ERLANG DISTRIBUTED ACTIVITY TIMES
IN STOCHASTIC ACTIVITY NETWORKS

Yousry H. Abdelkader

It is assumed that activity times in stochastic activity networks (SANs) are independent Erlang random variable (r.v.). A recurrence method of determining the $k$th moments completion time is presented. Applications are provided for illustration and are used to evaluate the applicability and appropriateness of the Erlang model to represent activity network.

Keywords: project planning, PERT, Erlang distribution

AMS Subject Classification: ???

1. INTRODUCTION

SANs are assumed a priori cumulative distribution function (c.d.f.). Normally distributed is proposed by Sculli [15] and Kamburovski [7], exponentially distributed is assumed by Kamburovski [8], Kulkarni and Adlakha [10] and Magott and Skudlarski [12]. Bendell et al [1] have considered the problem of using the Erlang distribution as a representation of activity times. Their method based on the moments approach which is more practical alternative to both the analytical and the numerical integration. However, they derived the first four central moments of max($X_1, X_2$) and $X_1 + X_2$ only where, $X_1$ and $X_2$ are independent r.v.’s. A survey of recent developments and complexity in SANs can be found in Elmaghraby [4] and [5]. This paper generalized the work of Bendell et al [1]. It presents the $k$th moments of the max($X_1, X_2, \cdot \cdot \cdot , X_n$) and the c.d.f. of the sum of $n$ independent r.v. ’s is also given.

SANs are defined as $(N, A, F(\cdot))$, where $N$ is the set of nodes $N = \{1, 2, \ldots , n\}$, $A$ is a set of arcs $A = \{a_1, a_2, \cdot \cdot \cdot , a_m\}$ and $F(t) = P(T_r \leq t)$, for $t > 0$ is the c.d.f., where $T_r$ is a r.v. which describes the duration of the arc $a_r$. The network has one starting and one ending node and is acyclic, i.e., the nodes are numbered in such a way that whenever there exists an arc$(i, j)$, then $i < j$. We shall use the names “activity” and “arc”, “event” and “node”, “project” and “network” synonymously.

The main problem in SANs (largest and shortest path) is divided into two categories. The first category is to find the distribution of

$$Y = X_1 + X_2 + \cdot \cdot \cdot + X_n$$

(1)
where $X_1, X_2, \ldots, X_n$ be r.v.‘s represent the time of the project activities. The distribution of this sum can be achieved through the convolution operation.

The second category is to find the distribution of

$$X_{n:n} = \max(X_1, X_2, \cdots, X_n) \quad \text{and} \quad X_{1:n} = \min(X_1, X_2, \cdots, X_n)$$

(2)

that is, finding $H_n(t) = P(X_{n:n} \leq t)$ and $L_n(t) = P(X_{1:n} \leq t)$ or finding the $k$th moments of $X_{n:n}$ and $X_{1:n}$ or approximating them.

We shall derive the distribution of (1) and the $k$th moments of (2) that is, $\mu_{1:n}^{(k)} = E(X_{1:n}^k)$ and $\mu_{n:n}^{(k)} = E(X_{n:n}^k)$, of the minimum and the maximum of a sample of size $n$ from Erlang distribution. Two applications are examined in light of the Erlang model. These applications are provided for illustration and are used to evaluate the applicability and appropriateness of the Erlang model to represent activity network.

We end this section with some definitions, see Ross [14]. Let $F(x)$ be the c.d.f. of the r.v. $X$, $F(x) = 1 - F(x)$ denotes the survival function.

**Definition 1.** A c.d.f. $F(x)$ is New Better than Used in Expectation (NBUE) if $F(x)$ has a finite mean $\mu$ and

$$\int_t^\infty F(x) \, dx \leq \mu F(t) \quad \text{for} \quad t \geq 0.$$

**Definition 2.** Let $X$ and $Y$ be two r.v.‘s with c.d.f.'s $F(x)$ and $F(y)$ respectively, $Y$ is stochastically larger than $X$, written $Y \succ X$, if $F(y) \geq F(x)$.

**Definition 3.** The r.v. $X$ is said to be more variable than $Y$, written $X \succeq_v Y$ with $E(X) = E(Y)$, when $E[\phi(X)] \geq E[\phi(Y)]$ for all increasing and convex functions $\phi$.

**Proposition 1.** If $F(x)$ is an NBUE distribution having mean $\mu$, then

$$F \leq_v \exp(\mu),$$

where $\exp(\mu)$ is the exponential distribution with mean $\mu$.

The above proposition states that the exponential distribution is maximal variability in the class of NBUE distributions.

2. MAIN RESULTS

Let $X_i$ be a r.v. obeys Erlang distribution with probability density function (p.d.f.) is given by

$$f_i(x) = \frac{\lambda_i^{m_i} x^{m_i-1} e^{-\lambda_i x}}{\Gamma(m_i)} \quad x \geq 0, \lambda_i > 0$$

(3)

and cumulative distribution function

$$F_i(x) = 1 - \sum_{t=0}^{m_i-1} \frac{(\lambda_i x)^t}{t!} e^{-\lambda_i x} \quad x \geq 0, \lambda_i > 0,$$

(4)
for \( i = 1, 2, \ldots, n \), where \( m_i \) denotes to the shape parameter, \( \lambda_i \) denotes to the scale parameter and \( \Gamma(\cdot) \) is the gamma function. The sum on the right-hand side involves the first \( m_i \) terms of Poisson probability mass function. There is a strong relation between the exponential, the Erlang, and the Gamma d.f.’s. The Erlang distribution is a special case of the Gamma distribution with p.d.f.,

\[
    f(x) = \frac{\lambda^n x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} x \geq 0, \lambda > 0, \alpha > 0.
\]

The sum on the right-hand side involves the first \( m_i \) terms of Poisson probability mass function. There is a strong relation between the exponential, the Erlang, and the Gamma d.f.’s. The Erlang distribution is obtained by letting \( n = 1 \). In addition, the Erlang r.v. results from the sum of \( m_i \) independent, exponentially distributed with parameter \( \lambda_i \).

The following lemma gives the c.d.f. of (1) and Theorem 1 gives the \( k \)th moments of (2).

**Lemma 1.** The cumulative distribution function of (1) is given by

\[
    F(y) = \prod_{i=1}^{n} (\lambda_i)^{m_i} \sum_{i=1}^{n} \sum_{j=1}^{m_i} B_{ij} \left( \frac{1}{\lambda_i} \right)^{m_i-j+1} \left( 1 - e^{-\lambda_i y} \sum_{r=0}^{m_i-j} \frac{(\lambda_i y)^r}{r!} \right)
\]

where,

\[
    B_{ij} = \frac{1}{(j-1)!} \frac{d^{j-1}}{ds^{j-1}} \left( \frac{(s + \lambda_i)^{m_i}}{\prod_{k=1}^{n} (s + \lambda_k)^{m_k}} \right)
\]

evaluated at \( s = -\lambda_i \) and \( \frac{d}{ds} (\cdot) \) stands for the derivative.

**Proof.** Laplace transform of the p.d.f. of (1) can be written as

\[
    f^*(s) = \prod_{i=1}^{n} \left( \frac{\lambda_i}{s + \lambda_i} \right)^{m_i}
\]

by using partial fraction expansion Kleinrock [9] (formulae (I40)and (I41)) showed that

\[
    f^*(s) = \prod_{i=1}^{n} (\lambda_i)^{m_i} \sum_{i=1}^{n} \sum_{j=1}^{m_i} B_{ij} \left( \frac{1}{s + \lambda_i} \right)^{m_i-j+1}
\]

where \( B_{ij} \) is defined in (6). Then the p.d.f. of (1) is given by

\[
    f(y) = \prod_{i=1}^{n} (\lambda_i)^{m_i} \sum_{i=1}^{n} \sum_{j=1}^{m_i} B_{ij} (m_i-j)! y^{m_i-j} e^{-\lambda_i y}.
\]

In consequence, the c.d.f. can be written as in (5).

**Theorem 1.** Let \( X_i \)’s be independent and non-identically r.v.’s obey the Erlang distribution. For \( n = 1, 2, \ldots \) and \( k = 1, 2, \ldots \),

\[
    \mu_{n,m}^{(k)} = k \sum_{j=1}^{n} (-1)^{j+1} I_j
\]
where,
\[
I_j = \sum_{1 \leq i_1 < i_2 \leq j \leq n} \sum_{t_1=0}^{m_{i_1}-1} \cdots \sum_{t_j=0}^{m_{i_j}-1} \frac{\Gamma(\sum_{s=1}^{j} t_s + k)}{(\sum_{s=1}^{j} \lambda_{i_s})^{\sum_{s=1}^{j} t_s + k}} \prod_{s=1}^{j} \frac{x_{i_s}^{t_s}}{t_s!},
\]
(8)

and
\[
\mu_{1:n}^{(k)} = I_n.
\]
(9)

**Proof.** By definition
\[
\mu_{n:n}^{(k)} = k \int_0^\infty x^{k-1} (1 - F_{n:n}(x)) dx
\]
\[
= k \int_0^\infty x^{k-1} \left( 1 - \prod_{i=1}^{n} \left( 1 - \sum_{t=0}^{m_i-1} \frac{\lambda_i x^t}{t!} e^{-\lambda_i x} \right) \right) dx
\]
\[
= k \int_0^\infty x^{k-1} \left( \sum_{i=1}^{n} \sum_{t_1=0}^{m_{i_1}-1} \sum_{t_2=0}^{m_{i_2}-1} \frac{\lambda_{i_1}^{t_1} \lambda_{i_2}^{t_2} x^{t_1+t_2}}{t_1! t_2!} e^{-(\lambda_{i_1} + \lambda_{i_2}) x} \right.
\]
\[
+ \sum_{1 \leq i_1 < i_2 < i_3 \leq n} \sum_{t_1=0}^{m_{i_1}-1} \sum_{t_2=0}^{m_{i_2}-1} \sum_{t_3=0}^{m_{i_3}-1} \frac{\lambda_{i_1}^{t_1} \lambda_{i_2}^{t_2} \lambda_{i_3}^{t_3} x^{t_1+t_2+t_3}}{t_1! t_2! t_3!} e^{-(\lambda_{i_1} + \lambda_{i_2} + \lambda_{i_3}) x}
\]
\[
+ \cdots + (-1)^{n-1} \sum_{t_1=0}^{m_{n-1}-1} \cdots \sum_{t_n=0}^{m_{n}-1} \prod_{i=1}^{n} \frac{\lambda_{i}^{t_i} x^{t_i}}{t_i!} e^{-\lambda_i x} \left. \sum_{i=1}^{n} \lambda_i \right) dx.
\]

Upon using
\[
\int_0^\infty x^{\beta+k-1} e^{-\alpha x} = \frac{\Gamma(k + \beta)}{\alpha^{k+\beta}}
\]
(10)
we get (7). The proof of (9) follows from the relation
\[
\mu_{1:n}^{(k)} = k \int_0^\infty x^{k-1} \prod_{i=1}^{n} (1 - F_i(x)) dx
\]
\[
= k \int_0^\infty x^{k-1} \prod_{i=1}^{n} \left( 1 - \sum_{t=0}^{m_i-1} \frac{\lambda_i x^t}{t!} e^{-\lambda_i x} \right) dx
\]
and using (10). Where \( F_{n:n}(x) = \prod_{i=1}^{n} F_i(x) \) and \( F_{1:n}(x) = 1 - \prod_{i=1}^{n} (1 - F_i(x)) \) are the c.d.f.'s of the \( \max(X_1, X_2, \cdots, X_n) \) and the d.f. of the \( \min(X_1, X_2, \cdots, X_n) \) respectively. □
Bounds for Erlang Estimate (EE). Increasing Failure Rate (IFR) distribution class (recall that \( F(x) \) belongs to IFR class if satisfies \( r(t) = \frac{f(t)}{1-F(t)} \) is increasing) contains all distributions proposed for describing the activity duration in SANs, see Kamburowski [8]. This class includes the exponential, the Weibull with parameter \( \alpha \geq 1 \), the Erlang, and all strongly unimodal distributions of non-negative r.v.’s. So, it is easy to argue that EE is located between a lower bound (LB) based on the critical path method (CPM) with deterministic model, see Devroye [2] and the upper bound (UB) with exponentially distributed times of activities which is given by Kamburowski [8]. Better lower bounds have been discussed by Fulkerson [6], Elmaghraby [3], and Robillard [13]. However, the computational effort for these procedures are increasing. A survey of the above methods, recent developments and complexity can be found in Elmaghraby [4] and [5], Tavares [16] and the papers cited herein.

**Lemma 2.** \( LB \leq EE \) The relation follows by using Jensen’s inequality
\[
\psi[E(X)] \leq E[\psi(X)]
\]
for each convex function \( \psi \) and each r.v. \( X \), the higher moments are also follow from Jensen’s inequality.

**Lemma 3.** \( EE \leq UB \) Since Erlang distribution belongs to IFR class and the exponential distribution is the maximal one on it, then the proof is followed.

3. APPLICATIONS

The Erlang distribution is used in this paper to represent the input distribution of activity duration. The thrust, the appropriateness and the advantages of using the Erlang distribution are discussed in Bendell et al [1] by providing some worked examples to demonstrate the accuracy of the method. This paper devoted to compare the Erlang estimate (when \( m = 2 \) and \( 3 \)) with the other distribution estimates such as: the exponential and the normal distribution.

In order to perform the comparison between EE and other distribution estimates, two example networks are given in Figures 1 and 2 with mean indicated on each arc. The comparison of the estimates are given in Tables 1 and 2.

<table>
<thead>
<tr>
<th>Table 1. Comparison between different estimates.</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Type of activity duration</strong></td>
</tr>
<tr>
<td>-----------------------------------------------</td>
</tr>
<tr>
<td>Exact</td>
</tr>
<tr>
<td>Sculli [15]</td>
</tr>
<tr>
<td>Exponential (UB) [8]</td>
</tr>
<tr>
<td>Erlang (m = 2)</td>
</tr>
<tr>
<td>Erlang (m = 3)</td>
</tr>
<tr>
<td>PERT(LB) [2]</td>
</tr>
</tbody>
</table>
where C.V denotes to the coefficient of variation \(=\frac{\text{variance}}{\text{mean}}\).

**Table 2. Comparison between different estimates.**

<table>
<thead>
<tr>
<th>Type of activity duration</th>
<th>Mean Estimate</th>
<th>Variance</th>
<th>C.V.</th>
<th>Appr. Error(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kulkarni and Adlakha [10]</td>
<td>40.985</td>
<td>58.803</td>
<td>0.187</td>
<td></td>
</tr>
<tr>
<td>Sculli [15]</td>
<td>42.61</td>
<td>12.03</td>
<td>0.081</td>
<td>3.96</td>
</tr>
<tr>
<td>Exponential (UB) [8]</td>
<td>315.75</td>
<td>56144</td>
<td>0.75</td>
<td>670.40</td>
</tr>
<tr>
<td>Erlang ((m = 2))</td>
<td>167.05</td>
<td>8137.3</td>
<td>0.54</td>
<td>307.58</td>
</tr>
<tr>
<td>Erlang ((m = 3))</td>
<td>122.30</td>
<td>2972.6</td>
<td>0.445</td>
<td>198.40</td>
</tr>
<tr>
<td>PERT(LB) [2]</td>
<td>26.1</td>
<td>70.05</td>
<td>0.321</td>
<td>-36.32</td>
</tr>
</tbody>
</table>

To compute the mean completion time of the project, let \(X_i\)'s \(i = 1, 2, \cdots, n\) be \(n\) r.v.'s with

\[
h(X_1, X_2, \cdots, X_n) = \mathbb{E} \max(X_1, X_2, \cdots, X_n). \tag{11}\]

The mean completion time of the node \(j\) is determined recursively by

\[
\begin{align*}
W_1 &= 0 \\
W_j &= h(W_i + X_{ij}, i\in B_j) \quad j = 2, 3, \cdots
\end{align*}
\tag{3}
\]

where \(B_j\) denotes the set of predecessors of arcs connected with the node \(j\) and \(X_{ij}\) denotes the random duration of the arc \((i, j)\) which is assumed to be Erlang distribution. The other moments can be easily computed by using equations (7) and (12) with a little bit modifications. Some special cases, for the maximum of two and three r.v.'s, are given in the Appendix.

To sum up the computations for obtaining the \(k\)th moment completion time of the project, we present the following algorithm:

**INPUT:** \(k, m \) and \(\mathbb{E}(X_{ij})\)

**OUTPUT:** The completion time of node \(j, j = 2, 3, \cdots, n\).

**Step 1.** Set \(W_1 = 0\) (starting node).

**Step 2.** Compute \(W_j, \) for \(j = 2, 3, 4, \cdots, n\) by using equations (12), (7) and (8).

**Step 3.** Go to Step 2.

**Example 1.** The network in Figure 1, for which the duration of each activity is exponentially distributed, is taken from Kamburowski [7]. Each activity duration is discretized by three values of equal probability. This was done in order to achieve the exact mean completion time through enumeration all possible realizations. The normal approach which is given by Sculli [15] is also presented. Table 1 summarizes the completion time for these estimates as well as the Erlang estimate for two different values \((m = 2 \text{ and } 3)\).
Example 2. The network in Figure 2, for which the activity times are assumed to be exponentially distributed, first introduced by Loulou and Beale [11] and used by Kulkarni and Adlakha [10] who developed analytical procedure known as Markov PERT networks (MPNs). They estimated the mean completion time of this network to be 40.985. But their method suffers from the exponential state explosion. Table 2 summarizes the completion time for different estimates.

The values of the percentage errors for the estimates are included in Tables 1 and 2. By considering the approximated exact values, which give good approximations for the mean completion time, for the networks in Example 1 and 2 to be 18.257 and 40.985. We note that from the Tables 1 and 2 the percentage errors of the Erlang estimate is smaller than the exponential upper bound [8].

Remark 1. Exponential (UB) is Erlang distribution with $m = 1$.

Remark 2. From the table one and two, we see that Erlang estimate is an upper bound much better than the best upper bound which is given by Kamburovski [8].

4. COMMENTS

Kamburovski [8] proposed a method to determine an upper bound on the mean completion time by replacing the individual activity duration with exponentially distributed r.v. with the same mean, and proceeding iteratively over the nodes. He computes the mean completion time of the node $j$ using equation (12). By replacing the distribution of $W_j$ (which is not exponentially distributed) with an exponential
of the same mean as $E(W_j)$. This compounded the error committed and resulted in a very loose UB. We follow the same procedure by considering the distribution of $W_j$ is Erlang distribution. It also compounded the error committed and resulted a little loose UB compared with Kamburowski [8]. There is also another reason caused the error to be less than Kamburowski [8] that is the Erlang distribution has less extreme compared with the exponential distribution.

Kulkarni and Adlakha [10] not follow the above strategy. They assumed that the PERT network as finite-state, absorbing, continuous-time Markov chain (CTMC) with single absorbing state. The state space of this chain is dependent on the structure of the PERT network. The special structure of the Chain allows them to develop the exact analysis of the network. These assumptions led them to avoid using the maximum operation to calculate the project completion time and hence there is no error committed of assuming that the distribution of $W_j$ is again exponential (and the independent assumption in this case may disappeared). Consequently, their estimate for the moments completion time is getting tighter than the UB given by Kamburowski [8] and the Erlang estimate which is located between them. However, their method suffers from the exponential state explosion and requires excessive computational effort to find the mean and the variance. Also, one must use a computer software to find these parameters even in small network, eg the examples which examined by them the in-degree of any node is at most two. If it is assumed the in-degree of some nodes are three or more, the computational

**Fig. 2.** Network (Loulou and Beale). The numbers on the arcs represent mean activity duration.
efforts could be impossible. On the other hand, these parameters can be derived manually using Erlang distribution. There is another important remark on their work. They claimed that their work is stable. It seems that this is not always true, eg consider the example 1 which cited in their paper. They estimated the mean and the variance to be (14.7787, 46.6981) whereas the estimation by Kamburovski [8] are (10.13, 60.46), ie Kamburovski’s [8] mean estimate is less than Kulkarni and Adlakha [10].

5. CONCLUDING REMARKS

The advantages of using Erlang distribution as a representation of activity times are stated in Bendell et al [10]. They have derived only the four central moments of the sum and the maximum of two independent Erlang r.v.’s. This paper generalized the work of Bendell et al [10]. The distribution of the sum of n independent nonidentically as well as the kth moments of the maximum and the minimum are derived. Bounds for the mean completion time of the project are also presented. The higher moments can be obtained easily by using equations (7), (8) and (12). Applications show that Erlang estimate is an upper bound which is less than obtained by Kamburovski [8]. Similar results for stochastic shortest-route networks can be obtained by using equations (8) and (9). Computational complexity which is given in the algorithm may be an interesting subject for future work.

APPENDIX

The rth moment of Erlang distribution about the origin is given by

\[ \mu_r = \int_0^\infty x^r f(x) \, dx = \frac{\Gamma(r + m)}{\lambda^r \Gamma(m)} \]  

(A4)

in particular,

\[ \mathbb{E}(X) = \frac{m}{\lambda} \quad \text{and} \quad \text{Var}(X) = \frac{m}{\lambda^2}. \]

By letting \( \lambda_i = \frac{m}{\mu_i} \). Some special cases are arose from equations (7) and (8).

For \( k = 1 \).

Case 1. \( n = 2 \) and \( m = 2 \).

\[ \mathbb{E}(\max(X_1, X_2)) = \mu_{2,2} = \mu_1 + \mu_2 - \left( \frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\mu_1\mu_2} \right). \]  

(A5)

Case 2. \( n = 3 \) and \( m = 2 \).

\[ \mathbb{E}(\max(X_1, X_2, X_3)) = \mu_{3,3} = \mu_1 + \mu_2 + \mu_3 - \left( \frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\mu_3} + \frac{1}{\mu_1\mu_2} + \frac{1}{\mu_1\mu_3} + \frac{1}{\mu_2\mu_3} + \frac{1}{\mu_1\mu_2\mu_3} \right). \]
Case 3. \( n = 2 \) and \( m = 3 \).

\[
\mu_{2:2} = \mu_1 + \mu_2 - \left( \frac{1}{\mu_1 + \mu_2} + \frac{1}{\mu_1 + \mu_3} + \frac{1}{\mu_2 + \mu_3} + \frac{2}{\mu_1 \mu_2 (\mu_1 + \mu_2)^3} \right). \tag{A7}
\]

Case 4. \( n = 3 \) and \( m = 3 \).

\[
\mu_{3:3} = \mu_1 + \mu_2 + \mu_3 - \left( \frac{1}{\mu_1 + \mu_2} + \frac{1}{\mu_1 + \mu_3} + \frac{1}{\mu_2 + \mu_3} + \frac{1}{\mu_1 \mu_2 (\mu_1 + \mu_2)^3} \right)
+ \frac{2}{\mu_1 \mu_3 (\mu_1 + \mu_3)^3} + \frac{2}{\mu_2 \mu_3 (\mu_2 + \mu_3)^3}
+ \frac{2}{\mu_1 \mu_2 \mu_3 (\mu_1 + \mu_2 + \mu_3)^3}
+ \frac{1}{\mu_1 \mu_2 \mu_3 (\mu_1 + \mu_2 + \mu_3)^3} \tag{A8}
\]

For \( k = 2 \).

Case 1. \( n = 2 \) and \( m = 2 \).

\[
\mu_{2:2}^{(2)} = \frac{3}{2} \left[ \mu_1^2 + \mu_2^2 - \left( \frac{1}{\mu_1^2 + \mu_2^2} + \frac{2}{\mu_1 \mu_2 (\mu_1 + \mu_2)^4} \right) \right]. \tag{A9}
\]

Case 2. \( n = 2 \) and \( m = 3 \).

\[
\mu_{2:2}^{(2)} = \frac{4}{3} \left[ \mu_1^2 + \mu_2^2 - \left( \frac{1}{\mu_1^2 + \mu_2^2} + \frac{2}{\mu_1 \mu_2 (\mu_1 + \mu_2)^4} + \frac{5}{\mu_1 \mu_2 \mu_3 (\mu_1 + \mu_2 + \mu_3)^6} \right) \right]. \tag{A10}
\]
Case 3. \( n = 3 \) and \( m = 2 \)

\[
\mu_{3:3}^{(2)} = \frac{3}{2} \left[ \mu_1^2 + \mu_2^2 + \mu_3^2 - \left( \frac{1}{\mu_1 + \mu_2} \right)^2 + \frac{1}{\mu_1 + \mu_3} + \frac{1}{\mu_2 + \mu_3} \right] \\
+ \frac{2}{\mu_1 \mu_2 \left( \frac{1}{\mu_1} + \frac{1}{\mu_2} \right)^4} + \frac{2}{\mu_1 \mu_3 \left( \frac{1}{\mu_1} + \frac{1}{\mu_3} \right)^4} + \frac{2}{\mu_2 \mu_3 \left( \frac{1}{\mu_2} + \frac{1}{\mu_3} \right)^4} + \frac{1}{\mu_1 + \mu_2 + \mu_3} \\
+ \frac{2(\mu_1 + \mu_2 + \mu_3)}{\mu_1 \mu_2 \mu_3 \left( \frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\mu_3} \right)^4} + \frac{8}{\mu_1 \mu_2 \mu_3 \left( \frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\mu_3} \right)^5} \right]. \quad (A11)
\]

ACKNOWLEDGEMENT

The author is very grateful to anonymous referees for their valuable comments and helpful suggestions that improved the presentation of the paper.

[Received February 22, 2002.]

REFERENCES


Dr. Yousry Hussein Abdelkader, Department of Mathematics, Faculty of Science, Alexandria University, Alexandria, Egypt.

Present address: Dept. of Mathematics, Faculty of Science, King Saud University, Al-Gassem Branch, P. O. Box 237, Buraidah, 81999. Kingdom of Saudi Arabia.

c-mail: yabdelkader@yahoo.com