theta \geq 0 \), for any nonnegative numbers \( a \) and \( b \) obeying the condition \( ab < 1 \), and for arbitrary \( \phi \in \mathcal{L}_a \) and \( f \in \mathcal{L}_b \), the function \( \phi(\Delta_\theta)f \) belongs to \( \mathcal{L}_c \) with \( c = b[1 - ab]^{-1} \). The map \( (\phi, f) \mapsto \phi(\Delta_\theta)f \), acting from \( \mathcal{L}_a \times \mathcal{L}_b \) into \( \mathcal{L}_c \), is continuous.

Theorem 3.3 may be proved, roughly speaking, by integration by parts. Comparing \( \Delta_\theta \) with \( \Lambda_N \) given by (3.10), one concludes that, by Proposition 3.1, the operator \( \phi(\Lambda_N) \) in (3.9) preserves the class \( \mathcal{L} \). Then one constructs a sequence of functions
\[ u_n(z) = (4D\phi(\Lambda_N)u_{n-1})(z), \quad n \in \mathbb{N}, \quad u_0 \in \mathcal{L} \]
which converges, uniformly on compact subsets of \( \mathbb{C} \), to \( f_\mu \). This proves Theorem 3.2 for \( h \equiv 1 \). To complete the proof one writes for the measure (3.7)
\[ f_\mu(z) = (h(\Lambda_N)u)(z), \]
where
\[ u((x, x)) = C \int_{\mathbb{R}^N} \exp((x, y) - \phi(y, y)) \, dy \]
which means \( u \in \mathcal{L} \). Since \( h(\Lambda_N) \) preserves the latter set, one has \( f_\mu \in \mathcal{L} \).

**References**