A Linear Time Algorithm for Solving the Incidence Coloring Problem on Strongly Chordal Graphs *

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Abstract

An incidence of G consists of a vertex and one of its incident edge in G. The incidence coloring problem is a variation of vertex coloring problem. The problem is to find the minimum number (called incidence coloring number) of colors assigned to every incidence of G so that the adjacent incidences are not assigned the same color. In this paper, we propose a linear time algorithm for incidence-coloring a strongly chordal graph when a strong elimination ordering is given. Further, we prove that the incidence coloring number of a strongly chordal graph is \(\Delta(G) + 1\), where \(\Delta(G)\) is the maximum degree of G.

Keywords: strongly chordal graphs, incidence coloring problem, strongly elimination ordering.

1. Introduction

The incidence set of a graph G = (V, E) is defined as \(I(G) = \{(v, e) : v \in V, e \in E, v\) is incident with \(e\}\), where V and E are the vertex and edge, respectively, sets of G. Two incidences \((v_1, e_1)\) and \((v_2, e_2)\) are adjacent if one of the following conditions holds: (i) \(v_1 = v_2\), (ii) \(e_1 = e_2\), or (iii) the edge \(v_1v_2\) equals to \(e_1\) or \(e_2\).

An incidence coloring function \(\sigma\) of G is a mapping from \(I(G)\) to a color set such that adjacent incidences of G are assigned different colors. For example, \(\sigma(v, e) = c\) means that the incidence \((v, e)\) is colored with \(c\). The incidence coloring number of G, denoted by \(\chi'_i(G)\), is the smallest size of the color set. The incidence coloring problem is to find the incidence coloring number of a given graph. In [3], Brualdi and Massey first defined the problem as a variation of vertex coloring problem.

Let \(\Delta(G)\) be the maximum degree of a graph G. Then, it is obvious that \(\chi'_i(G) \geq \Delta(G) + 1\) if G has at least one edge. Brualdi and Massey have proved that the incidence coloring number of a graph G is at most \(2\Delta(G)\) [3]. They also conjectured that any graph G can be incidence-colored with \(\Delta(G) + 2\) colors. However, their conjecture was disproved by Guiduli [8].

In [8], Guiduli also showed that the incidence coloring problem is a special case of directed star arboricity which was introduced by Algor and Alon [1]. Meanwhile, the directed star arboricity problem has applications in the WDM (Wavelength Division Multiplexing) of a star optical network [2].

As for the incidence coloring number of special classes of graphs, the following results are well-known:

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A chordal graph is Farber [6] form a subclass of chordal graphs, which were first introduced by chordal graphs. Strongly studied subclass of perfect graphs [7].

For every tree \( T \) of order \( n \geq 2 \), \( \chi_i(T) = \Delta(T) + 1 \) [3].

For every Halin graph \( G \) with \( \Delta(G) \geq 5 \), \( \chi_i(G) = \Delta(G) + 1 \) [13].

For every outerplanar graph \( G \) with \( \Delta(G) \geq 4 \), \( \chi_i(G) = \Delta(G) + 1 \) [13].

In [12], Shiu et al. showed that Brualdi’s conjecture holds for cubic Hamiltonian graphs and some other cubic graphs. In [10], Maydanskiy proved that \( \chi_i(G) \leq 5 \) for any graph with \( \Delta(G) = 3 \). In [9], Huang et al. showed that square mesh, hexagonal meshes and honeycomb meshes can be incidence-colored with \( \Delta(G) + 1 \) colors. In [5], Dolama et al. proved that incidence coloring of every \( k \)-degenerated graph \( G \) is at most \( \Delta(G) + 2k - 1 \).

A graph is chordal if and only if there is no induced cycle of length greater than three. Chordal graphs form an important and widely studied subclass of perfect graphs [7]. Strongly chordal graphs which were first introduced by Farber [6] form a subclass of chordal graphs. A chordal graph is strong if every even cycle of length at least 6 has an odd chord (i.e., a chord joining vertices that are separated by an odd number of edges in the cycle). In Figure 1(a), for example, vertices \( v_1, v_2, v_4, v_5, v_6 \) and \( v_3 \) form a 6-cycle. Edge \( v_3v_4 \) is an odd chord that divides the cycle into two parts and each part has 3 edges.

The family of strongly chordal graphs properly contains many special classes of graphs, such as directed path graphs, interval graphs, and power of trees. Also, we know that there are many physical problems that can be represented and characterized by the above classes of graphs.

In this paper, we shall propose a linear time algorithm for incidence-coloring a strongly chordal graph \( G \) in case that a strong elimination ordering is given. In addition, we also prove that the incidence coloring number of a strongly chordal graph \( G \) is \( \Delta(G) + 1 \).

The remaining part of this paper is organized as follows. In Section 2, we introduce properties of strongly chordal graphs and the main result of the paper. Section 3 contains our incidence coloring algorithm and an example. The last section gives our concluding remarks.

### 2. Preliminaries

Throughout this paper we assume that the graph \( G = (V,E) \) is undirected, connected and simple. Let \( N(v) \) denote the set of adjacent vertices (or neighbors) of \( v \) in \( G \) and \( N[v] = \{v\} \cup N(v) \) is the close neighbor set of \( v \). Then, the degree of \( v \), denoted by \( d(v) \), is the cardinality of \( N(v) \).

A vertex \( v \) is called simplicial if \( N(v) \) (as well as \( N[v] \)) induces a clique in \( G \). A perfect elimination ordering (PEO) \( v_1 \prec v_2 \prec \cdots \prec v_n \) is an ordering of vertices in \( G \) such that for each \( i (1 \leq i \leq n) \), \( N[v_i] \) induces a clique in \( G_i \). Note that \( G_i \) is the subgraph induced by the vertex set \( \{v_i, v_{i+1}, \ldots, v_n\} \). (Thus, \( G_1 = G \) and \( G_n \) is a trivial graph that only contains \( v_n \).) A well-known characteristic of chordal graphs is that a graph is chordal if and only if it has a perfect elimination ordering [7].

The ordering of vertices \( v_1 \prec v_2 \prec \cdots \prec v_n \) is called a strong elimination ordering (abbreviated SEO) if it is a perfect elimination ordering and, for each \( v_i \prec v_j \prec v_k \), if \( v_j, v_k \in N[v_i] \), then \( N[v_j] \subseteq N[v_k] \) in \( G_i \). The graph shown in Figure 1(a) is strongly chordal and the vertices are labeled in accord with an SEO. For example, \( G_5 \) is the subgraph induced by vertices \( v_6, v_7, v_8 \) and \( v_9 \). Both \( v_7 \) and \( v_8 \) are neighbors of \( v_5 \). Since \( v_7 \prec v_8 \), \( N[v_7] \subseteq N[v_8] \) in \( G_5 \).

**Theorem 1** [6] A graph is strongly chordal if and only if its vertices admit a strong elimination ordering.

There are many algorithms to find an SEO when the input graph is strongly chordal. The
fastest algorithm that takes $O(m \log n)$ time for recognizing a strongly chordal graph is proposed by Paige and Tarjan [11], where $m$ and $n$ are the size and order, respectively, of the graph. The algorithm also outputs an SEO when the input graph is strongly chordal.

The basic idea proposed in this paper is to assign color to every incidence based on a reversed SEO of the vertices in a strongly chordal graph. Assume that an SEO is given. We have known that $N[v_i]$ induces a clique in $G_i$. Applying a previous result proposed in [3], we can easily derive that $\chi_i(N[v_i]) = d(v_i) + 1$ in $G_i$.

By the definition of incidence coloring, it is feasible to assign the same color to incidence $(u, uv)$ for all $u \in N(v)$. The far incidence color set of vertex $v$, denoted by $A(v)$, is the set of colors assigned to such incidences $(u, uv)$ for all $u \in N(v)$. Further, the near incidence color set of vertex $v$, denote by $I(v)$, is the set of colors which are assigned to incidences $(v, uv)$. Due to the adjacent condition (i) of incidence coloring, $|I(v)| = d(v)$. In order not to violate the adjacent condition (ii) of incidence coloring, $A(v) \cap I(v) = \emptyset$. As a result, we define $C(v) = A(v) \cup I(v)$ as the incidence color set of vertex $v$, and obviously, $|C(v)| = |I(v)| + |A(v)| = d(v) + 1$ if $|A(v)| = 1$.

In a complete graph or a clique within a graph, every vertex $v$ owns the same incidence color set $C(v)$ and distinct far incidence color $A(v)$. As for a strongly chordal graph, we first derive the following lemma.

**Lemma 2** Let $G$ be a strongly chordal graph and the ordering of vertices $v_1 \prec v_2 \cdots \prec v_n$ is an SEO of $G$. If $|A(v_i)| = 1$ for $i = 1, 2, \ldots, n$, then $C(v_j) \subseteq C(v_k)$ in $G_i$ for $v_i \prec v_j \prec v_k$.

**Proof.** Based on the underlying property of SEO, if $v_j, v_k \in N[v_i]$ and $v_j \prec v_k$, then $N[v_j] \subseteq N[v_k]$ in $G_i$. Since $N[v_i]$ induced a clique in $G_i$, $A(v_j) \subseteq I(v_k)$ and $A(v_k) \subseteq I(v_j)$ for any $v_j, v_k \in N[v_i]$. However, $v_k$ might have more neighbors in $G_i$. That is, $I(v_j)$ may contain more colors than $I(v_j)$ contains. Therefore, $C(v_j) \subseteq C(v_k)$ in $G_i$. 

For each vertex $v$, the highest neighbor of $v$, denoted by $h(v)$, is $v$’s neighbor with the highest order in the SEO. We can easily derive from Lemma 2 that for all $v_j \in N[v_i]$, $C(v_j) \subseteq C(h(v_i))$ in $G_i$ when $|A(v_i)| = 1$, for $i = 1, 2, \ldots, n$, coloring approach is applied.

Now, we are in a position to present our main result.

**Theorem 3** Let $v_1 \prec v_2 \cdots \prec v_n$ be an SEO of strongly chordal graph $G$. Then, $\chi_i(G_i) = \Delta(G_i) + 1$ for $1 \leq i \leq n - 1$, where $n$ is the number of vertices in $G$.

**Proof.** We prove this theorem by mathematical induction on $G_{n-1}$. When $i = 1$, $G_{n-1}$ is clique $K_2$ that contains vertices $v_n$ and $v_{n-1}$. It is obviously true that $\chi_i(G_{n-1}) = \Delta(G_{n-1}) + 1 = 2$ since two incidence colors are required for $K_2$.

Suppose $i = k - 1$, $\chi_i(G_{n-(k-1)}) = \Delta(G_{n-(k-1)}) + 1$ is true. There are two conditions after $v_{n-k}$ is added to $G_{n-(k-1)}$. One condition is that the added vertex causes an increase of the maximum degree of $G_{n-(k-1)}$. That is, $\Delta(G_{n-k}) = \Delta(G_{n-(k-1)}) + 1$. Based on our incidence coloring approach, incidence $(v_{n-k}, v_{n-k}v_j)$ is colored with $A(v_j)$ for every vertex $v_j \in N(v_{n-k})$ and $v_{n-k} \prec v_j$. It is inevitable to assign a new color to $A(v_{n-k})$ since $C(h(v_{n-k}))$ contains all available incidence colors. As a result, $\chi_i(G_{n-k}) = \chi_i(G_{n-(k-1)}) + 1 = \Delta(G_{n-(k-1)}) + 2 = \Delta(G_{n-k}) + 1$.

The other condition is that the added vertex does not increase the maximum degree of $G_{n-(k-1)}$. That is, $\Delta(G_{n-k}) = \Delta(G_{n-(k-1)})$. Then, there must be a vertex $v_t (t > n-k)$ such that $v_t \notin N[v_{n-k}]$ and $d(v_t) = \Delta(G_{n-k})$. Since $\chi_i(G_{n-(k-1)}) = \Delta(G_{n-(k-1)}) + 1$, $d(h(v_{n-k})) < d(v_t)$ in $G_{n-(k-1)}$. Then, $C(h(v_{n-k})) \subseteq C(v_t)$ in $G_{n-(k-1)}$. On the other hand, by Lemma 2, $C(v_j) \subseteq C(h(v_{n-k}))$ for all $v_j \in N[v_{n-k}]$ in $G_{n-k}$. Therefore, at least one color exists in the difference set $C(v_t) \setminus C(h(v_{n-k}))$ that is available for $A(v_{n-k})$ in $G_{n-k}$. In this case, $\chi_i(G_{n-k}) = \chi_i(G_{n-(k-1)}) = \Delta(G_{n-(k-1)}) + 1 = \Delta(G_{n-k}) + 1$. 

Furthermore, since $G_1 = G$, we have the following corollary.
Corollary 4 For a strongly chordal graph $G$, $\chi_i(G) = \Delta(G) + 1$.

3. The Incidence Coloring Algorithm

In this section, we present our incidence coloring algorithm for a strongly chordal graph $G$ when an SEO of $G$ is given. Recall that $C(v) = A(v) \cup I(v)$ is the incidence color set of vertex $v$. Actually, it is sufficient to color all incidences if $A(v)$ is determined for every vertex $v$. The near incidence color set $I(v)$ is neglected in our design, since it can be obtained after all $A(v)$ in $G$ are determined. We use $k$ to count the number of used colors. Meanwhile, set $T$ contains all used colors. Further, current available colors for $A(v)$ are contained in set $S$. In addition, the highest neighbor $h(v)$ of all vertices in $G$ can be computed in $O(n)$ time [4]. Based on the spirit of Theorem 3, we build the following algorithm.

Algorithm InciColor_Strong
Input: A strongly chordal graph $G$ with an SEO $v_1 \prec v_2 \prec \cdots \prec v_n$.
Output: Incidence coloring number $\chi_i(G)$ and the far incidence color $A(v)$ of every vertex $v$ in $G$.

Step 1. Initialize the variables related to $v_n$:
\[ k \leftarrow 1, \quad T \leftarrow \{c_1\}, \quad A(v_n) \leftarrow \{c_1\}, \quad \Delta(G_n) \leftarrow 0, \quad d(v_n) \leftarrow 0. \]

Step 2. Determine $A(v_i)$ for $v_i$ whose attendance will increase the maximum degree in $G_i$.
For $i = n - 1$ downto 1 do
Substep 2.1 If the added vertex $v_i$ increases the maximum degree, i.e.,
\[ d(h(v_i)) = \Delta(G_{i+1}), \]
then assign a new color to $A(v_i)$ and update data:
\[ k \leftarrow k + 1, \quad T \leftarrow T \cup \{c_k\}, \quad A(v_i) \leftarrow \{c_k\}, \quad \Delta(G_i) \leftarrow \Delta(G_{i+1}) + 1. \]
Substep 2.2 Update the current degree of vertices in clique $N[v_i]$ (no matter whether the maximum degree is increased or not):
For each $v_j \in N(v_i)$ and $j > i$ do
\[ d(v_j) \leftarrow d(v_j) + 1, \quad d(v_i) \leftarrow d(v_i) + 1. \]
Enddo
\[ \chi_i(G) \leftarrow k; \]
Step 3. Determine $A(v_i)$ for remaining $v_i$ and update incidence coloring data:
For $i = n$ downto 1 do
Substep 3.1 Remove $A(v_i)$ from $T$, i.e., $S \leftarrow T \setminus A(v_i)$.
Substep 3.2 Process all neighbors of $v_i$ according to the reversed SEO.
For each $u \in N(v_i)$ (in decreasing $i$) do
Substep 3.2.1 If $A(u)$ is not assigned, then choose any color from $S$ and assign the color to $A(u)$.
Substep 3.2.2 Remove $A(u)$ from $S$, i.e., $S \leftarrow S \setminus A(u)$.
Substep 3.2.3 Assign colors to incidences:
If $\sigma(u, v_i)$ is not assigned, then $\sigma(u, v_i) \leftarrow A(v_i)$.
If $\sigma(v_i, v_i u)$ is not assigned, then $\sigma(v_i, v_i u) \leftarrow A(u)$.
Enddo

An example is given to illustrate Algorithm InciColor_Strong. Considering the strongly chordal graph $G$ shown in Figure 1(a), an SEO $v_1 \prec v_2 \prec v_3 \prec v_4 \prec v_5 \prec v_6 \prec v_7 \prec v_8 \prec v_9$ of vertices in $G$ is shown in Figure 1(b). We start with vertex $v_9$ and assign $\{c_1\}$ to $A(v_9)$. When vertex $v_8$ is processed, the maximum degree of subgraph $G_8$ is increased by one (comparing with $G_9$), and $\{c_2\}$ is assigned to $A(v_8)$. Subsequent vertices $v_7, v_6, v_5$ and $v_4$ are processed in a similar way due to Substep 2.1. So, we obtain $\Delta(G_4) = 6$. When vertex $v_3$ is processed, $h(v_3) = v_6$ and $d(v_6) < \Delta(G_4)$ in $G_4$. Thus, we only update the current degree of vertices in clique $N[v_3]$. Vertices $v_2$ and $v_1$ are processed in a similar way owing to Substep 2.2. At the end of Step 2, we obtain $\chi_i(G) = 6$ and $T = \{c_1, c_2, c_3, c_4, c_5, c_6\}$. In the following step, $v_9$ is processed. By Substep 3.1, $A(v_9)$ is removed from $S$. Then, $A(v_8)$
Algorithm InciColor\_Strong is an implementation of the constructive proof of Theorem 3. However, the computations of $C(v_i) \setminus C(h(v_i))$ is not efficient. To avoid redundant computations and achieve the linear time requirement, we first determine $A(v_i)$ for vertex $v_i$ whose attendance results in $\Delta(G_i) > \Delta(G_{i+1})$. Later on, when we update the incidence color data, the pendant $A(v_i)$’s are determined as a whole.

In Algorithm InciColor\_Strong, Substeps 2.1 and 2.2 take $O(n)$ and $O(e)$ time, respectively, where $e$ is the size of $G$. Step 3 also takes $O(e)$ time, and the overall time requirement is $O(e + n)$.

Theorem 5 Algorithm InciColor\_Strong can correctly incidence-color a strongly chordal graph in linear time.

Proof. Since an SEO can assure that $h(v_i)$ has the largest degree among $N[v_i]$ in the induced subgraph $G_i$, Step 2 of the algorithm can compute the maximum degree of $G_i$ as vertex $v_i$ is added into the subgraph. Then, the incidence coloring number, as well as the assigned colors, of $G_i$ are obtained.

In Step 3, we color every incidence of the graph according to the reversed SEO. When the far incidence color $A(v_i)$ of a vertex $v_i$ is not yet assigned, the degree of $h(v_i)$ in $G_i$ is at most $\Delta(G_i)$. It turns out that there are $\Delta(G_i) + 1$ colors available in $G_i$ and at least one color can be assigned to $A(v_i)$. Consequently, Algorithm InciColor\_Strong is correct. □

4. Concluding Remarks

We have proposed a linear time algorithm for incidence-coloring a strongly chordal graph when an SEO of the vertex set is given. The incidence coloring number of a strongly chordal graph $G$ is proved to be $\Delta(G) + 1$. Our future research work is focused on extending the solution to chordal graphs. Besides, another research direction is to find out other classes of graphs which have the property of $\chi_i(G) = \Delta(G) + 1$.

References


