A Laplace transform method for order statistics from nonidentical random variables and its application in Phase-type distribution

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ABSTRACT

In this paper, we derive a method for obtaining the Laplace transform of order statistics (o.s.) arising from general independent nonidentically distributed random variables (r.v.’s). A survey of the most important properties, applications and the o.s. of a Phase-type (PH) distribution are also presented. Two illustrative examples are provided.

1. Introduction

The need for the recurrence relations for the moment of o.s. is well established in statistical literature (see, e.g., Arnold and Balakrishnan, 1989, Bapat and Beg, 1989, Beg, 1991, Arnold et al., 1992, Balakrishnan, 1992 and Barakat and Abdelkader, 2004). Hlynka et al. (2010) have presented a path-counting method for deriving Laplace Transforms (LT) of order statistics of independent nonidentically distributed Erlang variables. Their technique is useful for a relatively small number of Erlang variables. Barakat and Abdelkader (2004) have derived recurrence relations of o.s. arising from general independent nonidentically distributed random variables (r.v.’s) using the moment technique. As noted in Hlynka et al. (2010), the moments may not uniquely define a probability distribution if the support is on an infinite interval such as (0, ∞) (see also, Casella and Berger, 2002, p. 64; and Rao, 1973, p. 106). Moreover, the LT includes more information than the moments. For example, the LT can be used to obtain the probability distribution of convolutions of an o.s. with an independent r.v. whose LT is also known. The power of the LT as a tool in distribution theory is the same as that of characteristic functions: the uniqueness and continuity theorems apply. Besides, most of the nice properties of characteristic functions are shared by the LT.

In this paper, we derive the moment of order statistics (o.s.) arising from general independent nonidentically distributed r.v.’s using LT. The moments of o.s. arising from a phase-type distribution are presented. These moments cannot be easily derived by the Barakat and Abdelkader (2004) technique. The rest of this section and Section 2 are devoted to provide a brief survey on the phase-type (PH) distribution, applications and some basic properties (e.g., its distribution function, density function, the kth moment, Laplace Transform and special cases). A method of obtaining LT of o.s. is presented in Section 3. The o.s. of the phase type distribution is presented in Section 4. Concluding remarks are drawn in Section 5.

A PH type distribution is the distribution of the time until absorption into state 0 in a Markov process. It can be represented by a r.v., X, describing the time until absorption of a Markov process with one absorbing state. Each of the states of the Markov

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process represents one of the phases (see, e.g., Botta and Harris, 1986). The following distributions are considered special cases of the PH type distribution: the exponential, hyperexponential, hypoexponential, Erlang and Coxian.


Coxian distribution has been used in the theory of network queues. Its importance is in large part due to its universality: any distribution function can be approximated arbitrarily closely to it (see, e.g., Wolff, 1989 and Johnson and Taaffe, 1989, 1990a,b, 1991, Johnson, 1993, Osogami and Harchol-Balter, 2002).

2. Some basic properties of the phase-type distribution

A r.v. $X$ is said to be a PH type distribution, $\text{PH}(\alpha, Q)$, if the distribution function (d.f.) is given by

$$F(x) = 1 - \alpha \exp(Qx).1, \quad x \geq 0$$

(1)

where $\alpha$ is a row vector of dimension $n$, $1$ is a column vector of all entries equal to 1, $Q$ is an $n \times n$ matrix and $\exp(.)$ is the matrix exponential $\left( \exp(X) = \sum_{i=0}^{\infty} \frac{X^i}{i!} \right)$. The density function is defined by

$$f(x) = \alpha \exp(Qx).Q^0,$$

(2)

where $Q^0 = -Q$. The $k$th non-central moment is given by

$$E[X^k] = (-1)^n \alpha \exp(Q^{-1}).Q^n.1.$$

The Laplace transform of the PH distribution function is given by

$$\tilde{F}(s) = \alpha_0 + \alpha.sI - Q^{-1}.Q^0,$$

(3)

where $\alpha_0 = 1 - \alpha.1$ and $I$ is an identity matrix.

The following phases are all considered special cases of the PH type distribution:

- Zero phases, Degenerate distribution (point mass at zero), that is $\alpha_{n+1} = 0$ where the state $(n + 1)$ is absorbing.
- One phase, the exponential distribution where the parameters of the PH distribution are: $Q = -\lambda$ and $\alpha = 1$. The distribution function is then

$$F(x) = 1 - \exp(-\lambda x).$$

- Two or more identical phases in sequence, the Erlang distribution with matrix $Q_{n \times n}$ and the vector $\alpha_{1 \times n}$ are given by

$$Q = \begin{bmatrix} -\lambda & \lambda & 0 & \cdots & 0 \\ 0 & -\lambda & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & -\lambda \end{bmatrix},$$

and $\alpha = (1, 0, 0, \ldots, 0)$. The distribution function is given by

$$F(x) = 1 - \sum_{i=0}^{n-1} \exp(-\lambda x).\frac{(\lambda x)^i}{i!}.$$

- Two or more non-identical phases (each phase has a probability of occurring in a mutually exclusive manner), Hyperexponential (or mixture of exponential) distribution with matrix $Q_{n \times n}$ and the vector $\alpha_{1 \times n}$ are given by

$$Q = \begin{bmatrix} -\lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & -\lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & -\lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & -\lambda_n \end{bmatrix},$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n > 0$ and $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n)$ with $\sum_{i=1}^{n} \alpha_i = 1$. The distribution function is given by

$$F(x) = 1 - \sum_{i=0}^{n} \alpha_i \exp(-\lambda_i x).$$

- Two or more (not necessarily identical) phases in sequence (absorbing state, reaching from any phase, after each phase), Coxian (or generalized hypoexponential) distribution. The representation of the matrix $Q_{n \times n}$ and the vector $\alpha_{1 \times n}$ are...
The acyclic PH distribution is called a Coxian distribution if its satisfies:
(a) the initial non-absorbing state (i.e., the initial Neuts, Johnson = 0),
(b) for each state, the next non-absorbing state is unique, and
(c) at each state, the sojourn time distribution is identical.

Notethat the exponential distribution is a special case of an Erlang distribution, therefore it is considered a subclass of an acyclic PH distribution.

The following distributions are subclasses of the acyclic PH distribution.

Definition 1. A PH type distribution is called acyclic if each state in the Markov chain is visited at most once. The following distributions are subclasses of the acyclic PH distribution.

• The acyclic PH distribution is called a hyperexponential (or mixture of exponential) distribution if: for any state, the next state is an absorbing state.
• The acyclic PH distribution is called an Erlang distribution (the sum of independent identically distributed exponential r.v.'s), if the following are satisfied: (a) the initial state is a unique non-absorbing one, (b) for each state, the next one is unique, and (c) at each state, the sojourn time distribution is identical.
• The acyclic PH distribution is called a Coxian distribution if it satisfies: (a) the initial non-absorbing state (i.e., the initial state is either absorbing or the unique non-absorbing state), and (b) for each state, the next non-absorbing state is unique (i.e., the next state is either absorbing or the unique non-absorbing state).

It is worth mentioning that the set of PH distributions is closed under some operations. In particular, a mixture of independent PH distributions is a PH distribution and the convolution of independent PH distributions is a PH distribution.

3. Main result

In this section, we derive the $k$th moment of the $r$th o.s, $u_r^{(k)}$, arising from general independent identically r.v.’s using LT. To lay the groundwork of our derivation Lemma 1 is introduced. It shows how we can express the distribution function of the $r$th o.s., $F_{r:n}$, in terms of the LT and we end this section with our main theorem.

Let $X_1, X_2, \ldots, X_n$ be independent non-identical r.v.’s with d.f.’s $F_1, F_2, \ldots, F_n$ respectively and let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ denote the corresponding order statistics. The distribution function of the $r$th o.s. $X_{r:n}(1 < r \leq n, n \geq 2)$ can be expressed as (see, e.g., Barakat and Abdelkader, 2004)

$$F_{r:n}(x) = F_{r-1:n}(x) - \sum_{P \in \mathcal{P}} \prod_{j=1}^{r-1} F_j(x) \prod_{j=1}^{n-r+1} (1 - F_{\ell_{r-j+1}}(x)),$$  \hspace{1cm} (4)

where the summation $\mathcal{P}$ extends over all permutations $(i_1, i_2, \ldots, i_n)$ of $(1, 2, \ldots, n)$ for which $1 \leq i_1 < i_2 < \cdots < i_{r-1} \leq n, 1 \leq i_r < i_{r+1} < \cdots < i_n \leq n$.

Lemma 1. Let $X$ be a nonnegative r.v. with d.f. $F(x)$. Then the LT of $X$ or $F(x)$ can be written as

$$\text{Lap}(s) = 1 - s \int_0^\infty \exp(-sx)(1 - F(x))dx, \hspace{1cm} s \geq 0.$$  \hspace{1cm} (5)

Proof. From the definition of LT

$$\text{Lap}(s) = E(e^{-sx}) = \int_0^\infty \exp(-sx)dF(x),$$

$$= - \int_0^\infty \exp(-sx)d(1 - F(x)).$$

Integrating by parts we get Eq. (5).
Let us denote $\text{Lap}_r(s)$ to the LT of the $r$th o.s., $X_{r:n}$, $1 \leq r \leq n$. In view of Lemma 1,

$$
\text{Lap}_r(s) = 1 - s \int_0^\infty \exp(-sx)(1 - F_{r:n}(x))dx.
$$

(6)

The $k$th moment of the $r$th o.s., $\mu_{r:n}^{(k)}$, can be obtained from Eq. (6) by differentiating $\text{Lap}_r(s)$ $k$ times with respect to $s$ and letting $s = 0$. That is

$$
(-1)^k \frac{d^k}{ds^k} \text{Lap}_r(0) = (-1)^k \text{Lap}_r^{(k)}(0) = \mu_{r:n}^{(k)}.
$$

(7)

This is the complete proof. □

The following theorem gives the $\mu_{r:n}^{(k)}$ in terms of Laplace transform.

**Theorem 1.** Let $X_1, X_2, \ldots, X_n$ be independent non-identical r.v.'s with survival functions $G_i(x) = 1 - F_i(x)$, the $k$th moment ($k = 1, 2, \ldots$) of the $r$th o.s. ($1 \leq r \leq n$) can be expressed in terms of LT as

$$
\mu_{r:n}^{(k)} = \sum_{j=n-r+1}^n (-1)^{j-(n-r+1)} \binom{j-1}{n-r} \text{L}_j^{(k)}(0),
$$

(8)

where

$$
\text{L}_j^{(k)}(s) = (-1)^k \sum_{1 \leq i_1 < i_2 < \cdots < i_j \leq n} \frac{d^k}{ds^k} \left( 1 - s \int_0^\infty e^{-sx} \prod_{i=1}^j G_i(x) \right) dx.
$$

(9)

and $\text{L}_j^{(k)}(0)$ is the $k$th derivative of $\text{L}_j^{(k)}(s)$ evaluated at $s = 0$.

**Proof.** By using Eq. (6) for $(1 < r \leq n, n \geq 2)$, we can write Eq. (4) as

$$
1 - s \int_0^\infty e^{-sx}(1 - F_{r:n}(x))dx = 1 - s \int_0^\infty e^{-sx}(1 - F_{r-1:n}(x))dx - s \int_0^\infty e^{-sx} \sum_{j=1}^{r-1} (1 - G_j(x)) \prod_{j=r}^n G_j(x)dx.
$$

Upon using Eqs. (6) and (7), we get

$$
(-1)^k \frac{d^k}{ds^k} \text{Lap}_r(0) = (-1)^k \frac{d^k}{ds^k} \text{Lap}_{r-1}(0) + Z_{r:n}^{(k)},
$$

$$
\mu_{r:n}^{(k)} = \mu_{r-1:n}^{(k)} + Z_{r:n}^{(k)},
$$

(10)

where

$$
Z_{r:n}^{(k)} = (-1)^k \sum_{j=1}^{r-1} \frac{d^k}{ds^k} \left. \left(-s \int_0^\infty e^{-sx} \sum_{j=1}^{r-1} \prod_{j=1}^{r-1} (1 - G_j(x)) \prod_{j=r}^n G_j(x)dx \right) \right|_{s=0}
$$

$$
= (-1)^k \sum_{j=1}^{r-1} \frac{d^k}{ds^k} \left. \left( 1 - s \int_0^\infty e^{-sx} \prod_{j=r}^n G_j(x)dx \right) \right|_{s=0}
$$

$$
- \frac{d^k}{ds^k} \left. \left( 1 - s \int_0^\infty e^{-sx} \sum_{j=1}^{r-1} G_{j_1}(x) \prod_{j=r}^n G_j(x)dx \right) \right|_{s=0}
$$

$$
+ \frac{d^k}{ds^k} \left. \left( 1 - s \int_0^\infty e^{-sx} \sum_{j_1=1}^{r-1} \sum_{j_2<j_1} G_{j_1}(x)G_{j_2}(x) \prod_{j=r}^n G_j(x)dx \right) \right|_{s=0}
$$

$$
+ \cdots + (-1)^{r-1} \frac{d^k}{ds^k} \left. \left( 1 - s \int_0^\infty e^{-sx} \prod_{j=1}^n G_j(x)dx \right) \right|_{s=0}
$$

$$
= \sum_{j=1}^{r-1} \left( \text{L}_j^{(k)}(0) - \text{L}_{n-r+2}^{(k)}(0) + \cdots + (-1)^{r-2} \text{L}_{n-1}^{(k)}(0) + (-1)^{r-1} \text{L}_n^{(k)}(0) \right)
$$

$$
= \sum_{j=1}^{r} (-1)^{j-1} a_j(r, n) \text{L}_{n-r+j}^{(k)}(0),
$$

(11)
where \( L_j \) is defined by Eq. (9) evaluated at \( s = 0 \) and \( a_j(r, n) \) is a suitable sequence of constants, which depends only on \( r \) and \( n \). On account of Eq. (11) and by using the obvious relations \( \sum_{p=1}^{n} (1) = \binom{n}{r-1} \) and \( \sum_{1 \leq j_1 < j_2 < \cdots < j_m \leq n} (1) = \binom{n}{m} \), for all \( n \geq m \), an application of the multiplication principle of the combinatorial analysis on the left and the right hand sides of the \( j \)th term of Eq. (11), yields the following combinatorial identity

\[
\binom{n}{r-1} \binom{r-1}{j-1} = a_j(r, n) \binom{n}{n-r+j}.
\]

Therefore, \( a_j(r, n) = \binom{n-r+j}{j-1} \), from which, by using Eq. (10), we get

\[
\mu^{(k)}_{r:n} = \mu^{(k)}_{r-1:n} + \sum_{j=1}^{r} \binom{n-r+j}{j-1} L^{(k)}_{m-n+j}(0), \quad 2 \leq r \leq n.
\] (12)

Finally, by induction over \( r \) we get Eq. (8) and the proof is completed. \( \square \)

**Corollary 1.** The \( k \)th moments \( \mu^{(k)}_{r:n} \) and \( \mu^{(k)}_{1:n} \) of the maximum and the minimum are given by

\[
\mu^{(k)}_{r:n} = \sum_{j=1}^{n} \binom{n}{j} \binom{n-j}{j}^{(k)}, \quad \mu^{(k)}_{1:n} = L^{(k)}_{n}(0).
\] (13)

**Proof.** Setting \( r = n \) and \( r = 1 \) in Eq. (8), we get the results. \( \square \)

### 4. Order statistics of the phase-type distribution

To lay the groundwork of deriving the \( k \)th moment of the \( r \)th order statistics (o.s.), \( \mu^{(k)}_{r:n} \), of the PH distribution, some conventions and notations will be used. First, we need to define the so-called Kronecker product and sum of two matrices. Let \( A_{m \times n} \) and \( B_{n \times q} \) be two rectangular matrices, their Kronecker product denoted by \( A \otimes B \) is the \( m \times nq \) block matrix, written in compact form as \( (a_{ij}B) \). The Kronecker sum of the square matrices \( A_{m \times n} \) and \( B_{n \times m} \) is defined by \( A \oplus B = A \otimes I_m + I_n \otimes B \), where \( I_k \) denotes the identity matrix of order \( k \). We shall also use the matrix operation \( (A \; \exp(Bx)) \otimes (C \; \exp(Dx)) = (A \; \otimes C) \; \exp((B \; \otimes D)x) \), for matrices \( A, B, C \) and \( D \) of appropriate dimensions and the following notations will be used

\[
\prod_{i=1}^{j} \otimes \alpha^{(k)}_{i} = \alpha^{(k)}_{1} \otimes \alpha^{(k)}_{2} \otimes \cdots \otimes \alpha^{(k)}_{j},
\]

\[
\sum_{i=1}^{j} \oplus \alpha^{(k)}_{i} = \alpha^{(k)}_{1} \oplus \alpha^{(k)}_{2} \oplus \cdots \oplus \alpha^{(k)}_{j}.
\] (15)

Details about the Kronecker product and sum can be found in Bellman (1970).

To get the o.s. of the Phase-type distribution, it is enough to derive \( L^{(k)}_{j}(0) \) and using Eq. (8) in Theorem 1. The following corollary gives an explicit form for \( L^{(k)}_{j}(0) \).

**Corollary 2.** Let \( X_1, X_2, \ldots, X_n \) be independent non-identical r.v.’s having Phase-type distribution with survival functions \( G_i(x) = \exp(\alpha^{(k)}_{i}x) \), then

\[
L^{(k)}_{j}(0) = (-1)^{k!} \sum_{1 \leq j_1 < j_2 < \cdots < j_m \leq n} \prod_{i=1}^{j} \otimes \alpha^{(k)}_{i} \left( \sum_{i=1}^{j} \oplus \alpha^{(k)}_{i} \right)^{-k}. \tag{17}
\]

where \( \prod_{i=1}^{j} \otimes \alpha^{(k)}_{i} \) and \( \sum_{i=1}^{j} \oplus \alpha^{(k)}_{i} \) are defined by Eqs. (15) and (16).

**Proof.** By applying Theorem 1, we have

\[
L^{(k)}_{j}(s) = \left( 1 - s \int_{0}^{\infty} \prod_{i=1}^{j} \otimes \alpha^{(k)}_{i} \cdot \sum_{i=1}^{j} \oplus \alpha^{(k)}_{i} \right) \exp(-s) \cdot \left( 1 - s \int_{0}^{\infty} \exp(-s) \cdot \prod_{i=1}^{j} \otimes \alpha^{(k)}_{i} \cdot \sum_{i=1}^{j} \oplus \alpha^{(k)}_{i} \right) ds.
\]

\[
\left( 1 - s \int_{0}^{\infty} \prod_{i=1}^{j} \otimes \alpha^{(k)}_{i} \cdot \sum_{i=1}^{j} \oplus \alpha^{(k)}_{i} \right) \exp(-s) \cdot \left( 1 - s \int_{0}^{\infty} \exp(-s) \cdot \prod_{i=1}^{j} \otimes \alpha^{(k)}_{i} \cdot \sum_{i=1}^{j} \oplus \alpha^{(k)}_{i} \right) ds.
\]

\[
\left( 1 - s \int_{0}^{\infty} \prod_{i=1}^{j} \otimes \alpha^{(k)}_{i} \cdot \sum_{i=1}^{j} \oplus \alpha^{(k)}_{i} \right) \exp(-s) \cdot \left( 1 - s \int_{0}^{\infty} \exp(-s) \cdot \prod_{i=1}^{j} \otimes \alpha^{(k)}_{i} \cdot \sum_{i=1}^{j} \oplus \alpha^{(k)}_{i} \right) ds.
\]
Let us consider the Erlang distribution with parameter $\lambda_1$, $\lambda_2$, ..., $\lambda_n$. The simplest nontrivial example of a PH distribution is the exponential distribution with parameter $\lambda$. The $k$th moments of a PH distribution are given by

$$m_\alpha^{(k)} = \sum_{i=1}^n \prod_{t=1}^n \alpha^{(t)} \left(1 - \frac{s}{\sum_{t=1}^n \alpha^{(t)}}\right)^{-1} \frac{d^k}{dx^k}\left(1 - \frac{s}{\sum_{t=1}^n \alpha^{(t)}}\right)^{-1} dx.$$

Differentiating $k$ times and letting $s = 0$, we get Eq. (17).

**Corollary 3.** The $k$th moments $\mu_{mn}^{(k)}$ and $\mu_1^{(k)}$ of the maximum and the minimum of the Phase-type distribution are given by

$$\mu_{mn}^{(k)} = \sum_{j=1}^n (-1)^{j-1} L_j^{(k)}(0)$$

$$\mu_1^{(k)} = L_n^{(k)}(0).$$

The rest of this section provides two examples for obtaining the moment of order statistics. The first one deals with the exponential distribution and the second deals with the Erlang distribution.

**Example 1.** The simplest nontrivial example of a PH distribution is the exponential distribution with parameters $\lambda_i$ ($i = 1, 2, \ldots, n$). The representations of the PH distribution in this case are: $\alpha = 1$ and $Q^{(i)} = (-\lambda_i)$. Using Corollary 3, the $k$th moments of $\mu_{mn}^{(k)}$ and $\mu_1^{(k)}$ are given by

$$\mu_{mn}^{(k)} = k! \left(\sum_{i=1}^n \lambda_i \right)^{-k} - \sum_{1 \leq i_1 < i_2 \leq n} \lambda_{i_1 + \lambda_{i_2}}^{-k} + \sum_{1 \leq i_1 < i_2 < i_3 \leq n} \lambda_{i_1 + \lambda_{i_2 + \lambda_{i_3}}}^{-k} + \cdots + (-1)^{n-1} \sum_{i=1}^n \lambda_i^{-k}.$$

$$\mu_1^{(k)} = k! \sum_{i=1}^n \lambda_i^{-k}.$$

**Example 2.** Let us consider the Erlang distribution with parameters $(m, \lambda_i)$, $i = 1, 2, 3$. The Mathematica 7 algorithm is written to obtain

$$\mu_{33} = \sum_{j=1}^3 (-1)^j L_j(0),$$

where

$$L_j(0) = \sum_{1 \leq i_1 < i_2 < i_3 \leq 3} \prod_{t=1}^3 \alpha^{(t)} \left(\sum_{t=1}^3 \beta^{(t)m}\right)^{-1} L_j^{(k)}. $$

$$\alpha = (1, 0, 0), \prod_{t=1}^3 \alpha^{(t)} = (1, 0, 0)$$

and

$$Q^{(i)} = \lambda_i \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}.$$

It gives, for $m = 2$, the following result

$$\mu_{33} = \sum_{j=1}^3 \frac{2}{\lambda_i} - 2 \left(\frac{1}{\lambda_1 + \lambda_2} + \frac{1}{\lambda_1 + \lambda_3} + \frac{1}{\lambda_2 + \lambda_3} + \frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2} + \frac{\lambda_1 \lambda_3}{\lambda_1 + \lambda_3} + \frac{\lambda_2 \lambda_3}{\lambda_2 + \lambda_3}\right)$$

$$+ 2 \left(\frac{1}{\lambda_1 + \lambda_2 + \lambda_3} + \frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2 + \lambda_3} + \frac{3 \lambda_1 \lambda_2 \lambda_3}{\lambda_1 + \lambda_2 + \lambda_3}\right).$$

In the case of an independent identical distribution, we have

$$\mu_{33} = 3.213 \frac{1}{\lambda}.$$
5. Conclusion

In this paper, we used the Laplace transform as well as the Kronecker product and sum to derive the order statistics arising from general independent nonidentically distributed random variables. The advantages of using the Laplace transform instead of the moment method (Barakat and Abdelkader, 2004) are demonstrated. Applications, properties and the order statistics of the Phase type distribution are presented. In future work, we can use the Laplace transform to derive the order statistics arising from discrete random variables and to obtain the order statistics of the discrete Phase-type distribution.

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