A LIFT OF THE SCHUR’S Q-FUNCTIONS TO THE PEAK ALGEBRA

NAIHUAN JING AND YUNNAN LI

Abstract. We construct a lift of the Schur’s Q-functions to the peak algebra of the symmetric group, called the non-commutative Schur’s Q-functions, and extract from them a new natural basis with many nice properties such as the positive right-Pieri rule, combinatorial expansion, etc. Dually, we get a basis of the Stembridge algebra of peak functions refining the Schur’s P-functions in a simple way.

Keywords: peak algebra, combinatorial Hopf algebras, noncommutative Schur’s Q-function, quasisymmetric Schur’s P-function

1. Introduction

The algebra of noncommutative symmetric functions (abbreviated as NSym) is the noncommutative lifting of that of symmetric functions (abbreviated as Sym) studied first in [12]. It is proved in [18] that the graded Hopf dual of NSym is the quasisymmetric functions (abbreviated as QSym), and NSym is isomorphic as Hopf algebras to the Solomon descent algebra of the symmetric group. As an important nonsymmetric generalization of Sym, QSym was introduced by Gessel as a source of generating functions for \( P \)-partitions [11]. After that, Stembridge developed this theory further to define the peak quasisymmetric functions as the weight enumerators of all enriched \( P \)-partitions of chains, which refine the classical Schur’s Q-functions [20]. The Stembridge algebra \( B \) of peak functions has been widely studied and found close relations to many topics in combinatorics, geometry and representation theory, including Eulerian enumeration [8], Schubert calculus [6, 10] and Kazhdan-Lusztig theory [5], etc.

On the other hand, the relations among these combinatorial Hopf algebras above are quite clear. The peak algebra \( \mathcal{P} \) of the symmetric group is naturally embedded into NSym, with the graded Hopf dual isomorphic to the Stembridge algebra \( B \). Meanwhile, \( \mathcal{P} \) can also serve as a Hopf quotient of NSym via the \((1-t)\)-transform at \( t = -1 \) introduced in [16], so is the case for \( B \) in QSym by duality. Moreover, it is shown in [11] that \( \mathcal{P} \) is the terminal object in the category of combinatorial Hopf algebras satisfying generalized Dehn-Sommerville relations (2.3). Such significant relation was first derived in [2] from the flag \( f \)-vectors of a ranked Eulerian poset by considering its M"obius function.

The main result of this paper is to find a noncommutative lifting of the Schur’s Q-functions in the peak algebra. We call them the noncommutative Schur’s Q-functions (abbreviated as NSQF). They are derived by a creation operator construction lifting the vertex...
operator defined by the first author in [14] to realize the Schur’s Q-functions. Under
the forgetful map \( \pi \) from \( \text{NSym} \) to \( \text{Sym} \), their image is the raising operator expression of the
Schur’s Q-functions. Such method has been applied by Berg et al. in [3] to construct a
noncommutative lift of Schur functions, called the immaculate basis, and also modified
Hall-Littlewood functions. However, we emphasize that the results in [3] can not specialize
at \( t = -1 \) to recover ours, just like the case for the vertex operators defined by the first author
in [15] and those in [14].

The innovation point of our work is that we can extract a new and quite natural basis
in the peak algebra \( \mathcal{P} \) indexed by the so-called peak compositions from the NSQF’s, which
has a positive right-Pieri rule (4.1). Note that the peak composition set naturally contains all
strict partitions, which parameterize the Schur’s Q-functions. Furthermore, in contrast with
the anti-symmetric relations satisfied by the Schur’s Q-functions, the NSQF’s obey more
subtle relations, which are still mysterious to us.

Dually, we find a new basis in the Stembridge algebra \( \mathcal{B} \), called the quasisymmetric
Schur’s Q-functions (abbreviated as QSQF), since they also nicely refine the Schur’s Q-
functions as the peak functions do (see (4.5), (4.6)). Moreover, several simple examples
convince us that there exists a positive, integral and unitriangular expansion of our QSQF’s
in peak functions (Conjecture 4.14), which in turn implies a positive expansion in monomial
or fundamental quasisymmetric functions (Prop. 4.15). It is also worthy to mention that
other interesting bases for \( \mathcal{B} \), and this result was applied to obtain a simpler and more explicit
combinatorial formula for the Kazhdan-Lusztig polynomials of a Coxeter group \( W \).

In [7], Bergeron et al. provided the peak algebra \( \mathcal{P} \) and its dual \( \mathcal{B} \) a representation
theoretical interpretation as Grothendieck rings of the tower of Hecke-Clifford algebras at
\( q = 0 \). In particular, the peak functions are realized as certain characters of simple super-
modules. Hence, if Conjecture 4.14 holds, then our QSQF’s may also have a nice character
realization for some certain modules. We note that similar work has been successively done
for the dual immaculate basis due to Berg et al. in [4].

The organization of the paper is as follows. In §2 we provide some notation, defini-
tions and mutual relations for all combinatorial Hopf algebras that we discuss, including
\( \text{NSym} \), \( \text{QSym} \), \( \text{Sym} \), \( \mathcal{P} \) and \( \mathcal{B} \). In §3 we lift the vertex operator realization of the Schur’s Q-
functions to the noncommutative level based on the \((1-t)\)-transform \( Q_r \)’s at \( t = -1 \). Then
we obtain the raising operator expression (3.2), a key relation (3.4) and a positive right-
Pieri rule (3.7) for the NSQF’s. In §4 we find a natural basis for the peak algebra from the
NSQF’s. A reformulated right-Pieri rule (4.1) and also a simple combinatorial expression
of the \( Q_\alpha \)’s in terms of such basis are given. In the last section, we obtain the dual basis
QSQF for the Stembridge algebra. For representation theoretical consideration, the positive
expansion of QSQF’s in terms of the peak functions is a further problem one would like to
solve (Conjecture 4.14).
2. Background

2.1. Notation and definitions. Denote by \( \mathbb{N} \) (resp. \( \mathbb{N}_0 \)) the set of positive (resp. nonnegative) integers. Given any \( m, n \in \mathbb{N} \), let \( [m,n] := \{m, m+1, \ldots, n\} \) and \([n] := \{1, n\}\) for short. Let \( \mathcal{C}(n) \) be the set of compositions of \( n \), consisting of ordered tuples of positive integers summed up to \( n \) and \( \mathcal{C} = \bigcup_{n \geq 1} \mathcal{C}(n) \). Write \( \alpha \nleq n \) when \( \alpha \in \mathcal{C}(n) \). Given \( \alpha = (\alpha_1, \ldots, \alpha_r) \nleq n \), let its length \( \ell(\alpha) := r \) and define its associated descent set as

\[
D(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_{r-1}\} \subseteq [n-1].
\]

Also, the refining order \( \leq \) on \( \mathcal{C}(n) \) is defined by

\( \alpha \leq \beta \) if and only if \( D(\beta) \subseteq D(\alpha) \), \( \forall \alpha, \beta \nleq n \).

In general, for \( \alpha \in \mathbb{N}_0^r \), let \( \ell(\alpha) = |\{i : \alpha_i > 0\}| \).

We highlight the subset \( \mathcal{C}_o(n) \) of \( \mathcal{C}(n) \), consisting of compositions of \( n \) with odd parts, and write \( \alpha \nleq o \) for \( \alpha \in \mathcal{C}_o(n) \). It is well-known that

\[
|\mathcal{C}_o(n)| = f_{n-1},
\]

where \( \{f_n\}_{n \geq 0} \) is the Fibonacci sequence defined recursively by

\[
f_0 = f_1 = 1, \ f_n = f_{n-1} + f_{n-2}, \ n \geq 2.
\]

Now fix an algebraically closed field \( \mathbb{K} \) of characteristic 0. Given the alphabet \( A = \{a_1, a_2, \ldots\} \), one has the free associative algebra \( \mathcal{F} = \mathbb{K}((a_1, a_2, \ldots)) \). Similarly define the following generating functions in \( \mathcal{F}[[t]] \),

\[
H(A, z) = \sum_{n \geq 0} H_n(A) z^n = \prod_{i \geq 1} \frac{1}{1 - a_i z}
\]

and

\[
E(A, z) = \sum_{n \geq 0} E_n(A) z^n = \prod_{i \geq 1} (1 + a_i z).
\]

Then \( \{H_n(A)\}_{n \geq 0} \) (or \( \{E_n(A)\}_{n \geq 0} \)) generates a subalgebra of \( \mathcal{F} \), called the algebra of noncommutative symmetric functions and denoted by \( \text{NSym} \). Let

\[
H_\alpha = H_{\alpha_1} \cdots H_{\alpha_r}, \ E_\alpha = E_{\alpha_1} \cdots E_{\alpha_r}, \ \alpha = (\alpha_1, \ldots, \alpha_r) \nleq n.
\]

Both \( \{H_\alpha\}_{\alpha \nleq n} \) and \( \{E_\alpha\}_{\alpha \nleq n} \) are \( \mathbb{Z} \)-bases of \( \text{NSym}_n \), called the noncommutative complete and elementary symmetric functions respectively. There exists another important \( \mathbb{Z} \)-basis \( \{R_\alpha\}_{\alpha \nleq n} \) of \( \text{NSym}_n \), called the noncommutative ribbon Schur functions and defined by

\[
R_\alpha = \sum_{\beta \nleq \alpha} (-1)^{\ell(\beta) - \ell(\alpha)} H_\beta.
\]

Let

\[
Q(A, z) = \sum_{n \geq 0} Q_n(A) z^n = E(A, z) H(A, z) = \prod_{i \geq 1} (1 + a_i z) \prod_{i \geq 1} \frac{1}{1 - a_i z}.
\]

\[
\begin{align*}
(2.1) & \quad H(A, z) = \sum_{n \geq 0} H_n(A) z^n = \prod_{i \geq 1} \frac{1}{1 - a_i z} \\
(2.2) & \quad E(A, z) = \sum_{n \geq 0} E_n(A) z^n = \prod_{i \geq 1} (1 + a_i z).
\end{align*}
\]
When $n$ is even, we have

**Proposition 2.1.**

It can also be deduced from the even part. In fact, iterated use of the even part implies that

\[
Q_0 = 1, \quad Q_n = \sum_{k=0}^{n} E_k H_{n-k}, \quad n \geq 1.
\]

We also let $Q_n = 0$, $n < 0$ for convenience. It should be noticed that $Q(A, z) \neq \prod_{i \geq 1} \frac{1 + a_i z}{1 - a_i z}$ or $\prod_{i \geq 1} \frac{1 + a_i z}{1 - a_i z}$. By definition it follows that $Q(A, z)Q(A, -z) = 1$, that is, the Euler relations

\[
\sum_{r+s=n} (-1)^r Q_r Q_s = 0, \quad \forall n \geq 1.
\]

Or equivalently,

\[
\sum_{i=1}^{n-1} (-1)^{i-1} Q_i Q_{n-i} = \begin{cases} 2Q_n, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd.} \end{cases}
\]

When $Q_r$’s commute, the odd part of identity (2.3) is trivial. For the noncommutative case, it can also be deduced from the even part. In fact, iterated use of the even part implies that

**Proposition 2.1.** When $n$ is even, we have

\[
Q_n = \sum_{\alpha \in \mathbb{P}^{\text{odd}} n} (-1)^{\ell(\alpha)/2 - 1} C_{\ell(\alpha)/2 - 1} 2^{-\ell(\alpha) + 1} Q_{\alpha},
\]

where $C_k = \frac{1}{k+1} \binom{2k}{k}$ ($k \geq 0$), the $k$th Catalan number.

**Proof.** Fix $\alpha \equiv n$ with odd parts. In order to divide $n$ into $\alpha$, we can first divide $n$ into $\ell(\alpha)/2$ even parts, and then split each part into two odd ones to get $\alpha$. In our case, the first step contributes coefficient $C_{\ell(\alpha)/2 - 1} (-2)^{-\ell(\alpha)/2 + 1}$, while the second step gives $2^{-\ell(\alpha)/2}$. They combine to give the desired coefficient of $Q_{\alpha}$ on the RHS of (2.5). \qed

Now for $n = 2k + 1$, $k \geq 0$,

\[
\sum_{i=1}^{n-1} (-1)^{i-1} Q_i Q_{n-i} = \sum_{j=0}^{k-1} Q_{2i+1} Q_{2(k-i)} + \sum_{i=1}^{k} Q_{2i} Q_{2(k-i)+1}.
\]

When we expand those $Q_r$’s with $r$ even above by (2.5), the terms will cancel in pairs.

In general, let $t$ be an indeterminate, $\mathbb{K}(t)$ be the rational field and $\mathcal{F}_t = \mathbb{K}(t)[\{a_1, a_2, \ldots\}]$. Define the generating sequence in $\mathcal{F}_t[[z]]$,

\[
Q_t(A, z) = \sum_{n \geq 0} Q_n(A, t)z^n = E(A, -tz)H(A, z) = \prod_{i \geq 1} (1 - a_i tz) \prod_{i \geq 1} \frac{1}{1 - a_i z}.
\]

Note that $Q_n(A, 0) = H_n(A)$, thus $(Q_n(t) := Q_n(A, t))_{n \in \mathbb{N}}$ also forms a $\mathbb{Z}[t]$-basis of NSym. We remark that $Q_n(A, t)$ is just the $(1 - t)$-transform $H_n((1 - t)A)$ of $H_n(A)$ discussed
A NONCOMMUTATIVE LIFT OF THE SCHUR’S Q-FUNCTIONS

in [16, §5]. Moreover,

\[
\Delta(Q_n(t)) = \sum_{k=0}^{n} Q_k(t) \otimes Q_{n-k}(t).
\]

In particular, \(Q_n(A)\) is just the \((1-t)\)-transform at \(t = -1\).

2.2. **The peak subalgebra and its Hopf dual.** Let \(P\) be the Hopf subalgebra of \(NSym\) generated by \(Q_n\) \((n \geq 1)\). Then \(P_n := P \cap NSym_n\) is isomorphic to the **peak algebra** of the symmetric group \(\Xi_n\) when endowed with the internal product. According to (2.6), one can define a Hopf algebra projection

\[
\Theta : NSym \to P, \quad H_n \mapsto Q_n, \quad n \geq 1.
\]

From [9, Theorem 5.4], we know that \(\text{Ker } \Theta\) is the Hopf ideal of \(NSym\) generated by \(H_{2n}\) \((n \geq 1)\), which correspond to the (even) Euler relations (2.3). Equivalently, \(\{Q_\alpha\}_{\alpha \in C_n}\) forms a linear basis of \(P_n\), according to [19, Main Theorem 3].

It is well-known that the graded Hopf dual of \(NSym\) is the algebra of **quasisymmetric functions**, denoted by \(QSym\) [18]. It is a subring of the power series ring \(\mathbb{K}[[x_1, x_2, \ldots]]\) in the commuting variables \(x_1, x_2, \ldots\) and has a linear basis, the **monomial quasisymmetric functions**, defined by

\[
M_\alpha := M_\alpha(x) = \sum_{i_1 < \cdots < i_r} x_{i_1} \cdots x_{i_r},
\]

where \(\alpha = (\alpha_1, \ldots, \alpha_r)\) varies over the composition set \(\mathcal{C}\). There is another important basis, **fundamental quasisymmetric functions**, defined by

\[
F_\alpha := F_\alpha(x) = \sum_{i_1 \leq \cdots \leq i_n \atop i_k \in D(\alpha)} x_{i_1} \cdots x_{i_n}, \quad \alpha \vdash n.
\]

That means \(F_\alpha = \sum_{\beta \leq \alpha} M_\beta\). Meanwhile, the canonical pairing \(\langle \cdot, \cdot \rangle\) between \(NSym\) and \(QSym\) is defined by

\[
\langle H_\alpha, M_\beta \rangle = \langle R_\alpha, F_\beta \rangle = \delta_{\alpha, \beta}
\]

for any \(\alpha, \beta \in \mathcal{C}\).

Let \(\Lambda\) be the graded ring of symmetric functions in the commuting variables \(x_1, x_2, \ldots\), with integer coefficients, and \(\Omega\) be the subring of \(\Lambda\) generated by the symmetric functions \(q_n\) \((n \geq 1)\) defined by

\[
\sum_{n \geq 0} q_n z^{n} = \prod_{i \geq 1} \frac{1 + x_i z}{1 - x_i z}.
\]

For the basics of this subring \(\Omega\) and the Schur’s Q-functions, one can refer to [17, Ch. III, §8], where \(\Omega\) is denoted as \(\Gamma\). There exists a surjective Hopf algebra homomorphism

\[
\theta : \Lambda \to \Omega, \quad h_n \mapsto q_n, \quad n \geq 1.
\]
Then $\theta(p_n) = (1 - (-1)^n)p_n$, $n \geq 1$. Also let

$$\pi : \text{NSym} \to \Lambda, \ H_n \mapsto h_n$$

be the forgetful map.

Now we introduce the famous Stembridge algebra $\mathcal{B}$ of peak functions defined in [20]. It is a Hopf subalgebra of $\text{QSym}$. In order to define the usual bases of $\mathcal{P}$ and $\mathcal{B}$, we need the concept of peak subsets of $[n]$. Recall that $P \subseteq [n]$ is a peak set in $[n]$ if $P \subseteq [2, n - 1]$ and $i \in P \Rightarrow i - 1 \notin P$. Denote by $\mathcal{P}_n$ the collection of peak sets in $[n]$ and $\mathcal{P} = \bigcup_{n \geq 1} \mathcal{P}_n$.

Meanwhile, we denote by $\emptyset_n$ the empty set $\emptyset$ in $\mathcal{P}_n$. Given $\alpha = (\alpha_1, \ldots, \alpha_r) \ni n$, let $P(\alpha) := D(\alpha) \cap ([n - 1] \setminus D(\alpha)) + 1 \subseteq [2, n - 1]$ be its associated peak set in $[n]$. For any $P \in \mathcal{P}_n$, define

$$\Pi_P = \sum_{P(\alpha) = P} R_{\alpha} \in \text{NSym}. \quad (2.7)$$

Then $\{\Pi_P\}_{P \in \mathcal{P}_n}$ forms a linear basis of $\mathcal{P}_n$. Note that

$$Q_n = 2\Pi_{\emptyset_n} = 2 \sum_{k=0}^{n-1} R_{[1, n-k], n \geq 1}. \quad (2.8)$$

On the other hand, Stembridge’s peak functions in $\mathcal{B}$ are defined by

$$K_P = \sum_{P(\alpha) \subseteq P} F_{\alpha}, \ P \in \mathcal{P}_n \quad \text{where} \quad D \Delta (D + 1) = D \setminus (D + 1) \cup (D + 1) \setminus D \text{ for any } D = \{D_1 < \cdots < D_r\} \subseteq [n - 1].$$

Then $\{K_P\}_{P \in \mathcal{P}_n}$ forms a linear basis of $\mathcal{B}_n$ and there also exists a surjective Hopf algebra homomorphism

$$\theta : \text{QSym} \to \mathcal{B}, \ F_{\alpha} \mapsto K_{P(\alpha)}.$$ 

The coproduct formula of peak functions can be found in [10, Lemma 1.4]. Note that

$$K_P = \sum_{P(\alpha) \subseteq P} 2^{f(\alpha)} M_{\alpha}, \ P \in \mathcal{P}_n$$

and in particular,

$$K_{\emptyset_n} = q_n = \sum_{\alpha \ni n} F_{\alpha} = \sum_{\alpha \ni n} 2^{f(\alpha)} M_{\alpha}. \quad (2.9)$$

Now one can define a graded Hopf dual pairing

$$[\cdot, \cdot] : \mathcal{P} \times \mathcal{B} \to \mathbb{K}, \quad [\Pi_P, K_Q] = \delta_{P, Q}, \ P, Q \in \mathcal{P},$$

which satisfies the following property [19, Cor. 5.6.],

$$\langle \Theta(F), f \rangle = \langle F, \theta(f) \rangle = [\Theta(F), \theta(f)], \ F \in \text{NSym}, f \in \text{QSym}.$$ \hspace{1cm} (2.10)

In particular, when $f \in \Lambda$, then

$$[\Theta(F), \theta(f)] = [\Theta(F), f] = \langle \pi \Theta(F), f \rangle = [\pi \Theta(F), \theta(f)].$$
where the rightmost one is the canonical inner product [ , ] on Ω defined by

\[ [\rho_\lambda, \rho_\mu] = z_1 2^{-f(\lambda)} \delta_{\lambda \mu} \]

for any strict partitions \( \lambda, \mu \). Hence, \( \mathcal{P} \) can be regarded as a noncommutative lift of \( \Omega \), in which we shall find a lift of the Schur’s Q-functions. Moreover, the following commutative diagrams hold:

\[
\begin{align*}
\text{NSym} & \xrightarrow{\pi} \mathcal{P}, \\
\Lambda & \xrightarrow{\theta} \mathcal{B},
\end{align*}
\]

where the vertical maps in the second diagram are inclusions.

3. The noncommutative lift of the Schur’s Q-functions

Now we are in the position to give our main construction. Given \( f \in \mathcal{B} \), we define the adjoint operator \( f^\dagger \in \text{End}(\mathcal{P}) \) by

\[ [f^\dagger(H), g] = [H, fg] \]

for any \( H \in \mathcal{P}, g \in \mathcal{B} \). Similarly for \( f \in \Omega \), define \( f^\dagger \in \text{End}(\Omega) \) by

\[ [f^\dagger(h), g] = [h, fg], g, h \in \Omega. \]

**Definition 3.1.** We define the formal power series \( \Upsilon(z) \) in \( \text{End}(\mathcal{P})[[z, z^{-1}]] \) via

\[ \Upsilon(z) = \sum_{n \in \mathbb{Z}} \Upsilon_n z^{-n} = \left( \sum_{n \geq 0} Q_n z^n \right) \left( \sum_{n \geq 0} K^\perp_{\emptyset}(z^{-n}) \right), \]

i.e.

\[ \Upsilon_n = \sum_{i \geq 0} (-1)^i Q_{-\ell+i} K^\perp_{\emptyset}, n \in \mathbb{Z}. \]

For \( \alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{Z}^r \), define

\[ Q_\alpha := \Upsilon_{-\alpha}(1) = \Upsilon_{-\alpha_1} \cdots \Upsilon_{-\alpha_r}(1). \]

The following lemma shows that \( Q_\alpha \)'s are a lift of the Schur’s Q-functions onto NSym, thus we call them the noncommutative Schur’s Q-functions (abbreviated as NSQF).

**Lemma 3.2.** (1) For any \( f \in \Omega \subset \mathcal{B} \), \( \pi f^\dagger = f^\dagger \pi \).

(2) \( \pi \Upsilon(z) = Y(z) \pi \), where \( Y(z) \) is the twisted vertex operator on \( \Omega \) defined in [14],

\[ Y(z) = \sum_{n \in \mathbb{Z}} Y_n z^{-n} = \left( \sum_{n \geq 0} q_n z^n \right) \left( \sum_{n \geq 0} q_n^\perp(z^{-n}) \right) = \exp \left( \sum_{n \in \mathbb{N}_{\text{odd}}} 2p_n z^n \right) \exp \left( - \sum_{n \in \mathbb{N}_{\text{odd}}} 2p_n^\perp z^{-n} \right). \]

(3) For any \( \alpha \in \mathcal{C} \), \( \pi(Q_\alpha) = Q_\alpha \), where \( Q_\alpha \) is the Schur’s Q-function indexed by \( \alpha \).
Lemma 3.3. For any $n \geq 1$ follows from (2).

Proof. For (1), according to (2.10),

$$[\pi f^\perp(H), g] = [f^\perp(H), g] = [H, fg] = [\pi(H), fg] = [f^\perp\pi(H), g]$$

for any $H \in \mathcal{P}$, $g \in \Omega$. Hence, $\pi f^\perp = f^\perp\pi$. Combining (1) with the identities $\pi(Q_n) = q_n$, $n \geq 0$ and (2.9), one gets (2).

On the other hand, we know that $Q_n = Y_n(1)$ by [14] Theorem 5.9. Using the relation $\{Y_n, Y_m\} = 0$, one gets the basis of Schur’s Q-functions indexed by strict partitions. Now (3) follows from (2). \qed

Lemma 3.4. For any $n \geq 1$ and $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{C}$, we have

$$K^\perp_{\emptyset_n}(Q_\alpha) = \sum_{\beta \in \mathbb{N}^r \atop |\beta| = n} 2^{f(\beta)} Q_{\alpha - \beta}.$$

Proof. Since $\cdot$, $\cdot$ is a Hopf dual pair, one can easily see that

$$f^\perp(GH) = \sum f^\perp_1(G)f^\perp_2(H)$$

for any $f \in \mathcal{B}$ and $G, H \in \mathcal{P}$. Meanwhile, for any $f \in \mathcal{B}$,

$$[K^\perp_{\emptyset_n}(Q_m), f] = [Q_m, K^\perp_{\emptyset_n}f] = \sum_{i=0}^m [Q_i, K^\perp_{\emptyset_n}]Q_{m-i}, f] = 2^{1-\delta_{n,0}}[Q_{m-n}, f]$$

as $Q_m = 2^{1-\delta_{n,0}}\Pi_{\emptyset_n}$. It means that

$$K^\perp_{\emptyset_n}(Q_m) = 2^{1-\delta_{n,0}} Q_{m-n}.$$

In particular, the formula

$$\Delta^{(r-1)}(K^\perp_{\emptyset_n}) = \sum_{\beta \in \mathbb{N}^r \atop |\beta| = n} K^\perp_{\emptyset_1} \otimes \cdots \otimes K^\perp_{\emptyset_r}, \ r \geq 2$$

gives the desired result. \qed

Definition 3.5. For an ordered set $X = \{x_1, x_2, \ldots\}$ of commuting variables, we define the noncommutative analogue of Cauchy kernel associated with the Schur’s Q-functions.

$$\Xi_X := \prod_{i \geq 1} \left( \prod_{j \geq 1} \left( 1 + x_j a_j \right) \prod_{j \geq 1} \frac{1}{1 - x_j a_j} \right) = \sum_{\alpha \in \mathbb{C}} M_\alpha(X) Q_\alpha(A) = \sum_{P \in \mathcal{P}} K_P(X) \Pi_P(A).$$

In particular, when $X$ has only one variable $z$, then $\Xi_z = \sum_{n \geq 0} Q_n(A) z^n$.

Lemma 3.5. The noncommutative Cauchy kernel $\Xi_X$ has the following properties:

1. $\Xi_z \Xi_X = \Xi_z \Xi_X$, where $z, X$ denotes the alphabet $\{z, x_1, x_2, \ldots\}$.
2. For any $f \in \mathcal{B}$, $f^\perp(\Xi_X) = f(X) \Xi_X$. 

Proof. (1) follows from the identity $M_\alpha(z, X) = M_\alpha(X) + z^{\sigma_1} M_{\sigma_2, \ldots, \sigma_r}$, $\alpha = (\alpha_1, \ldots, \alpha_r)$.

For (2),

$$f^\perp(\Xi_X) = \sum_{P \in \mathcal{P}} K_P(X) f^\perp(\Pi_P) = \sum_{P \in \mathcal{P}} K_P(X) \sum_{Q \in \mathcal{Q}} [\Pi_P, f K_Q] \Pi_Q = \sum_{Q \in \mathcal{Q}} (f K_Q)(X) \Pi_Q = f(X) \Xi_X.$$

□

In particular, let $\mathcal{K}_z^\perp = \sum_{i \geq 0} z^i K_0^\perp$. Then from Lemma 3.5 (2), we have

$$\mathcal{K}_z^\perp \Xi_X = \sum_{i \geq 0} z^i K_0^\perp(X) \Xi_X = \Xi_X \prod_{x \in X} \frac{1 + zx}{1 - zx}.$$

Using the notations above, we know that

$$\mathcal{Y}(z) = \left( \sum_{n \geq 0} Q_n z^n \right) \left( \sum_{n \geq 0} K_{0_n}^\perp(-z)^{-n} \right) = \Xi_c \mathcal{K}_z^\perp.$$

Hence,

$$\mathcal{Y}(z) \Xi_X = \Xi_c \mathcal{K}_z^\perp \Xi_X = \Xi_c \Xi_X \prod_{x \in X} \frac{1 - x/z}{1 + x/z} = \Xi_c X \prod_{x \in X} \frac{1 - x/z}{1 + x/z}.$$

More generally,

$$\mathcal{Y}(z_1) \cdots \mathcal{Y}(z_r) \Xi_X = \Xi_{z_1, \ldots, z_r} X \prod_{i=1}^r \left( \prod_{x \in \{z_1, \ldots, z_r\} \cup X} \frac{1 - x/z_i}{1 + x/z_i} \right).$$

Now let $X = \emptyset$ and take the coefficient of $z_1^{\alpha_1} \cdots z_r^{\alpha_r}$ in the above series, we get

**Proposition 3.6.** For $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{Z}^r$,

$$Q_\alpha := \mathcal{Y}_{-\alpha_1} \cdots \mathcal{Y}_{-\alpha_r}(1) = \prod_{1 \leq i < j \leq r} \frac{1 - R_{ij}}{1 + R_{ij}} Q_\alpha,$$

where $R_{ij}$ is the usual raising operator acting on $\mathbb{Z}^r$ by

$$R_{ij}(\alpha_1, \ldots, \alpha_r) = (\alpha_1, \ldots, \alpha_i + 1, \ldots, \alpha_j - 1, \ldots, \alpha_r).$$

Recall that for any anti-symmetric matrix $A = (a_{ij})_{1 \leq i, j \leq 2n}$ over a commutative ring $R$, the Pfaffian $\text{Pf}(A) \in R$ is the unique square root of $\det(A)$ up to a sign defined by

$$\text{Pf}(A) := \sum_{\sigma \in \mathbb{S}_{2n}} a_{\sigma(1), \sigma(2)} \cdots a_{\sigma(2n-1), \sigma(2n)},$$

where $\mathbb{S}_{2n} = \{ \sigma \in \mathbb{S}_{2n} : \sigma(2i - 1) < \sigma(2i), 1 \leq i \leq n; \sigma(2j - 1) < \sigma(2j + 1), 1 \leq j \leq n - 1 \}.$
By definition, for any $\alpha = (\alpha_1, \ldots, \alpha_{2n}) \in \mathbb{Z}^{2n}$, $\Omega_\alpha$ is the coefficient of $z^\alpha$ in 
\[
\Xi_{z_1, \ldots, z_{2n}} \prod_{1 \leq i < j \leq 2n} \left( \frac{1 - z_j/z_i}{1 + z_j/z_i} \right) = \Xi_{z_1, \ldots, z_{2n}} \text{Pr} \left( \frac{z_i - z_j}{z_i + z_j} \right)
\]
\[
= \sum_{\alpha \in \mathbb{Z}_{2n}} (-1)^{\ell(\alpha)} \left( \frac{z_{0}(1) - z_{0}(2)}{z_{0}(1) + z_{0}(2)} \right) \cdots \left( \frac{z_{0}(2n-1) - z_{0}(2n)}{z_{0}(2n-1) + z_{0}(2n)} \right) \sum_{\beta \in \mathbb{Z}_{2n}} \beta_1 \cdots \beta_{2n} Q_{\beta_1} \cdots Q_{\beta_{2n}}
\]
\[
= \sum_{\beta \in \mathbb{Z}_{2n}} \sum_{\alpha \in \mathbb{Z}_{2n}} \sum_{i_1, i_2, i_3 = 1} \left( \frac{z_{0}(1) + \sum_{i_k} i_{2k-1} 2^{n-i_{2k-1}} \sum_{i_{2k-1}} i_{2k-1}} {z_{0}(1) + \sum_{i_{2k-1}} i_{2k-1}} \right) \sum_{\alpha \in \mathbb{Z}_{2n}} \beta_1 \cdots \beta_{2n} Q_{\beta_1} \cdots Q_{\beta_{2n}}
\]
\[
= \sum_{\beta \in \mathbb{Z}_{2n}} \sum_{\alpha \in \mathbb{Z}_{2n}} \left( (-1)^{\ell(\alpha) + \sum_{i_k} i_{2k-1} 2^{n-i_{2k-1}} \sum_{i_{2k-1}} i_{2k-1}} \right) \sum_{\alpha \in \mathbb{Z}_{2n}} \beta_1 \cdots \beta_{2n} Q_{\beta_1} \cdots Q_{\beta_{2n}}
\]
where we use the identity \( \frac{1-R}{1+R} = 1 + 2 \sum_{i \geq 1} (-1)^i R^i \). Based on (3.2), we also have

**Corollary 3.7.** For $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathcal{C}$, let $\alpha' := (\alpha_1, \ldots, \alpha_{r-1})$, then $\Omega_\alpha$ satisfies the following recursive relation

\[
Q_\alpha = \sum_{i=0}^{\alpha_r} \left( \sum_{\beta \geq 0} \beta \right) 2^{n-i} \delta_{\alpha' + \beta} Q_{\alpha' + \beta} \]  
\[
(3.3)
\]

Next we give the following key relations for NSQF.

**Proposition 3.8.** For any $\alpha \in \mathcal{C}$ and $n \geq 2$,

\[
(3.4)
\]

\[
\sum_{i=1}^{n-1} Q_{\alpha, i, n-i} = 0.
\]

**Proof.** By the definition of $Q_\alpha$, we only need to show that

\[
(3.5)
\]

\[
\sum_{i=1}^{n-1} Q_{i, n-i} = 0, \quad n \geq 2,
\]

as $Q_{i, n-i} = \mathcal{Y}_{\alpha}(Q_{i, n-i})$.

Indeed using the recursive formula (3.3), we have

\[
\sum_{i=1}^{n-1} Q_{i, n-i} = \sum_{i=1}^{n-1} \left( Q_{i, n-i} + 2 \sum_{j=1}^{n-i} (-1)^j Q_{i+j, n-i-j} \right) = \sum_{i=1}^{n-1} Q_{i, n-i} + 2 \sum_{i=1}^{n-1} \sum_{k=i+1}^{n} (-1)^{k-i} Q_{k, n-k}
\]
\[
= \sum_{i=1}^{n-1} Q_{i, n-i} + 2 \sum_{k=2}^{n} \sum_{i=1}^{k-1} (-1)^{k-i} Q_{k, n-k} = \sum_{k=0}^{n} (-1)^{k-1} Q_{k, n-k} = 0.
\]
\[\Box\]
From the proof above, we know that relation (3.5) is equivalent to the Euler relation (2.3). In particular, we know that those $Q_\alpha$’s, with $\alpha$ ranging over strict compositions, are not linearly independent by letting $n$ to be odd in (3.5). Instead, we will find another index set for a natural basis of $P$ from NSQF later.

### 3.1. The right-Pieri rule for NSQF.

Next we relate the operators $\Upsilon_m (m \in \mathbb{Z})$ with those $Q_s (s \geq 1)$ to obtain a right-Pieri rule for NSQF.

**Lemma 3.9.** For $s \geq 1$ and $m \in \mathbb{Z}$, and for $F \in P$,

\[(3.6) \quad \Upsilon_m(F)Q_s = \Upsilon_m(FQ_s) + 2 \sum_{i=0}^{s-1} \Upsilon_{m-s+i}(FQ_i).\]

**Proof.** We first prove that

\[\Upsilon_m(F)Q_s = \Upsilon_m(FQ_s) + 2 \sum_{j \geq 1} (-1)^{j-1} \Upsilon_{m-j}(F)Q_{s-j}.\]

By definition,

\[
\Upsilon_m(FQ_s) = \sum_{i \geq 0} (-1)^i Q_{m+i} K^\perp_{\emptyset_i}(FQ_s) = \sum_{i \geq 0} (-1)^i Q_{m+i} \sum_{j=0}^i K^\perp_{\emptyset_{i-j}}(F) K^\perp_{\emptyset_j}(Q_s)
\]

\[= \sum_{j \geq 0} \sum_{k \geq 0} (-1)^{j+k} Q_{m+j+k} K^\perp_{\emptyset_j}(F) K^\perp_{\emptyset_j}(Q_s)
\]

\[= \sum_{j \geq 0} (-1)^j \left( \sum_{k \geq 0} (-1)^k Q_{m+j+k} K^\perp_{\emptyset_j}(F) \right) K^\perp_{\emptyset_j}(Q_s)
= \Upsilon_m(FQ_s) + 2 \sum_{j \geq 1} (-1)^j \Upsilon_{m-j}(F)Q_{s-j},
\]

where we use Lemma 3.3 to get the equalities.
Now we prove (3.6) by induction on $s$: The above formula clarifies the case when $s = 1$. Furthermore,

\[
\mathcal{Y}_{-m}(F)Q_s = \mathcal{Y}_{-m}(F)Q_0 + 2 \sum_{j=1}^{s} (-1)^{j-1} \mathcal{Y}_{-m-j}(F)Q_{s-j}
\]

\[
= \mathcal{Y}_{-m}(F)Q_s + 2 \sum_{j=1}^{s} (-1)^{j-1} \left( \mathcal{Y}_{-m-j}(F)Q_{s-j} + 2 \sum_{k=0}^{s-j-1} \mathcal{Y}_{-m-s+k}(F)Q_k \right)
\]

\[
= \mathcal{Y}_{-m}(F)Q_s + 2 \sum_{j=0}^{s-1} (-1)^{j-1} \mathcal{Y}_{-m-s+j}(F)Q_j + 4 \sum_{k=1}^{s-2} \left( \sum_{j=1}^{s-k-1} (-1)^{j-1} \right) \mathcal{Y}_{-m-s+k}(F)Q_k
\]

\[
= \mathcal{Y}_{-m}(F)Q_s + 2 \sum_{j=0}^{s-1} \mathcal{Y}_{-m-s+j}(F)Q_j.
\]

\[
\square
\]

In order to describe our Pieri rule, we need to introduce the following notion: For any compositions $\alpha, \beta$, we write $\alpha \subset_s \beta$, $s \in \mathbb{N}$ if

1. $|\beta| = |\alpha| + s$,
2. $\alpha_i \leq \beta_i$ for all $1 \leq i \leq \ell(\alpha)$,
3. $\ell(\beta) \leq \ell(\alpha) + 1$.

By the formula (3.6) and the definition of $\mathcal{Q}_\alpha$, it is easy to deduce that

**Theorem 3.10.** For any composition $\alpha$, $\mathcal{Q}_\alpha$ satisfies the following right Pieri rule for multiplication by $Q_s$:

\[
\mathcal{Q}_\alpha Q_s = \sum_{\alpha \subset \beta} 2^{\ell(\beta' - \alpha)} \mathcal{Q}_\beta,
\]

where $\beta' = (\beta_1, \ldots, \beta_{\ell(\alpha)})$, the truncation of $\beta$, such that $\beta' - \alpha \in \mathcal{Y}_0^{\ell(\alpha)}$.

**Remark 3.11.** Applying the forgetful map $\pi$ to (3.7), one gets the pieri rule of the Schur’s Q-functions [17] Ch. III, (8.15)]:

\[
\mathcal{Q}_\mu q_s = \sum_\lambda 2^{b(\lambda/\mu)} \mathcal{Q}_\lambda
\]

for any strict partition $\mu$, where the sum is over all strict $\lambda \supset \mu$ such that $\lambda/\mu$ is an $s$-horizontal strip, and $b(\lambda/\mu)$ is the number of $i \geq 1$ such that $\lambda/\mu$ has a box in the $(i + 1)$th column but not in the $i$th one.

Indeed, the special case for $\alpha$ with one part can illustrate the point. Assuming that $\alpha = (r)$, $r \in \mathbb{N}$, we need to consider two kinds of $\beta$ with $\alpha \subset_s \beta$:

1. $(r + s)$;
2. $(r + i, s - i)$, $i = 0, \ldots, s - 1$. 

Case (1) gives the right coefficient 2. For (2) when \( s - i < r \) gives the right coefficient \( 2 - \delta_{i,1} \). Now for (2) with \( s - i \geq r \), we have the pairs \( (r+i, s-i) \) and \( (s-i, r+i) \). When \( i = 0 \) and \( s > r \), then \( Q_{(r,s)} \) has coefficient 1 while \( Q_{(s,r)} \) has \( -2 \), thus gives \( Q_{(s,r)} \) the right 1. When \( i > 0 \) and \( s - i > r \), then both \( Q_{(r+i,s-i)} \) and \( Q_{(s-i,r+i)} \) have coefficient 2, thus cancel with each other when degenerating to the commutative case.

**Corollary 3.12.** For any composition \( \alpha = (\alpha_1, \ldots, \alpha_r) \), \( Q_\alpha \) has the following expansion in terms of NSQF:

\[
Q_\alpha = \sum_{\beta^{(i)} = (\alpha_1) \subset (\alpha_2, \ldots, \alpha_r), \beta^{(i)} = \beta} 2^{\sum_{i=2}^r 2(\beta^{(i)}_r - \beta^{(i-1)}_r)} Q_\beta,
\]

where \( \beta^{(i)} = (\beta_1^{(i)}, \ldots, \beta_{(\alpha_i) - 1}^{(i)}) \), the truncation of \( \beta^{(i)} \).

4. A natural basis in the peak algebra from NSQF

By the expansion (3.9), we know that the NSQF’s also linearly span the peak subalgebra \( \mathcal{P} \). In this section we extract a natural basis for \( \mathcal{P} \) from NSQF.

First we note that the Schur’s Q-functions satisfy the anti-symmetric relation:

\[ Q_\alpha = -Q_{\alpha_{ij}} \]

for any \( i < j \), where \( \alpha_{ij} \) is the composition derived from \( \alpha \) by transposing its \( i \)th and \( j \)th parts. Naturally, \( Q_\alpha \)'s with \( \alpha \) ranging over the strict partitions form a linear basis of \( \Omega \). On the other hand, since NSQF serve as the noncommutative lift of Schur’s Q-functions, we know that the kernel of \( \pi_\mathcal{P} \) is spanned by those \( Q_\alpha + Q_{\alpha_{ij}} \).

Now in order to define a natural basis from NSQF, we need the following notion.

**Definition 4.1.** All the compositions of \( n \) with peak subsets of \( [n] \) as descent sets are those with 1 only possibly appearing in the last part, and we call them the *peak compositions*. We denote by \( \mathcal{P}_n \) the set of peak compositions of \( n \) and \( \mathcal{P} = \bigcup_{n \geq 1} \mathcal{P}_n \).

For example,

\[ \mathcal{P}_5 = \{\emptyset_5, \{4\}, \{3\}, \{2\}, \{2, 4\}\} \]

and correspondingly

\[ \mathcal{P}_5 = \{5, 41, 32, 23, 2^21\} \].

**Theorem 4.2.** For \( n \in \mathbb{N} \), those \( Q_\alpha \)'s with \( \alpha \in \mathcal{P}_n \) form a linear basis of \( \mathcal{P}_n \).

**Proof.** Since \( |\mathcal{P}_n| = 2^n \), the \((n - 1)\)th Fibonacci number, we only need to show that any \( Q_\beta, \beta \equiv \mathcal{F}_n \), can be spanned by these \( Q_\alpha \)'s with \( \alpha \in \mathcal{P}_n \).

Now for the right-Pieri rule (5.7), if \( \alpha \in \mathcal{P}_n \), then there exist two situations for the composition \( \beta \) with \( \alpha \subset \beta \):

1. \( \beta \) is still a peak composition.
2. \( \beta \) has the second last part equal to 1.
For case (2), we can use relation (3.4) to cancel such $\Omega_\beta$ with a linear combination of NSQF indexed by peak compositions appearing in case (1). Explicitly, when $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathcal{PC}$, the right-Pieri rule can be reformulated as

$$Q_\alpha Q_s = \sum_{\mu \in \mathcal{PC}, \nu \in \mathcal{PC}} 2^{(\mu - \alpha) \cdot \delta_{\mu,1}} 2^{(1 - \delta_{\mu,1})} Q_{\beta},$$

(4.1)

By induction, we know that any $Q_\alpha$ can be spanned by those $Q_\beta$'s with $\beta \in \mathcal{PC}_{n}$. \hfill $\square$

Example 4.3. For those small $n$, we write down the natural basis and $2^{n-1} - f_{n-1}$ complete linear relations for $Q_\alpha$'s, $\alpha \neq n$. We also underline those elements derived from the natural basis by the complete relations. The computation for these relations heavily relies on the key relation (3.4) and the right-Pieri rule for NSQF.

<table>
<thead>
<tr>
<th>$n$</th>
<th>basis</th>
<th>relations</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$Q_1$</td>
<td>( \setminus )</td>
</tr>
<tr>
<td>2</td>
<td>$Q_2$</td>
<td>$Q_{12}$</td>
</tr>
<tr>
<td>3</td>
<td>$Q_3, Q_{21}$</td>
<td>$Q_{11}, Q_{21} + Q_{12}$</td>
</tr>
<tr>
<td>4</td>
<td>$Q_4, Q_{31}, Q_{22}$</td>
<td>$Q_{14}, Q_{21,2}, Q_{12,1} + Q_{12,2}, Q_{13} + Q_{22} + Q_{13}$</td>
</tr>
<tr>
<td>5</td>
<td>$Q_5, Q_{41}, Q_{32}, Q_{23}, Q_{21}$</td>
<td>$Q_{15}, Q_{21,1}, Q_{31,2}, Q_{12,1} + Q_{12,2}, Q_{13} + Q_{22} + Q_{12,1}$</td>
</tr>
<tr>
<td>6</td>
<td>$Q_6, Q_{51}, Q_{42}, Q_{24}, Q_{32}, Q_{31,2}, Q_{231}, Q_{22}$</td>
<td>$Q_{16}, Q_{21,1}, Q_{31,2}, Q_{13,2} + Q_{13,1}, Q_{12,1} + Q_{12,2}, Q_{13} + Q_{22} + Q_{12,1}$</td>
</tr>
</tbody>
</table>

4.1. Combinatorial description for NSQF. In order to convince the readers that our basis from NSQF is quite natural, we give some interesting combinatorial description here. First let us introduce the following combinatorial objects.

In [4], the authors defined the notion of immaculate tableaux to discuss the relations between the classical bases of NSym and their immaculate basis. Let $\alpha, \beta$ be compositions. Recall that an immaculate tableau of shape $\alpha$ and content $\beta$ is a labeling of the boxes of the diagram of $\alpha$ by positive integers such that:

(a) the number of boxes labeled by $i$ is $\beta_i$;
(b) the sequence of entries in each row, from left to right, is weakly increasing;
(c) the sequence of entries in the first column, from top to bottom, is increasing.

An immaculate tableau is said to be standard if it has content $1^{|\alpha|}$. Correspondingly, they created a labeled poset $\mathcal{C}$ on $\mathcal{C}$, called the immaculate poset, such that $\beta$ covers $\alpha$ if $\alpha \subset_1 \beta$.

Write $\beta \rightarrow_m \alpha$ if $\beta$ is obtained by adding 1 to the $m$th part of $\alpha$. Note that standard immaculate tableaux of shape $\beta/\alpha$ one-to-one correspond to maximal chains on this poset from $\alpha$ to $\beta$.

Now we just need to focus on the subposet of $\mathcal{C}$ on $PC$ and call it the peak composition poset, denoted by $\mathfrak{C}$. The first few levels of $\mathfrak{C}$ are portrayed as follows.

![Diagram of peak composition poset]

**Definition 4.4.** An immaculate tableau of shape $\alpha$ and content $\beta$ is called a peak composition tableau (abbreviated as PCT) if it satisfies the following extra condition:

Let $r = \ell(\beta)$. For any $i : 1 \leq i \leq r$, the subdiagram of $\alpha$ with labeling in $\{1, \ldots, i\}$ gives a peak composition. In particular, necessarily $\alpha \in \mathfrak{C}$.

Let $PCT(\alpha, \beta)$ denote the set of peak composition tableaux of shape $\alpha$ and content $\beta$, and $p_{\alpha, \beta}$ its cardinality. Then one can easily see that

1. $p_{\alpha, \alpha} = 1$ for any $\alpha \in \mathfrak{C}$. Such unique tableau consists of $\alpha_i$ many $i$'s in the $i$th row.
2. $p_{\alpha, \beta} = 0$ unless $\beta \leq_\ell \alpha$, where $\leq_\ell$ represents the lexicographic order on compositions.

For example,

$$PCT(342, 2214) = \left\{ \begin{array}{c}
\begin{array}{ccc}
1 & 1 & 2 \\
2 & 2 & 4 \\
3 & 4 & 4
\end{array} & 
\begin{array}{ccc}
1 & 1 & 3 \\
2 & 2 & 4 \\
3 & 3 & 4
\end{array} & 
\begin{array}{ccc}
1 & 1 & 4 \\
2 & 3 & 4 \\
3 & 4 & 4
\end{array} & 
\begin{array}{ccc}
1 & 1 & 4 \\
2 & 4 & 4 \\
3 & 4 & 4
\end{array}
\end{array}\right\}.$$  

By contrast,

$$T = \begin{array}{ccc}
1 & 1 & 2 \\
3 & 4 & 4
\end{array}$$

is only an immaculate tableau, but not a peak composition tableau.

Note that any standard peak composition tableau is just a standard immaculate tableau with a peak composition as its shape. Hence, standard peak composition tableaux of shape $\beta/\alpha$, $\alpha, \beta \in \mathfrak{C}$, one-to-one correspond to maximal chains on $\mathfrak{C}$ from $\alpha$ to $\beta$.
Given \( T \in \text{PCT} \), we also define \( p_i(T) \), \( i \geq 1 \) as the number of integers appearing in the \( i \)th row of \( T \) minus one. Let \( p(T) = \sum_{i \geq 1} p_i(T) \). Meanwhile, let \( m(T) \) be the number of boxes in the leftmost column whose right and bottom adjacent boxes are labeled with the same integer. For example,

\[
T = \begin{pmatrix}
1 & 1 & 3 \\
2 & 3 & 3 \\
3 & 1 & 3 \\
4 & 1 & 3 \\
3 & 1 & 3 \\
5 & 1 & 3 \\
\end{pmatrix}
\]

Then \( p_1(T) = p_3(T) = p_4(T) = 1 \), \( p_2(T) = 2 \), \( p_5(T) = 0 \), thus \( p(T) = 5 \). Meanwhile, \( m(T) = 2 \).

Now using the reformulated Pieri rule \([4.1]\) and the terminology of PCT, we can simplify formula \([3.9]\) as follows:

**Theorem 4.5.** For any composition \( \alpha \), \( Q_{\alpha} \) has the following expansion in terms of the natural basis from NSQF:

\[
Q_{\alpha} = \sum_{\mu, \nu \in \text{PCT} \atop \alpha \leq \beta} \left( \sum_{T \in \text{PCT}^{(\beta, \alpha)}} 2^{p(T) - m(T)} \right) Q_{\beta}.
\]

Meanwhile, we also find that

**Lemma 4.6.** For any \( T \in \text{PCT}(\beta, \alpha) \), \( p(T) - m(T) + \ell(\beta) - \ell(\alpha) \geq 0 \).

**Proof.** Note that by definition, \( p(T) + \ell(\beta) \) is the sum of numbers of integers appearing in each row of \( T \). Meanwhile, if there exists a box in the leftmost column whose right and bottom adjacent boxes are both labeled with some integer \( i \), then such \( i \) should be counted at least twice. Hence, we have \( p(T) + \ell(\beta) \geq m(T) + \ell(\alpha) \).

**Example 4.7.** For \( \alpha = 2^3 \), we have

\[
\text{PCT}(\cdot, 2^3) = \left\{ \begin{array}{c}
1111, 1112, 1113, 1114, 1115, 1122, 1123, 1124, 1125, 1133, 1134, 1135, 1144, 1145, 1155, 1222, 1223, 1224, 1225, 1233, 1234, 1235, 1244, 1245, 1255, 1333, 1334, 1335, 1344, 1345, 1355, 1444, 1445, 1455, 1555, 2222, 2223, 2224, 2225, 2233, 2234, 2235, 2244, 2245, 2255, 2333, 2334, 2335, 2344, 2345, 2355, 2444, 2445, 2455, 2555, 3333, 3334, 3335, 3344, 3345, 3355, 3444, 3445, 3455, 3555, 4444, 4445, 4455, 4555, 5555 \end{array} \right\}.
\]

Then \( p(T) = 0, 1, 1, 2, 1, 2, 1, 2, 1, 3, 1, 2, 2, 2 \) successively and all \( m(T) = 0 \) except the fourth one equal to 1. Hence,

\[
Q_{2^3} = Q_{2^3} + 2(Q_{231} + Q_{24}) + 4(Q_{321} + Q_{6}) + 8(Q_{32} + Q_{51}) + 12Q_{42}.
\]

**Corollary 4.8.** For \( n \in \mathbb{N} \), \( \{ Q_{\alpha} \}_{\alpha \in \mathcal{P}^n} \) is also a linear basis of \( \mathcal{P}_n \).

**Proof.** Note that for any \( \alpha \in \mathcal{P}^n \), the unique \( T \in \text{PCT}(\alpha, \alpha) \) satisfies \( p(T) = m(T) = 0 \). In particular, by \([4.12]\) the transition matrix between \( \{ Q_{\alpha} \}_{\alpha \in \mathcal{P}_n} \) and \( \{ Q_{\alpha} \}_{\alpha \in \mathcal{P}_n} \) is upper unitriangular with respect to the lexicographic order, and thus the former is also a basis.
Remark 4.9. Via the bijection between peak sets and peak compositions, we can take $\mathcal{PC}$ as the index set for bases of the peak algebra and let

$$\Pi_\alpha := \Pi_{D(\alpha)}, \ \alpha \in \mathcal{PC}.$$ 

By [19, Prop. 3.4.], we have

$$Q_\alpha = 2^{\ell(\alpha)} \sum_{\beta \in \mathcal{PC} \cap D(\alpha) \cup (D(\alpha) + 1)} \Pi_\beta$$

for any composition $\alpha$. Obviously, for $\beta \in \mathcal{PC}$ such that $D(\beta) \subseteq D(\alpha) \cup (D(\alpha) + 1)$, we must have $\alpha \leq \beta$. In particular, the transition matrix between $\{Q_\alpha\}_{\alpha \in \mathcal{PC}_n}$ and $\{\Pi_\alpha\}_{\alpha \in \mathcal{PC}_n}$ is also upper triangular.

For small $n \in \mathbb{N}$, we list the transition matrices between $\{Q_\alpha\}_{\alpha \in \mathcal{PC}_n}$ and $\{\Pi_\alpha\}_{\alpha \in \mathcal{PC}_n}$ (resp. $\{\Pi_\alpha\}_{\alpha \in \mathcal{PC}_n}$), denoted by $M_n(Q, Q)$ (resp. $M_n(Q, \Pi)$).

<table>
<thead>
<tr>
<th>$n$</th>
<th>index</th>
<th>$M_n(Q, Q)$</th>
<th>$M_n(Q, \Pi)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>21, 3</td>
<td>1 2</td>
<td>4 4</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0 1</td>
<td>0 2</td>
</tr>
<tr>
<td>4</td>
<td>2^2, 31, 4</td>
<td>1 2 2</td>
<td>4 4 4</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0 1 2</td>
<td>0 4 4</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0 0 1</td>
<td>0 0 2</td>
</tr>
<tr>
<td>5</td>
<td>2^1, 23, 32, 41, 5</td>
<td>1 2 6 6 4</td>
<td>8 8 8 8 8</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0 1 2 2 2</td>
<td>0 4 0 4 0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0 0 1 2 2</td>
<td>0 0 4 4 4</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0 0 0 1 2</td>
<td>0 0 0 4 4</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0 0 0 0 1</td>
<td>0 0 0 0 2</td>
</tr>
<tr>
<td>6</td>
<td>2^3, 231, 24, 321, 3^2, 42, 51, 6</td>
<td>1 2 2 4 8 12 8 4</td>
<td>8 8 8 8 8 8 8 8</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0 1 2 2 6 8 6 4</td>
<td>0 8 8 8 8 8 8 8</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0 0 1 0 2 2 2 2</td>
<td>0 0 4 0 4 0 0 4</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0 0 0 1 2 6 6 4</td>
<td>0 0 0 8 8 8 8 8</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0 0 0 0 1 2 2 2</td>
<td>0 0 0 0 4 4 0 4</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0 0 0 0 0 1 2 2</td>
<td>0 0 0 0 0 4 4 4</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0 0 0 0 0 0 1 2</td>
<td>0 0 0 0 0 0 4 4</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0 0 0 0 0 0 0 1</td>
<td>0 0 0 0 0 0 0 2</td>
</tr>
</tbody>
</table>

Next we give the following example to show that the recursive formula (3.3) is quite efficient for calculation of $M_n(Q, Q)$.

Example 4.10. For $n = 2$:

$$Q_2 = Q_2 = 2\Pi_2.$$
For $n = 3$:
\[
\Omega_3 = Q_3 = 2\Pi_3,
\Omega_{21} = \Omega_2Q_1 - 2Q_3 = Q_{21} - 2Q_3 = 4\Pi_{21}.
\]

For $n = 4$:
\[
\Omega_4 = Q_4 = 2\Pi_4, \Omega_{31} = \Omega_3Q_1 - 2Q_4 = Q_{31} - 2Q_4 = 4\Pi_{31},
\Omega_{22} = \Omega_2Q_2 - 2\Omega_3Q_1 + 2Q_4 = Q_{22} - 2Q_{31} + 2Q_4 = 4(\Pi_{22} - \Pi_{31}).
\]

For $n = 5$:
\[
\Omega_5 = Q_5 = 4\Pi_5, \Omega_{41} = \Omega_4Q_1 - 2Q_5 = Q_{41} - 2Q_5 = 4\Pi_{41},
\Omega_{32} = \Omega_3Q_2 - 2\Omega_4Q_1 + 2Q_5 = Q_{32} - 2Q_{41} + 2Q_5 = 4(\Pi_{32} - \Pi_{41}),
\Omega_{23} = \Omega_2Q_3 - 2\Omega_3Q_2 + 2\Omega_4Q_1 - 2Q_5 = Q_{23} - 2Q_{32} + 2Q_{41} - 2Q_5 = 4(\Pi_2 - \Pi_3),
\Omega_{22} = \Omega_2Q_2 - 2(\Omega_{23} + \Omega_{32}) = Q_{22} - 2Q_{23} - 2Q_{32} + 2Q_{41} = 8(\Pi_{22} - \Pi_{32} + \Pi_{41}).
\]

For $n = 6$:
\[
\Omega_6 = Q_6 = 4\Pi_6, \Omega_{51} = \Omega_5Q_1 - 2Q_6 = Q_{51} - 2Q_6 = 4\Pi_{51},
\Omega_{42} = \Omega_4Q_2 - 2\Omega_5Q_1 + 2Q_6 = Q_{42} - 2Q_{51} + 2Q_6 = 4(\Pi_{42} - \Pi_{51}),
\Omega_{32} = \Omega_3Q_2 - 2\Omega_4Q_1 + 2\Omega_5Q_1 - 2Q_6
= Q_{32} - 2Q_{42} + 2Q_{51} - 2Q_6 = 4(\Pi_{32} - \Pi_{42}),
\Omega_{32} = \Omega_3Q_2 - 2\Omega_5Q_1 - 2Q_6
= Q_{32} - 2\Omega_{42} + 2Q_5Q_1 - 2Q_6 = 4(\Pi_{32} - \Pi_{42}),
\Omega_{23} = \Omega_2Q_3 - 2\Omega_4Q_1 - 2Q_5 + 2Q_6
= Q_{23} - 2\Omega_{42} + 2Q_{52} - 2Q_5 + 2Q_6 = 4(\Pi_{23} - \Pi_{42}),
\Omega_{23} = \Omega_2Q_3 - 2\Omega_{23}Q_1 + 2Q_{42} + 2Q_5 + 4Q_6
= Q_{23} - 2\Omega_{23} + 2Q_{42} + 2Q_5 + 4Q_6 = 8(\Pi_{23} - \Pi_{32} + \Pi_{42}).
\]

4.2. Quasisymmetric Schur’s P-functions. With respect to $[\cdot, \cdot]$, the dual of Schur’s Q-functions are the Schur’s P-functions, denoted by $Q^\prime$ (The usual notation $P_\lambda$ is given up to avoid confusion). Then $Q^\prime_\lambda = 2^{(\lambda)}Q^*_\lambda$ for any strict partition $\lambda$. Let $Q^\prime_\alpha$ be the dual of $Q_\alpha$ in $\mathcal{B}$ with respect to the pairing $[\cdot, \cdot]$. We call them the quasisymmetric Schur’s P-functions, since they are nice refinements of the Schur’s P-functions stated as follows.

**Theorem 4.11.** For any strict partition $\lambda$, we have
\[
Q^\prime_\lambda = \sum_{\alpha \in \lambda} (-1)^{f(\alpha(\alpha))} Q^*_\alpha,
\]
where $\alpha$ is the unique partition as the rearrangement of $\alpha$, and $\alpha(\alpha)$ is the permutation for this rearrangement.

**Proof.** Let $Q^\prime_\lambda = \sum_{\beta \in \mathcal{P}_E} c_{\lambda, \beta} Q^*_\beta$. Then by (2.10), for any peak composition $\beta$,
\[
c_{\lambda, \beta} = [Q^\prime_\lambda, Q^\prime_\beta] = [\pi(Q_\beta), Q^\prime_\lambda] = [Q_\beta, Q^\prime_\lambda] = \delta_{\beta, \lambda} (-1)^{f(\alpha(\beta))}.
\]
Remark 4.12. In [20], the author also gave the following refinement of the Schur’s Q-functions in terms of peak functions:

\[ Q_{\alpha} = \sum_{T \in S(\lambda)} K_{A(T)}, \]

where \( S(\lambda) \) is the set of standard shifted tableaux of shape \( \lambda \), and \( A(T) \) is the peak set of \( T \).

Let \( \tilde{Q}_{\alpha} = 2^{-\ell(\alpha)}Q_{\alpha}, \tilde{Q}_{\alpha}^* = 2^{\ell(\alpha)}Q_{\alpha}^* \), \( \alpha \in \mathcal{P} \mathcal{C} \). Then \( \pi(\tilde{Q}_{\alpha}) = 2^{-\ell(\alpha)}Q_{\alpha} = Q'_{\alpha} \) and

\[ Q_{\alpha} = \sum_{\alpha \in \mathcal{P} \mathcal{C}} (-1)^{\ell(\alpha)}\tilde{Q}_{\alpha}^*. \]

Therefore, we call \( \tilde{Q}_{\alpha}^* \)'s the quasisymmetric Schur’s Q-functions.

Example 4.13. Let \( K_{\alpha} = K_{D(\alpha)}, \alpha \in \mathcal{P} \mathcal{C} \). Denote by \( M_n(\Pi, \tilde{Q}) \) and \( M_n(\tilde{Q}^*, K) \) the corresponding transition matrices. Then \( M_n(\Pi, \tilde{Q}) = M_n(\tilde{Q}, \Pi)^{-1} = M_n(\tilde{Q}^*, K)^T \), and we have

<table>
<thead>
<tr>
<th>( n )</th>
<th>index</th>
<th>( M_n(\tilde{Q}, \Pi) )</th>
<th>( M_n(\Pi, \tilde{Q}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>21, 3</td>
<td>1 0</td>
<td>1 0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0 1</td>
<td>0 1</td>
</tr>
<tr>
<td>4</td>
<td>2^2, 31, 4</td>
<td>1 -1 0</td>
<td>1 1 0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0 1 0</td>
<td>0 1 0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0 0 1</td>
<td>0 0 1</td>
</tr>
<tr>
<td>5</td>
<td>2^21, 23, 32, 41, 5</td>
<td>1 0 -1 1 0</td>
<td>1 0 1 0 0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0 1 -1 0 0</td>
<td>0 1 1 0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0 0 1 -1 0</td>
<td>0 0 1 1 0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0 0 0 1 0</td>
<td>0 0 0 1 0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0 0 0 0 1</td>
<td>0 0 0 0 1</td>
</tr>
<tr>
<td>6</td>
<td>2^3, 231, 24, 321, 3^2, 42, 51, 6</td>
<td>1 -1 0 -1 0 1 -1 0</td>
<td>1 1 0 2 1 1 0 0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0 1 0 -1 -1 1 0</td>
<td>0 1 0 1 1 1 0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0 0 1 0 -1 1 0</td>
<td>0 0 1 1 1 0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0 0 0 1 0 -1 1</td>
<td>0 0 0 1 1 0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0 0 0 0 1 -1 0</td>
<td>0 0 0 1 0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0 0 0 0 0 1</td>
<td>0 0 0 0 0 1</td>
</tr>
</tbody>
</table>
Conjecture 4.14. \([\Pi_\alpha]_{\alpha \in \mathbb{P}C}\) has a positive, integral and unitriangular expansion in \([\bar{Q}_\alpha]_{\alpha \in \mathbb{P}C}\). Dually, \([\bar{Q}^*_\alpha]_{\alpha \in \mathbb{P}C}\) has such an expansion in peak functions \([K_\alpha]_{\alpha \in \mathbb{P}C}\).

Finally we discuss the expansion of quasisymmetric Schur’s \(P\)-functions in terms of the monomial \(M_\nu\)'s or fundamental \(F_\nu\)'s. Note that by the dual of the expansion formula (4.2), we have

\begin{equation}
Q^*_\alpha = \sum_{\beta \in \mathbb{P}C} \left( \sum_{T \in \mathbb{P}C(\alpha, \beta)} 2^{p(T) - m(T)} \right) Q_\beta^*
\end{equation}

for any \(\alpha \in \mathbb{P}C\), where \(Q_\beta^*\) is the dual of \(Q_\beta\) in \(\mathbb{B}\). Meanwhile,

\[Q^*_\alpha = \sum_{\beta \in \mathbb{P}C} \langle H_\beta, Q^*_\alpha \rangle M_\beta = \sum_{\beta \in \mathbb{P}C} [Q^*_\beta, Q^*_\alpha] M_\beta\]

\[= \sum_{\beta \in \mathbb{P}C} \langle R_\beta, Q^*_\alpha \rangle F_\beta = \sum_{\beta \in \mathbb{P}C} [\Theta(R_\beta), Q^*_\alpha] F_\beta.
\]

In order to compute the coefficient \([Q_\beta, Q^*_\alpha]\) via (4.7), one needs to rewrite \(Q_\beta\) as an integral linear combination of \(Q_\gamma\), \(\gamma \in \mathbb{P}C\) via relation (2.3). Note that \(\beta \leq \gamma\) for all these \(\gamma \in \mathbb{P}C\). Therefore, combining with (4.7), we know that \([Q_\beta, Q^*_\alpha]\) = 0 unless \(\beta \leq \alpha\). Furthermore, since \(\Theta(R_\beta) = \sum_{\gamma \geq \beta} (-1)^{\ell(\beta) - \ell(\gamma)} Q_\gamma\) and \(\gamma \geq \beta \Rightarrow \gamma \geq \beta\), \([\Theta(R_\beta), Q^*_\alpha]\) = \(\sum_{\gamma \geq \beta} (-1)^{\ell(\beta) - \ell(\gamma)} [Q_\gamma, Q^*_\alpha]\) = 0 unless \(\beta \leq \alpha\).

Now we assume that Conjecture 4.14 holds. Since

\[K_\alpha = \sum_{\beta \in \mathbb{P}C} 2^{\ell(\beta)} M_\beta, \ \alpha \in \mathbb{P}C_n,
\]

we know that \([Q_\beta, Q^*_\alpha]\), \(\beta \in \mathbb{C}\) are all nonnegative integers. On the other hand, by the formula

\[\Theta(R_\alpha) = \sum_{\beta \in \mathbb{P}C_n} 2^{\ell(\beta)} \Pi_\beta, \ \alpha \in \mathbb{C}_n
\]

in [7, (6)], the coefficients \([\Theta(R_\beta), Q^*_\alpha]\), \(\beta \in \mathbb{C}\) are also nonnegative integers. In summary, we have

**Proposition 4.15.** If Conjecture 4.14 holds, then for any \(\alpha \in \mathbb{P}C\), \(Q^*_\alpha\) has a positive, integral expansion in both \(M_\beta\)'s and \(F_\beta\)'s with \(\beta \leq \alpha\) and the leading coefficient equal to 1.

For example, \(Q_{123} = 2Q_{24} - Q_{231} + Q_{213} = 4Q_{23} + Q_{231} + Q_{213}\), thus \([Q_{123}, Q^*_{321}] = 2\). Similarly, one can compute all the coefficients and get that \(Q^*_{321} = M_{321} + M_{312} + 2(M_{31^2} + M_{231}) + 4M_{23} + 8M_{21^2} + 2M_{213} + 8(2M_{21^2} + M_{21^2}) + 16M_{21^3} + M_{14^2} + 3M_{13^2} + 6M_{131^2} + 2M_{123} + 8(4M_{13^2} + M_{12^2} + 16M_{121^3} + 4M_{1^23} + 8M_{1^22} + 16M_{1^21^2} + 4M_{1^33} + 16(M_{1^2^2} + M_{1^2}) + 32M_{1^6}

= F_{321} + 2F_{312} + 2F_{231} + 4F_{23} + 2(2F_{2^21^2} + F_{213}) + 3F_{2121} + F_{21^2} + F_{141} + 3F_{132} + 2(F_{131^2} + F_{123}) + 4F_{12^21} + 2F_{1212} + F_{1^231} + F_{1^2^2}.

\]
Acknowledgments

We would like to thank Weiqiang Wang for stimulative discussions. NJ thanks the partial support of Simons Foundation grant 198129 and NSFC grant 11271138 during this work.

References


Department of Mathematics, North Carolina State University, Raleigh, NC 27695-8205, USA
E-mail address: jing@math.ncsu.edu

Department of Mathematics, South China University of Technology, Guangzhou 510640, China
E-mail address: scynli@scut.edu.cn