Spanning local tournaments in locally semicomplete digraphs

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Abstract

We investigate the existence of a spanning local tournament with possibly high connectivity in a highly connected locally semicomplete digraph. It is shown that every \((3k - 2)\)-connected locally semicomplete digraph contains a \(k\)-connected spanning local tournament. This improves the result of Bang-Jensen and Thomassen for semicomplete digraphs and of Bang-Jensen [1] for locally semicomplete digraphs.

Keywords: Digraph; Connectivity

1. Terminology and preliminaries

We denote by \(V(D)\) and \(E(D)\) the vertex set and the arc set of a digraph \(D\), respectively. If \(xy\) is an arc of \(D\), then we say that \(x\) dominates \(y\). More generally, if \(A\) and \(B\) are two disjoint subdigraphs of \(D\) such that every vertex of \(A\) dominates every vertex of \(B\), then we say that \(A\) dominates \(B\), denoted by \(A \rightarrow B\). In addition, if \(A \rightarrow B\), but there is no arc from \(B\) to \(A\), then we say that \(A\) strictly dominates \(B\), denoted by \(A \rightarrow B\).

The outset of a vertex \(x \in V(D)\) is the set \(N^+(x) = \{y \mid xy \in E(D)\}\). Similarly, \(N^-(x) = \{y \mid yx \in E(D)\}\) is the inset of \(x\). More generally, for a subdigraph \(A\) of \(D\), we define its outset by \(N^+(A) = \bigcup_{x \in V(A)} N^+(x) - A\) and its inset by \(N^-(A) = \bigcup_{x \in V(A)} N^-(x) - A\) (if necessary, we write \(N^+_D(A)\) and \(N^-_D(A)\) instead of \(N^+(A)\) and \(N^-(A)\), respectively). Every vertex of \(N^+(A)\) is called an out-neighbour of \(A\) and every vertex of \(N^-(A)\) is an in-neighbour of \(A\).

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In this paper we shall consider finite digraphs without loops and multiple arcs. The numbers \(d^+(x) = |N^+(x)|\) and \(d^-(x) = |N^-(x)|\) are called outdegree and indegree of \(x \in V(D)\), respectively (if necessary, we write \(d^+_D(x)\) and \(d^-_D(x)\) instead of \(d^+(x)\) and \(d^-(x)\), respectively). If \(d^+(x) = d^-(x) = r\) holds for every vertex \(x\) of \(D\), then we say that \(D\) is \(r\)-regular.

Paths and cycles in a digraph are always directed. A path from \(x\) to \(y\) is called an \((x,y)\)-path. Two \((x,y)\)-paths are internally disjoint if they have only the vertices \(x\) and \(y\) in common.

The subdigraph of \(D\) induced by a subset \(A\) of \(V(D)\) is denoted by \(D(A)\). In addition, \(D - A = D(V(D) - A)\).

A strong component \(H\) of \(D\) is a maximal subdigraph such that for any two vertices \(x,y \in V(H)\), the subdigraph \(H\) contains a path from \(x\) to \(y\) and a path from \(y\) to \(x\). The digraph \(D\) is strong or strongly connected, if it has only one strong component, and \(D\) is \(k\)-connected if for any set \(A\) of at most \(k - 1\) vertices, the subdigraph \(D - A\) is strong.

If \(D\) is strong and \(S\) is a subset of \(V(D)\) such that \(D - S\) is not strong, then we say that \(S\) is a separating set of \(D\). A separating set \(S\) of \(D\) is minimal if for any proper subset \(S'\) of \(S\), the subdigraph \(D - S'\) is strong.

A digraph is connected, if its underlying graph is connected. In this paper, we only consider connected digraphs.

If we replace every arc \(xy\) of \(D\) by \(yx\), then we call the resulting digraph (denoted by \(D^{-1}\)) the converse digraph of \(D\).

A subdigraph of \(D\) is called a spanning subdigraph if it contains all vertices of \(D\).

A digraph is semicomplete if for any two different vertices \(x\) and \(y\), there is at least one arc between them. A tournament is a semicomplete digraph without a cycle of length two.

In 1990, Bang-Jensen [1] introduced a very interesting generalization of tournaments – the class of locally semicomplete digraphs. A digraph \(D\) is locally semicomplete, if for every vertex \(x\), \(D(N^+(x))\) and \(D(N^-(x))\) are both semicomplete. A local tournament is a locally semicomplete digraph without a cycle of length two. It is obvious that the class of locally semicomplete digraphs is a superclass of local tournaments.

We note that every induced subdigraph of a locally semicomplete digraph is locally semicomplete. In addition, the converse of a locally semicomplete digraph is also locally semicomplete.

It has been shown that locally semicomplete digraphs have many properties in common with tournaments. The first of them is the following:

**Theorem 1.1** (Bang-Jensen [1]). Every strong locally semicomplete digraph contains a hamiltonian cycle.

In other words, every strong locally semicomplete digraph contains a spanning local tournament that is also strong.
In general, the existence of a spanning local tournament with possibly high connectivity in a highly connected locally semicomplete digraph is an interesting problem.

Bang-Jensen and Thomassen (cf. [1]) proved that every $5k$-connected semicomplete digraph contains a $k$-connected spanning tournament. Bang-Jensen [1] generalized this result to locally semicomplete digraphs and proved that every $5k$-connected locally semicomplete digraph contains a $k$-connected spanning local tournament. However, he conjectured the following:

**Conjecture 1.2** (Bang-Jensen [1]). Every $2k$-connected locally semicomplete digraph contains a $k$-connected spanning local tournament.

In this paper we prove that a $(3k - 2)$-connected locally semicomplete digraph contains a $k$-connected spanning local tournament (see Theorem 2.2). So, Conjecture 1.2 is also true for $k = 2$.

To prove our main results we need the following results.

**Theorem 1.3** (Bang-Jensen [1]). Let $D$ be a connected locally semicomplete digraph that is not strong. Then the following holds:

(a) If $A$ and $B$ are two strong components of $D$, then either there is no arc between them or $A \Rightarrow B$ or $B \Rightarrow A$.

(b) If $A$ and $B$ are two strong components of $D$ such that $A$ dominates $B$, then $A$ and $B$ are both semicomplete digraphs.

(c) The strong components of $D$ can be ordered in a unique way $D_1, D_2, \ldots, D_p$ such that there are no arcs from $D_j$ to $D_i$ for $j > i$, and $D_i$ dominates $D_{i+1}$ for $i = 1, 2, \ldots, p - 1$.

The unique sequence $D_1, D_2, \ldots, D_p$ of the strong components of $D$ in Theorem 1.3 (c) is called the strong decomposition of $D$ with the initial component $D_1$ and the terminal component $D_p$.

**Theorem 1.4** (Guo and Volkmann [3]). Let $D$ be a connected locally semicomplete digraph that is not strong and let $D_1, \ldots, D_p$ be the strong decomposition of $D$. Then $D$ can be decomposed in $r \geq 2$ subdigraphs $D_1', D_2', \ldots, D_r'$ as follows:

- $D_1' = D_p$, $\lambda_1 = p$,
- $\lambda_{i+1} = \min\{ j \mid N^+(D_j) \cap V(D_i') \neq \emptyset \}$

and

- $D_{i+1}' = D(V(D_{\lambda_{i+1}}) \cup V(D_{\lambda_{i+1}+1}) \cup \cdots \cup V(D_{\lambda_1-1})).$

Furthermore, the subdigraphs $D_1', D_2', \ldots, D_r'$ satisfy the following:

(a) $D_i'$ consists of some strong components of $D$ and it is semicomplete for $i = 1, 2, \ldots, r$.
(b) $D_{i+1}'$ dominates the initial component of $D_i'$ and there exists no arc from $D_i'$ to $D_{i+1}'$ for $i = 1, 2, \ldots, r - 1$;
(c) if $r \geq 3$, then there is no arc between $D_i'$ and $D_j'$ for $i, j$ satisfying $|j - i| \geq 2$.

For a connected, but not strongly connected locally semicomplete digraph $D$, the unique sequence $D_1', D_2', \ldots, D_r'$ defined in Theorem 1.4 is called the semicomplete decomposition of $D$.

Lemma 1.5 (Bang-Jensen [1]). Let $D$ be a strong locally semicomplete digraph and let $S$ be a minimal separating set of $D$. Then $D - S$ is connected.

Lemma 1.6. Let $D$ be a $k$-connected digraph and let $A$ be a vertex set with $A \cap V(D) = \emptyset$. If we add some arcs between $A$ and $D$ such that $\min\{d^+(v), d^-(v)\} \geq k + 1$ for every $v \in A$, then the resulting digraph $H$ is also $k$-connected. Moreover, if $H$ is not $(k + 1)$-connected, then every minimum separating set of $H$ is also a minimum separating set of $D$.

Proof. Suppose that $H$ is not $(k + 1)$-connected. Let $S$ be a minimum separating set of $H$. Then $|S| \leq k$. Because of $d^+(v), d^-(v) \geq k + 1$ for each $v \in A$, every vertex of $A - S$ has at least one out-neighbour and at least one in-neighbour in the subdigraph $D - S$. It follows that $D - S$ is not strong. Since $D$ is $k$-connected, we see that $|S| = k$. Therefore, $H$ is $k$-connected and every minimum separating set of $H$ is also a minimum separating set of $D$. □

Lemma 1.7. Let $D$ be a $k$-connected digraph and $xy \in E(D)$. If $D$ contains at least $k + 1$ internally disjoint $(x, y)$-paths, each of which is of length at least 2, then the digraph $D'$ obtained from $D$ by replacing $xy$ with $yx$ is also $k$-connected. Furthermore, if $D'$ is not $(k + 1)$-connected, then every minimum separating set of $D'$ is also a separating set of $D$.

Proof. Suppose that $D'$ is not $(k + 1)$-connected. Let $S$ be a minimum separating set of $D'$ with $|S| \leq k$. There are two vertices $a$ and $b$ such that $D' - S$ contains no $(a, b)$-path. If $S \cap \{x, y\} = \emptyset$, then it is obvious that $D - S$ also contains no $(a, b)$-path. Thus $D - S$ is not strong. If $S \cap \{x, y\} = \emptyset$, then it is easy to see that $x$ and $y$ are in the same strong component of $D' - S$, and hence $D - S$ contains no $(a, b)$-path. This means that $D - S$ is not strong. Therefore, every minimum separating set of $D'$ is also a separating set of $D$. Since $D$ is $k$-connected, $|S| = k$ holds. Thus $D'$ is $k$-connected. □

2. Main results

We first consider the existence of a spanning local tournament with possibly high connectivity in a highly connected locally semicomplete digraph which is not semicomplete.
Theorem 2.1. Let $D$ be a $(2k - 1)$-connected locally semicomplete digraph which is not semicomplete. Then $D$ contains a $k$-connected spanning local tournament.

Proof. We shall show the statement by induction on the number $k$. If $k = 1$, then, by Theorem 1.1, $D$ has a Hamiltonian cycle and we are done. So we may assume that $k \geq 2$. Since $D$ is not semicomplete, $D$ has a minimal separating set $S$ such that $D - S$ is not semicomplete. According to Lemma 1.5, $D - S$ is connected. Let $D_1, D_2, \ldots, D_p$ be the strong decomposition and let $D'_1, D'_2, \ldots, D'_r$ be the semicomplete decomposition of $D - S$, respectively. Obviously, $r \geq 3$ and $|S|, |V(D'_j)| \geq 2k - 1 \geq 3$. Using the definition of locally semicomplete digraphs and Theorem 1.4, it is not difficult to show that $D'_j \Rightarrow D'_l \Rightarrow S \Rightarrow D_1$.

Let $x_1 \in V(D_1), x_2 \in V(D_p)$ and $D' = D - \{x_1, x_2\}$. It is clear that $D'$ is $(2(k - 1) - 1)$-connected. By the induction hypothesis, $D'$ contains a $(k - 1)$-connected spanning local tournament $T'$. Now we consider the subdigraph $D''$ of $D$ obtained from $T'$ by adding the two vertices $x_1, x_2$ and all the arcs between $T'$ and $\{x_1, x_2\}$. From the choice of the two vertices $x_1, x_2$ and the assumption that $D$ is $(2k - 1)$-connected, we can destroy in $D''$ all cycles of length 2 between $T'$ and $\{x_1, x_2\}$ such that every vertex of $\{x_1, x_2\}$ has at least $k$ out-neighbours and $k$ in-neighbours in $T'$. Thus, we obtain a spanning local tournament $T$ of $D$. By Lemma 1.6, $T$ is $(k - 1)$-connected.

Suppose that $T$ is not $k$-connected. Then $T$ has a separating set $S'$ with $|S'| = k - 1$. By Lemma 1.6, $S'$ is also a separating set of $T'$. It follows that $x_1$ and $x_2$ do not belong to $S'$. Let $T_1, T_2, \ldots, T_q$ be the strong decomposition and let $T'_1, T'_2, \ldots, T'_r$ be the semicomplete decomposition of $T - S'$, respectively. Since $x_1$ and $x_2$ are not adjacent, $T - S'$ is not semicomplete. It follows that $t \geq 3$. Thus, $S'$ is the outset of $V(T_q)$ in $T$. Because $T$ is a spanning local tournament of $D$, $S'$ is also the outset of $V(T_q)$ in $D$. This contradicts the assumption that $D$ is $(2k - 1)$-connected. Therefore, $T$ is $k$-connected.

In the proof of the next theorem, we will use some ideas of Bang-Jensen (see the proof of Theorem 4.5 in [1]).

Theorem 2.2. A $(3k - 2)$-connected locally semicomplete digraph contains a $k$-connected spanning local tournament.

Proof. Let $D$ be a $(3k - 2)$-connected locally semicomplete digraph on $n$ vertices. If $D$ is not semicomplete, then we are done by Theorem 2.1, since $3k - 2 \geq 2k - 1$ for all $k \geq 1$. So we assume that $D$ is semicomplete. We shall prove the statement by induction on the number $k$.

If $k = 1$, then $D$ has a Hamiltonian cycle by Theorem 1.1. Assume thus $k \geq 2$. If $D$ is complete, then it is a simple matter that $D$ contains a $k$-connected spanning tournament. So we only need consider the case that $D$ is not complete. This implies that $n \geq 3k$. Because of $3k - 2 > 3(k - 1) - 2$, $D$ contains a $(k - 1)$-connected spanning tournament by the induction hypothesis.
Suppose that $D$ does not contain a $k$-connected spanning tournament. Let $T$ be chosen among all $(k - 1)$-connected spanning tournaments of $D$ such that the following holds:

1. The number of minimum separating sets of $T$ is as small as possible.
2. $T$ has a minimum separating set $S$ such that the number of strong components of $T - S$ is as small as possible.

Let $T_1, T_2, \ldots, T_p$ be the strong components of $T - S$. Suppose without loss of generality that $|V(T_1)|$ is as small as possible (otherwise, we consider the converse $D^{-1}$ of $D$). Clearly, the semicomplete decomposition of $T - S$ has exactly two components $T'_1$ and $T'_2$ and

$$2|V(T'_2)| \geq |V(T'_1)| + |V(T'_2)| = n - |S| \geq 3k - (k - 1) = 2k + 1.$$ 

It follows that $|V(T'_2)| \geq k + 1$. Let $A_1 = N^+_D(T'_2) \cap V(T'_1)$ and $A_2 = N^+_D(T'_1) \cap V(T'_2)$. Since $D$ is $(3k - 2)$-connected, $|A_i| \geq \min\{2k - 1, |V(T'_i)|\}$ holds for $i = 1, 2$. We shall show that there are two vertices $x \in V(T_p)$, $y \in V(T_v)$ satisfying the following conditions:

(a) $1 \leq \mu < v \leq p$ and $|V(T_p) \cup \cdots \cup V(T_v)| \geq 3$.
(b) $yx$ is an arc of $D$.
(c) $T$ contains at least $k$ internally disjoint $(x, y)$-paths each of length at least 2.

Suppose $|V(T'_1)| \geq 2k$. Let $A'_1$ be a subset of $A_1$ with $|A'_1| = 2k - 1$. Since $T'_1$ is strong, there is at least one arc from $T'_1 - A'_1$ to $A'_1$. So we have

$$\sum_{v \in A'_1} d^-_{T'_1}(v) = \frac{(2k - 1)(2k - 2)}{2} + 1.$$

It follows that $A'_1$ contains a vertex $y$ having at least $k$ in-neighbours $y_1, y_2, \ldots, y_k \in V(T'_1)$. Let $x$ be a vertex of $A_2$ such that $yx$ is an arc of $D$. Because of $x \Rightarrow T'_1$, $T$ contains $k$ internally disjoint $(x, y)$-paths $x \rightarrow y_i \rightarrow y$ for $i = 1, 2, \ldots, k$.

Suppose now that $|V(T'_1)| = x \leq 2k - 1$. It follows that $A_1 = V(T'_1)$.

Assume that $T'_1$ contains a vertex $y$ which has at least $[x/2] + 1$ in-neighbours in $T'_1$. Let $A'_2 = N^+_D(y) \cap A_2$. Since $D$ is $(3k - 2)$-connected, $|A'_2| \geq (3k - 2) - |S| - (x - 1) = 2k - x$. It is easy to see that $A'_2$ contains a vertex $x$ having at least

$$\left\lfloor \frac{(2k - x)(2k - x - 1)}{2} \cdot \frac{1}{2k - x} \right\rfloor = k - \left\lfloor \frac{x - 1}{2} \right\rfloor$$

out-neighbours in $T'_2$. Because of $k - [(x - 1)/2] + ([x/2] + 1) \geq k$, we see that $T$ has at least $k$ internally disjoint $(x, y)$-paths of length 2.

Assume thus that every vertex of $T'_1$ has at most $[x/2]$ in-neighbours in $T'_1$. This implies that $T'_1$ is regular, and hence the number $x = |V(T'_1)|$ is odd. Let $x = 2t + 1$ with $t \geq 0$. Furthermore, we may assume that every vertex of $A_2$ has at most $k - t - 1$ out-neighbours in $T'_2$. By the observation $|A_2| \geq 2k - x$, it is easy to check that $A_2 = V(T_{p-1})$, $|A_2| = 2k - x$ and $T_{p-1}$ is also regular. Now we note that there are exactly $k - 1$ internally disjoint $(x, y)$-paths in $T(V(T_{p-1}) \cup V(T_p))$ for any two vertices $x \in V(T_{p-1})$ and $y \in V(T_p)$.

Since $n \geq 3k$ and $|V(T_p)| = |V(T_{p-1})| = 2k$, we have $p \geq 3$. Suppose that $T$ contains an arc $xy$ from $T_{p-1}$ to $S$. Because of $s \to z \to T_p$ for some $z \in V(T_1)$, $T$ contains $k$
internally disjoint \((x, y)\)-paths for any \(y \in V(T_p)\). So we may assume that \(S \rightarrow T_{p-1}\).

Since \(D\) is \((3k - 2)\)-connected and \(A_2 - V(T_{p-1})\), \(\lvert N_D(T'_2 - V(T_{p-1})) \rvert \geq 3k - 2\). It follows that \(x = 1\), and hence \(T_{p-1}\) is a \((k - 1)\)-regular tournament. Thus \(N_D(T'_2 - V(T_{p-1})) = S \cup V(T_{p-1})\). Let \(x\) be a vertex of \(T'_2 - V(T_{p-1})\) and \(y \in V(T_{p-1})\) such that \(yx\) is an arc of \(D\). Then we see that the \(k - 1\) in-neighbours of \(y\) in \(T_{p-1}\) and the path \(x \rightarrow T_p \rightarrow s \rightarrow y\) with \(s \in S\) form \(k\) internally disjoint \((x, y)\)-paths.

Altogether, we have shown that \(T - S\) contains two vertices \(x\) and \(y\) satisfying the conditions (a)–(c) above.

Now we consider the tournament \(T'\) obtained from \(T\) by replacing the arc \(xy\) with \(yx\). From the assumption that \(D\) contains no spanning \(k\)-connected tournament, we see that \(T'\) is not \(k\)-connected. By Lemma 1.7, \(T'\) is \((k - 1)\)-connected and every minimum separating set of \(T'\) is also one of \(T\).

If \(T' - S\) is strong, then the number of minimum separating sets of \(T'\) is less than that of \(T\), a contradiction to the choice of \(T\) in view of the condition (1) above. So we assume that \(T' - S\) is not strong. Since \(T'(V(T'_2) \cup \cdots \cup V(T'_i))\) is strong, the number of the strong components of \(T' - S\) is less than that of \(T - S\). This contradicts the conditions (1) and (2).

Therefore, \(D\) contains a \(k\)-connected spanning tournament. \(\Box\)

**Corollary 2.3** (Bang-Jensen [1]). A \(5k\)-connected locally semicomplete digraph has a \(k\)-connected spanning local tournament.

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**References**

