Anti-periodic solutions for fully nonlinear first-order differential equations

Yuqing Chen\textsuperscript{a}, Juan J. Nieto\textsuperscript{b,\textdagger}, Donal O’Regan\textsuperscript{c}

\textsuperscript{a} Faculty of Applied Mathematics, Guangdong University of Technology, Guangzhou, Guangdong 510006, PR China
\textsuperscript{b} Departamento de Análisis Matemático, Facultad de Matemáticas, Universidad de Santiago de Compostela, Santiago de Compostela 15782, Spain
\textsuperscript{c} Department of Mathematics, National University of Ireland, Galway, Ireland

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Abstract

In this paper, we study the anti-periodic boundary value problems for nonlinear first-order differential equations both in finite and in infinite dimensional spaces. Several new existence results are obtained.

Keywords: Anti-periodic solution; Gradient mapping; Subdifferential

1. Introduction

The study of anti-periodic solutions for nonlinear evolution equations is closely related to the study of periodic solutions, and it was initiated by Okochi [1]. During the past fifteen years, anti-periodic problems have been extensively studied by many authors; see [2–20] and references therein. For example anti-periodic trigonometric polynomials are important in the study of interpolation problems [21,22], and anti-periodic wavelets are discussed in [23]. Recently, anti-periodic boundary conditions have been considered for the Schrödinger and Hill differential operator [24,25]. Also anti-periodic boundary conditions appear in the study of difference equations [26,27]. Moreover, anti-periodic boundary conditions appear in physics in a variety of situations; see [28–31]. In this paper, we first consider anti-periodic solutions of the following fully nonlinear equation:

\[
\begin{align*}
F(t, u(t), u'(t)) &= 0, \quad t \in \mathbb{R}, \\
u(t) &= -u(t + T), \quad t \in \mathbb{R}
\end{align*}
\]  \quad (E 1.1)

where \( F : \mathbb{R}^3 \rightarrow \mathbb{R} \) is a continuous function. We introduce the concept of an anti-periodic viscosity solution, and then we prove a Massera-type theorem for the existence of an anti-periodic solutions of (E 1.1). (For periodic viscosity

\textdagger Corresponding author. Tel.: +34 981 563 100; fax: +34 981 597 054.
E-mail addresses: yqchen@foshan.net (Y. Chen), amnieto@usc.es (J.J. Nieto), donal.oregan@nuigalway.ie (D. O’Regan).

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solutions, we refer the reader to [32]). Then we consider the following anti-periodic boundary value problem:

\[
\begin{align*}
    u'(t) + \partial G(u(t) + f(t, u(t)) &= 0, \quad \text{a.e. } t \in [0, T], \\
    u(0) &= -u(T),
\end{align*}
\]  

(E 1.2)

where \( G : R^n \to R^n \) is an even continuously differentiable function, and \( f : [0, T] \times R^n \to R^n \) is a Caratheodory function, i.e. \( f \) satisfies

1. for every \( x \in R^n \), \( f(\cdot, x) \) is Lebesgue measurable on \( t \);
2. for a.e. \( t \in [0, T] \), \( f(t, \cdot) \) is continuous on \( R^n \).

By imposing a suitable growth condition on \( f \), we prove an existence result for (E 1.2). Finally with \( H \) a real Hilbert space we consider the problem

\[
\begin{align*}
    u'(t) &\in -\partial \phi(u(t)) + \partial G(u(t)) + f(t), \quad \text{a.e. } t \in [0, T], \\
    u(0) &= -u(T)
\end{align*}
\]  

(E 1.3)

where \( \phi : D(\phi) \subseteq H \to R \cup \{+\infty\} \) is a proper lower semi-continuous convex function, \( G : H \to H \) is a continuously differentiable mapping such that \( \partial G \) is a bounded mapping, i.e. \( \partial G \) maps bounded subsets to bounded subsets and \( f(\cdot) \in L^2([0, T]; H) \). Under a compact condition on the level set \( \{x : \phi(x) \leq \alpha\} \), where \( \alpha > 0 \), we prove an existence result for (E 1.3).

2. Anti-periodic viscosity solutions for first-order fully nonlinear equations

In this section, we study the existence problem (E 1.1). We will use the following concept introduced by Crandall and Lions [33]. Let \( \Omega \subset R^n \) be an open subset and let \( F : \Omega \times R \times R^n \to R \) be a continuous function. Consider the following equation:

\[ F(x, u(x), Du(x)) = 0, \quad x \in \Omega, \]

where \( x = (x_1, x_2, \ldots, x_n) \), and \( Du(x) = (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n}) \).

**Definition 2.1** ([33,34]). A continuous function \( u(\cdot) \in C(\Omega) \) is said to be a viscosity solution of equation \( F(x, u(x), Du(x)) = 0 \) if it satisfies the following conditions:

1. \( u(\cdot) \) is a viscosity subsolution of equation \( F(x, u(x), Du(x)) = 0 \), i.e.
   \[
   F(x, u(x), p) \leq 0, \quad \text{for all } x \in \Omega, \ p \in D^+ u(x); \text{ here}
   \]
   \[
   D^+ u(x) = \left\{ p \in R^n : \limsup_{y \to x, y \in \Omega} \frac{u(y) - u(x) - p(y - x)}{|y - x|} \leq 0 \right\}
   \]

2. \( u(\cdot) \) is a viscosity supersolution of \( F(x, u(x), Du(x)) = 0 \), i.e.
   \[
   F(x, u(x), p) \geq 0, \quad \text{for all } x \in \Omega, \ p \in D^- u(x); \text{ here}
   \]
   \[
   D^- u(x) = \left\{ p \in R^n : \liminf_{y \to x, y \in \Omega} \frac{u(y) - u(x) - p(y - x)}{|y - x|} \geq 0 \right\}
   \]

**Remark.** Let \( u(\cdot) \in C^1(\Omega) \) be a continuous function. Then:

1. \( p \in D^+ u(x) \) if and only if there exists a function \( \phi \in C^1(\Omega) \) such that \( u - \phi \) achieves a maximum at \( x \) and \( p = D\phi(x) \).
2. \( p \in D^- u(x) \) if and only if there exists a function \( \phi \in C^1(\Omega) \) such that \( u - \phi \) achieves a minimum at \( x \) and \( p = D\phi(x) \).

**Remark.** For viscosity solutions of fully nonlinear equations, one may see [33–35] for more references.

We use the concept of a viscosity solution and introduce the concept of anti-periodic viscosity solutions.
Definition 2.2. Let \( u(\cdot) \in C(R) \) satisfy \( u(t) = -u(t + T) \) for all \( t \in R \). If \( u \) is a viscosity subsolution of \( F(t, u(t), u'(t)) = 0, t \in R \), then \( u(\cdot) \) is said to be an anti-periodic viscosity subsolution, and similarly, if \( u(\cdot) \) is a viscosity supersolution of \( F(t, u(t), u'(t)) = 0, t \in R \), we call \( u(\cdot) \) an anti-periodic viscosity supersolution. If \( u(\cdot) \) is both an anti-periodic subsolution and an anti-periodic supersolution, then \( u(\cdot) \) is said to be an anti-periodic viscosity solution of (E 1.1).

Definition 2.3. We say \( u(\cdot) \in C(R) \) is a viscosity solution of

\[
\begin{align*}
F(t, u(t), u'(t)) &= 0, \quad \forall t \in R, \\
u(t_0) &= u_0, \quad u_0 \in R.
\end{align*}
\]  

(E 2.1)

if \( u(t_0) = u_0 \), and \( u(\cdot) \) is a viscosity solution of \( F(t, u(t), u'(t)) = 0, t \in R \).

The following result extends the classical Massera theorem on the existence of a periodic solution to viscosity problems with anti-periodicity (see [6]).

Theorem 2.4. Let \( F(t, x, y) : R \times R^2 \to R \) be a continuous function satisfying \( F(t + T, -x, -y) = -F(t, x, y) \) for all \((t, x, y) \in R \times R^2\). Suppose that Eq. (E 2.1) has a unique viscosity solution which depends continuously on the initial value \( u_0 \). If also there exists a bounded and uniformly continuous viscosity solution to (E 2.1), then

\[
\begin{align*}
F(t, u(t), u'(t)) &= 0, \quad t \in R, \\
u(t) &= -u(t + T), \quad t \in R.
\end{align*}
\]  

(E 2.2)

has an anti-periodic viscosity solution.

Proof. Let \( y(\cdot) \) be a bounded and uniformly continuous viscosity solution of (E 2.1). We may also assume that \( y(0) \neq -y(T) \).

Set \( u_{2k}(t) = y(t + 2kT) \), and \( u_{2k+1}(t) = -y(t + (2k + 1)T) \), \( k = 1, 2, \ldots \). We claim that \( u_{2k}(t) \) is a viscosity solution of (E 2.1) with initial value \( u_{2k}(0) = y(2kT) \), and \( u_{2k+1}(t) \) is a viscosity solution of (E 2.1) with initial value \( u_{2k+1}(0) = -y((2k + 1)T) \) for \( k = 1, 2, \ldots \).

Let \( \phi \in C^1(R) \) be such that \( u_{2k}(t) - \phi(t) \) achieves a local maximum at \( t_0 \), i.e. \( y(t) - \phi(t - 2kT) \) achieves a local maximum at \( t_0 + 2kT \). Since \( y(\cdot) \) is a viscosity solution of (E 2.1), we have

\[
F(t_0 + 2kT, y(t_0 + 2kT), \phi'(t_0)) \leq 0.
\]

This and \( F(t + 2kT, x, y) = F(t, x, y) \) for all \((t, x, y) \in R \times R^2\) imply that

\[
F(t_0, u_{2k}(t_0), \phi'(t_0)) \leq 0.
\]

Hence \( u_{2k}(t) \) is a viscosity subsolution of \( F(t, u(t), u'(t)) = 0, t \in R \). Similarly, we can show that \( u_{2k}(t) \) is a viscosity supersolution of \( F(t, u(t), u'(t)) = 0, t \in R \). Therefore, \( u_{2k}(t) \) is a viscosity solution of \( F(t, u(t), u'(t)) = 0, t \in R \).

Let \( \phi \in C^1(R) \) be such that \( u_{2k+1}(t) - \phi(t) \) achieves a local maximum at \( t_0 \), i.e. \( -y(t) - \phi(t - (2k + 1)T) \) achieves a local minimum at \( t_0 + (2k + 1)T \). Thus \( y(t) + \phi(t - (2k + 1)T) \) achieves a local maximum at \( t_0 + (2k + 1)T \). Since \( y(\cdot) \) is a viscosity solution of (E 2.1), we have

\[
F(t_0 + (2k + 1)T, y(t_0 + (2k + 1)T), -\phi'(t_0)) \geq 0.
\]

Now using the assumption \( F(t + T, -y, -z) = -F(t, y, z) \) we get

\[
F(t_0, u_{2k+1}(t_0), \phi'(t_0)) \leq 0.
\]

Hence \( u_{2k+1}(t) \) is a viscosity subsolution of \( F(t, u(t), u'(t)) = 0, t \in R \). Similarly, we can show that \( u_{2k+1}(t) \) is a viscosity supersolution of \( F(t, u(t), u'(t)) = 0, t \in R \).

We may assume \( y(T) \leq y(3T) \). (The proof is similar if \( y(T) > y(3T) \).) Then it follows from the uniqueness that

\[
u_1(t) \geq u_3(t) \geq u_5(t) \geq \cdots \geq u_{2k+1}(t) \geq \cdots.
\]

Thus \( x_1(t) = \lim_{k \to \infty} u_{2k+1}(t) \) exists. Similarly \( x_2(t) = \lim_{k \to \infty} u_{2k}(t) \) exists.

Next, we show that \( x_1(t), x_2(t) \) are viscosity solutions of \( F(t, u(t), u'(t)) = 0, t \in R \). Let \( \psi \in C^1(R) \) be such that \( x_1(t) - \psi(t) \) attains a strict local maximum at \( t_0 \). (One may replace \( \psi(t) \) by \( \psi(t) + (t - t_0)^2 \) if it is not strict, as
noted in [33].) Let \( t_k \) be the point where \( u_{2k+1}(t) - \psi(t) \) attains its maximum in a neighbourhood around \( t_0 \). We may assume \( t_k \to t_1 \) as \( k \to \infty \). From the definition of \( u_{2k+1}(t) \), we have

\[
|u_{2k+1}(t_k) - x_1(t_1)| \leq | - y((2k + 1)T + t_k) + y((2k + 1)T + t_1)| + | - y((2k + 1)T + t_1) - x_1(t_1)|.
\]

Thus we have \( x_1(t_1) = \lim_{k \to \infty} u_{2k+1}(t_k) \) by the uniform continuity of \( y(\cdot) \). Hence \( t_1 = t_0 \). Now since \( u_{2k+1}(t) \) is a viscosity subsolution, we have

\[
F(t_k, u_{2k+1}(t_k), \psi'(t_k)) \leq 0, \quad n = 1, 2, \ldots
\]

By letting \( k \to \infty \), we get

\[
F(t_0, x_1(t_0), \psi'(t_0)) \leq 0.
\]

Therefore \( x_1(t) \) is a viscosity subsolution. Similarly, we can show that \( x_1(t) \) is also a viscosity supersolution. Therefore \( x_1(t) \) is a viscosity solution of \( F(t, u(t), u'(t)) = 0, t \in R \). Similarly one can prove that \( x_2(t) \) is a viscosity solution of \( F(t, u(t), u'(t)) = 0, t \in R \). Moreover \( x_1(t), x_2(t) \) are \( 2T \) periodic. If \( x_1(0) = -x_1(T) \), then \( x_1(t) \) is an anti-periodic viscosity solution of (E.2.2). As a result we may assume \( x_1(0) \neq -x_1(T) \). Now we define a mapping \( K : R \to R \) by

\[
Kz = z + y(T, z), \quad \forall z \in R,
\]

where \( y(t, z) \) is the unique viscosity solution of (E.2.1) with initial value \( z \). From the assumptions in the statement of Theorem 2.4, we know that \( K : R \to R \) is a continuous mapping. Clearly \( Kx(0) = x(0) + y(T, x(0)) = x(0) + x_1(T) \). Since \( x_1(0) \neq x_1(T) \), we may assume \( x_1(0) + x_1(T) > 0 \). From the definition of \( x_1(t) \) and \( x_2(t) \), it follows that \( x_1(t + T) = -x_2(t) \). Thus we have \( K(-x_1(T)) = -x_1(T) - x_1(2T) = -x_1(T) - x_1(0) < 0 \). Hence there exists \( z_0 \) between \( x_1(0) \) and \( -x_1(T) \) such that

\[
Kz_0 = 0.
\]

That is, \( z_0 = -y(T, z_0) \). Thus \( y(t, z_0) \) is an anti-periodic viscosity solution of (E.2.2).

The proof is complete. \( \square \)

3. Anti-periodic boundary value problems in \( R^n \)

In this section, let \( (\cdot, \cdot) \) be the inner product in \( R^n \), and \( |\cdot| \) the norm in \( R^n \). We prove an existence result for (E.1.2).

**Theorem 3.1.** Let \( G : R^n \to R^n \) be an even continuously differentiable function, and let \( f : [0, T] \times R^n \to R^n \) be a Caratheodory function. Suppose the following conditions are satisfied:

1. \( |f(t, x)| \leq M|x| + g(t) \) for a.e. \( (t, x) \in [0, T] \times R^n \), where \( M > 0 \) is a constant, \( g(\cdot) \in L^2(0, T) \);
2. \( MT < 2 \).

Then

\[
\begin{cases}
u'(t) + \partial G(u(t) + f(t, u(t))) = 0, & \text{a.e. } t \in [0, T], \\
u(0) = -u(T),
\end{cases}
\]

has a solution \( u(\cdot) \in C([0, T]; R^n) \) with \( u'(\cdot) \in L^2([0, T]; R^n) \).

**Proof.** Put \( C_a = \{v(\cdot) \in C([0, T]; R^n) : v(0) = -v(T)\} \). Then \( C_a \) is a Banach space under the norm \( |v(\cdot)|_\infty = \max_{t \in [0, T]} |u(t)| \). For each \( v(\cdot) \in C_a \), consider the following equation:

\[
\begin{cases}u'(t) + \partial G(u(t) + f(t, v(t))) = 0, & \text{a.e. } t \in [0, T], \\
u(0) = -u(T).
\end{cases}
\]

It is easy to see that \( u(t) = -\int_0^T [G(v(s) + f(s, v(s))) \, ds + \frac{1}{2} \int_0^T |G(v(s) + f(s, v(s))) \, ds \] is the unique solution of (E.3.2).

We define a mapping \( K : C_a \to C_a \) as follows:

\[
Kv(\cdot) = u(\cdot), \quad v(\cdot) \in C_a, u(\cdot) \text{ is the solution of (E.2.2)}.
\]
First we prove that $K$ is a continuous compact mapping. Now assume $v_n(\cdot) \in C_d$, $n = 1, 2, \ldots$, and $v_n(\cdot) \to v(\cdot) \in C_d$; then $|\partial G v_n(\cdot) - \partial G v(\cdot)| \to 0$ as $n \to \infty$ since $\partial G$ is continuous. Now assumption (1) and Lebesgue's dominated convergence theorem guarantee that $f(\cdot, v_n(\cdot)) \to f(\cdot, v(\cdot))$ in $L^2([0, T]; R^n)$ as $n \to \infty$.

Since $(K v_n(t))' - (K v(t))' \to \partial G v(t)$ + $\partial G v(t) + f(t, v_n(t)) - f(t, v(t)) = 0$, a.e. $t \in [0, T]$, if we multiply both sides by $(K v_n(t))' - (K v(t))'$ and integrate over $[0, T]$, we get

$$\int_0^T |(K v_n(t))' - (K v(t))'|^2 dt + \int_0^T (\partial G v_n(t) - \partial G v(t), (K v_n(t))' - (K v(t))') dt + \int_0^T (f(t, v_n(t)) - f(t, v(t)), (K v_n(t))' - (K v(t))') dt = 0.$$ 

Thus

$$\left(\int_0^T |(K v_n(t))' - (K v(t))'|^2 dt \right)^{1/2} \leq \sqrt{T} |\partial G v_n(\cdot) - \partial G v(\cdot)| + \left(\int_0^T |f(\cdot, v_n(\cdot)) - f(\cdot, v(\cdot))|^2 dt \right)^{1/2},$$

and therefore $\int_0^T |(K v_n(t))' - (K v(t))'|^2 dt \to 0$ as $n \to \infty$.

We have $K v_n(t) - K v(t) = \frac{1}{2} [\int_0^T (K v_n(s))' - (K v(s))' ds - \int_0^T (K v_n(s))' - (K v(s))' ds]$, and so $K v_n(\cdot) \to K v(\cdot)$ in $C_d$. Now since $(K v(t))' + \partial G v(t) + f(t, v(t)) = 0$, a.e. $t \in [0, T]$, it is easy to see that

$$\left(\int_0^T |(K v(t))'|^2 dt \right)^{1/2} \leq \sqrt{T} |\partial G v(\cdot)| + M \sqrt{T} |v(\cdot)| + \left(\int_0^T |g(t)|^2 dt \right)^{1/2}.$$

Thus $K$ maps a bounded subset of $C_d$ to a bounded equicontinuous subset in $C_d$, so therefore $K$ is compact.

Next take $r_0 > (1 - MT)^{-1} \frac{\sqrt{T}}{2} \left(\int_0^T |g(t)|^2 dt \right)^{1/2}$. We show that $K v(\cdot) \neq \lambda v(\cdot)$ for all $\lambda \geq 1$, and $|v(\cdot)|_\infty = r_0$. If this is not true, there exist $\lambda_0 \geq 1$, $v_0(\cdot) \in C_d$ with $|v_0(\cdot)|_\infty = r_0$ such that $K v_0(\cdot) = \lambda v_0(\cdot)$, i.e. $v_0(0) = v_0(-T)$ and

$$\lambda v_0'(t) + \partial G v_0(t) + f(t, v_0(t)) = 0, \quad \text{a.e. } t \in [0, T]. \quad (3.1)$$

Multiply (3.1) by $v_0'(t)$ and integrate over $[0, T]$, and note that $\int_0^T (\partial G v_0(t), v_0'(t)) dt = 0$ to get

$$\lambda_0 \left(\int_0^T |v_0'(t)|^2 dt \right)^{1/2} \leq M \sqrt{T} |v_0(\cdot)| + \left(\int_0^T |g(t)|^2 dt \right)^{1/2}. \quad (3.2)$$

Notice that $v_0(t) = \frac{1}{2} [\int_0^T v_0'(s) ds - \int_0^T v_0'(s)] ds$, so we have

$$\left| v_0(\cdot) \right|_\infty \leq \frac{\sqrt{T}}{2} \left(\int_0^T |v_0'(t)|^2 dt \right)^{1/2}. \quad (3.3)$$

From (3.2) and (3.3), we have

$$\lambda_0 \left| v_0(\cdot) \right|_\infty \leq \frac{MT}{2} \left| v_0(\cdot) \right|_\infty + \frac{\sqrt{T}}{2} \left(\int_0^T |g(t)|^2 dt \right)^{1/2}. \quad (3.4)$$

From assumption (2), we get

$$\left| v_0(\cdot) \right|_\infty \leq \left(1 - \frac{MT}{2} \right)^{-1} \frac{\sqrt{T}}{2} \left(\int_0^T |g(t)|^2 dt \right)^{1/2},$$

which contradicts $|v_0(\cdot)|_\infty = r_0$.

Thus the Leray–Schauder degree $\text{deg}(I - K, B(0, r_0), 0) = 0$, where $B(0, r_0)$ is the open ball centered at 0 with radius $r_0$ in $C_d$. Consequently, $K$ has a fixed point in $B(0, r_0)$, i.e. (E 3.1) has a solution. □
4. Anti-periodic boundary value problem in Hilbert spaces

In this section, $H$ is a real Hilbert space, $(\cdot, \cdot)$ is the inner product of $H$, and the norm of $H$ is denoted by $\| \cdot \|$. Let $C([0, T]; H)$ be all the continuous functions from $[0, T]$ to $H$ with the max norm. Also $L^2([0, T]; H) = \{ f(t) : [0, T] \rightarrow H; f(t) = 0 \text{ } \int_0^T \| f(s) \|^2 \, ds < +\infty \}$, and the norm in $L^2([0, T]; H)$ is denoted by $\| f(\cdot) \|_{L^2} = (\int_0^T \| f(s) \|^2 \, ds)^{\frac{1}{2}}$.

We let $C_a = \{ v(\cdot) \in C([0, T]; H) : v(0) = -v(T) \}$, and $W_a = \{ u(\cdot) \in C_a : u(\cdot) \in L^2([0, T]; H) \}$. $C_a$ is a Banach space under the norm $|v(\cdot)|_\infty = \max_{t \in [0, T]} \| u(t) \|$. 

**Lemma 4.1** ([5]). Let $\phi : D(\phi) \subseteq H \rightarrow R \cup \{+\infty\}$ be an even proper lower semi-continuous convex function. Then $A : D(A) \subseteq L^2([0, T]; H) \rightarrow L^2([0, T]; H)$ defined by

$$ Au(t) = u'(t) + \partial \phi(u(t)), \quad u(\cdot) \in D(A) $$

is a surjective maximal monotone mapping, where $D(A) = W_a$.

**Theorem 4.2.** Let $\phi : D(\phi) \subseteq H \rightarrow R \cup \{+\infty\}$ be an even proper lower semi-continuous convex function; $G : H \rightarrow R$ is a continuously differentiable function such that $\partial G$ is a bounded mapping, i.e. $\partial G$ maps bounded subsets to bounded subsets in $H$, and $f(\cdot) \in L^2([0, T]; H)$. Suppose for each $\alpha > 0$, $\{ x : \phi(x) \leq \alpha \}$ is compact in $H$. Then the following anti-periodic boundary value problem:

$$ \begin{align*}
  u'(t) &\in -\partial \phi(u(t)) + \partial G(u(t)) + f(t), \quad \text{a.e. } t \in [0, T], \\
  u(0) &= -u(T).
\end{align*} $$

(E 4.1)

has a solution $u(\cdot) \in W_a$.

**Proof.** First, if $u(\cdot) \in C([0, T]; H)$ with $u'(\cdot) \in L^2([0, T]; H)$ is a solution to (E 4.1), then

$$ \| u'(\cdot) \|_{L^2} \leq \| f(\cdot) \|_{L^2}, \quad \text{and} \quad |u(\cdot)|_\infty \leq \frac{\sqrt{T}}{2} \| f(\cdot) \|_{L^2}. $$

Let $M = \frac{\sqrt{T}}{2} \| f(\cdot) \|_{L^2}$. Since the norm of $H$ is differentiable, we take an even continuously differentiable function $\psi : H \rightarrow \tilde{R}$ such that $\psi(x) = 1$ for $\| x \| \leq M$, and $\partial \psi(x) = 0$ for $x \geq 2M$ with $\psi, \partial \psi$ uniformly bounded on $H$. For each $v(\cdot) \in C_a$, we consider the following anti-periodic boundary value problem:

$$ \begin{align*}
  u'(t) + 2u(t) &\in -\partial \phi(u(t)) + \partial \psi(u(t))[G(u(t)) + \| v(t) \|^2] + f(t), \quad \text{a.e. } t \in [0, T], \\
  u(0) &= -u(T).
\end{align*} $$

(E 4.2)

By Lemma 4.1, (E 4.2) has a unique solution in $W_a$, and we denote it by $Kv(\cdot)$. From the uniform boundedness of $\psi, \partial \psi$ and the boundedness of $\partial G$, we may assume that $\| \partial \psi(v)(G(v) + \| v \|^2) \| \leq L$ for all $v \in H$, where $L > 0$ is a constant.

Multiply both sides of the first equality of (E 4.2) by $(Kv)'(t)$ and integrate over $[0, T]$ to get

$$ \| (Kv)'(\cdot) \|_{L^2} \leq L\sqrt{T} + \| f(\cdot) \|_{L^2}. $$

(4.1)

Thus there exists $N > 0$ such that for all $v(\cdot) \in C_a$,

$$ |Kv(\cdot)|_\infty \leq N. $$

(4.2)

Now (E 4.2), (4.1) and (4.2) yield

$$ \| \partial \phi(Kv(\cdot)) \|_{L^2} \leq r $$

(4.3)

for some $r > 0$.

Notice that $\phi(Kv(t)) \leq (\partial \phi(Kv(t)), Kv(t))$, a.e. $t \in [0, T]$, and $\phi$ is an even function, so we get from (4.2) and (4.3) that

$$ \int_0^T |\phi(Kv(t))| \, dt \leq N\sqrt{T}r. $$

(4.4)
Since \( \frac{d}{dt} \phi(Kv(t)) = \langle (Kv)'(t), \partial \phi(Kv(t)) \rangle \), a.e. \( t \in [0, T] \), we conclude from (4.1) and (4.3) that
\[
\int_0^T \left| \frac{d}{dt} \phi(Kv(t)) \right| \leq r [L \sqrt{T} + \| f(\cdot) \|_{L^2}].
\] (4.5)

From (4.4) and (4.5), we know that there exists a constant \( \alpha_0 > 0 \) such that
\[
|\phi(Kv(t))| \leq \alpha_0, \quad t \in [0, T].
\] (4.6)

Now from our assumption that for each \( \alpha > 0 \), \( \{ x : \phi(x) \leq \alpha \} \) is compact in \( H \), and (4.6), we know that there exists a compact subset \( D \in H \) such that \( Kv(t) \in D \) for all \( t \in [0, T] \), \( v(\cdot) \in C_a \). Also from (4.1), we know that \( \{ Kv(\cdot) : v(\cdot) \in C_a \} \) is equicontinuous. Thus \( \{ Kv(\cdot) : v(\cdot) \in C_a \} \) is relatively compact in \( C_a \).

Finally, we prove that \( K : C_a \to C_a \) is continuous. Let \( v_n(\cdot) \to v(\cdot) \in C_a \) as \( n \to \infty \). (Consequently, \( v_n(\cdot) \to v(\cdot) \in L^2([0, T]; H) \).) By the above argument, \( Kv_n(t) \in D \) for \( t \in [0, T] \). Thus \( \{ Kv_n(\cdot) \}_{n=1}^\infty \) has a subsequence \( Kv_{n_k}(\cdot) \to u(\cdot) \) in \( C_a \). Thus \( Kv_{n_k}(\cdot) \to u(\cdot) \in L^2([0, T]; H) \). From Lemma 4.1, we know that \( u(\cdot) = Kv(\cdot) \). Therefore \( K \) is continuous. Schauder’s fixed point theorem guarantees that \( K \) has a fixed point in \( C_a \), which is easily seen to be a solution of (E 4.1). \( \square \)

**Remark.** Theorem 4.2 was proved in [5] with the assumption that \( \| \partial G(u) \| \leq k(\| u \| + 1) \), \( u \in H \) for some \( k > 0 \). (See Lemma 3.7 in [5].) The example \( G(x) = e^{x^2} \) in \( R \) shows that \( \partial G \) is continuous but does not satisfy this condition.

**Corollary 4.3.** Let \( \beta(x) = |x| \) for \( x \in R \), and \( f(\cdot) \in L^2([0, T], R) \). Then the following boundary value problem:
\[
\begin{align*}
  u'(t) & \in -\beta(u(t)) + 2u(t)e^{u^2(t)} + f(t), & \text{a.e. } t \in [0, T], \\
  u(0) & = -u(T),
\end{align*}
\] (E 4.3)

has a solution.

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