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Games with Unawareness

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Abstract

We provide a tool to model and solve strategic situations where players' perceptions are limited, in the sense that they may only be aware of, or model, some of the aspects of the strategic situations at hand, as well as situations where players realize that other players' perceptions may be limited. We define normal, repeated, incomplete information, and dynamic (extensive) form games with unawareness using a unified methodology. A game with unawareness is defined as a collection of standard games (of the corresponding form). The collection specifies how each player views the game, how she views the other players' perceptions of the game and so on. The modeler's description of perceptions, the players' description of other players' reasoning, etc. are shown to have consistent representations. We extend solution concepts such as rationalizability and Nash equilibrium to these games and study their properties. It is shown that while unawareness in normal form games can be mapped to incomplete information games, the extended Nash equilibrium solution is not mapped to a known solution concept in the equivalent incomplete information games, implying that games with unawareness generate novel types of behavior. **JEL Classification: C72,D81,D82.**

1 Introduction

As game theoretical modeling becomes more prevalent as a practical modeling tool a central problem arises due to a multitude of models that can be employed. The reasoning players in a particular strategic interaction seem likely to come up with different games representing

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the situation at hand. This discrepancy may be the result of players being unaware of some aspects of the situation, or that the mere act of modeling and reasoning about the situation leads the players to incorporate only a selection of aspects they deem most crucial, or both. In all cases the player obtain a restricted perception of the game. The objective of this paper is to provide a model that allows the players to have differing models for their interaction and analyze reasoning and behavior in this setting.

Players who model the game differently should also be allowed to recognize that other players may well have different models for the strategic interaction. We provide a new game form that allows the representation of economic agents with varying restricted perception of the environment at hand. The game form that we call – games with unawareness – is a collection of standard games describing the perception of each player, their perception of other players' perceptions and so on. The standard games in the collection are then related based on properties of reasoning about others reasoning in a manner that guarantees that how each player views all interactive perceptions of the game is itself a game with unawareness – players perceive the situation much like the modeler does. We use the same methodology to extend normal (strategic) form games, repeated games, incomplete information games and dynamic (extensive) form games, providing a unified framework for reasoning about limited perceptions, or unawareness. We then turn to solution concepts and extend rationalizability and Nash equilibria to normal form games with unawareness, as well as the other game forms.

Our main results include the consistency of the representation of games with unawareness for the aforementioned game forms, i.e. that every player in every game form views the game as a game with unawareness that all players perceive other players' view of the game as a game with unawareness, and so on. We prove the existence of the extended Nash equilibrium solution. Finally, we show that while games with unawareness can be naturally mapped to games with incomplete information, the latter cannot replace the explicit modeling of games with limited perceptions since the mapping cannot preserve the new solution concept. In particular, the extended Nash solution concept generates a tighter prediction about behavior than the Bayesian Nash equilibrium of the equivalent incomplete information game.

1.1 Two Examples

We operationalize restricted perception of the game as follows: A player – Alice – can be unaware of some of the actions available to another player – Bob. This will imply that the game she perceives does not contain some of Bob's actions. Alternatively, Alice may not include a third player in the game at all. This again represents a restricted game. It might also be the case that Bob perceives that Alice is unaware of some aspect of the game, in this case his view of her view of the game is restricted but it does not imply that Alice is indeed restricted in this manner. It might well be that Alice *is* aware of all aspects of the interaction, yet Bob's perception of her perception is limited. The two examples below provide some variations on these type of situations. They also shed some light on how the solutions to these games are constructed. The guiding principle for behavior is that every player chooses a strategy in the game they perceive which reacts (e.g. with a best response) to strategies the player believes others will be playing in the game she thinks they perceive.

Our first example is of a normal form game with unawareness. We begin with the game depicted in (1) below. This game represents all the actions available to the players – Alice and Bob – and the payoffs associated with each action profile. Assume that in the situation we are modeling Alice and Bob are both aware of all the actions available in the game, so when *they* write a description of the actions and payoffs it corresponds to (1) below. Assume that they are commonly aware of each other's existence, i.e. they are aware that the other player is aware that they both participate in the game, they are aware of that, and so on. However, consider a situation where Alice is unaware that Bob is aware of *all* her actions. In particular, She is only aware that he is aware of the actions $\{a_1, a_2, b_1, b_2, b_3\}$ – She is unaware that he is aware of her third action. This situation may arise if Alice perceives that action a_3 is secret, or if, say, Bob is new to the environment in which this interaction occurs and there is no reason to think he would model action a_3.

The strategy profile (a_2, b_1) is the unique Nash equilibrium of the game depicted in (1), hence if such games have been played in the past (even if fully perceived by all players) there is no reason that a data point containing a_3 would appear. Hence, even if Alice believes an inexperienced Bob will study similar past situations, she may well conclude that he will not model this action. In this example we further assume that Bob realizes all this. Not only does he consider a_3 he also deduces that Alice does not realize that he is considering it. Hence, Bob is aware that Alice perceives Bob's perception of the game to consist only of the actions $\{a_1, a_2, b_1, b_2, b_3\}$. We also assume that Bob is aware that Alice is aware of the whole action set $\{a_1, a_2, a_3, b_1, b_2, b_3\}$.

Turning to higher levels of interactive views of the game; since Alice is only aware of Bob being aware of $\{a_1, a_2, b_1, b_2, b_3\}$, she cannot be aware that he is aware that she is aware of anything beyond this set, otherwise, she would be aware that he is able to reason about her reasoning about the additional action a_3, so he must be able to reason about a_3 as far as Alice can deduce, which would contradict our assumption that Alice is unaware that Bob is aware of a_3. In particular, any higher order iteration of awareness of Alice and Bob which is not considered above is assumed to be associated with the set $\{a_1, a_2, b_1, b_2, b_3\}$.

		Bob		
		b_1	b_2	b_3
Alice	a_1	0,2	3,3	0,2
	a_2	2,2	2,1	2,1
	a_3	1,0	4,0	0,1

(1)

As note above, the initial game we started out with in (1) has a unique Nash equilibrium (a_2, b_1) obtained by iteratively eliminating strictly dominated strategies. However, while both players are aware that this is the game being played, we assumed that Alice perceives that Bob is aware of two of her actions. In other words, Alice is unaware that Bob is aware of a_3. While Alice and Bob both view the game as in (1) Alice perceives that Bob finds the game being played as depicted in (2).

		Bob		
		b_1	b_2	b_3
Alice	a_1	0,2	3,3	0,2
	a_2	2,2	2,1	2,1

(2)

Hence, Alice also finds that Bob finds that *she* perceives the game as in (2), and so on for every higher order awareness. We obtain that Alice finds that Bob views the game as a standard normal form game with complete awareness – a game where all participating players are aware of all aspects of the game, they are aware that all other players are aware of the same game, and so on. Taking Nash equilibria as the solution concept for normal form games, Alice may deduce that Bob plays according to the Nash equilibrium (a_1, b_2) of the game in (2) which is also the Pareto dominant outcome of this normal form game. Alice, who sees herself as being more aware than Bob, will be inclined to choose her best response to b_2 which is a_3. Bob can make the exact same deduction that we, as modelers, just made, since he is aware of all the actions and he fully realizes how Alice perceives *his* awareness. Hence, Bob can deduce that Alice, being unaware of his full awareness of her action set, will assume he plays b_2 and will play a_3. This would lead Bob to play his best response to a_3 which is b_3. We will have that Alice chooses a_3 and Bob chooses b_3 as a result of this higher order unawareness. Starting with a Nash equilibrium and considering best responses, we end up with the worst possible payoff for Alice and a low payoff for Bob, even though both are aware of the full extent of the game, both are commonly aware of the action profile of the unique Nash equilibrium (a_2, b_1) and both act rationally given their perceived view of the

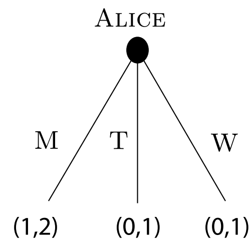

Figure 1: Alice's Shipping Decision.

game.

This example illustrates the general formulation of a game with unawareness as a collection of standard normal form games: each game describing how players view the game, how they view how others view the game and so on. The discussion above indicates some of the properties linking these viewpoints, e.g. that all aspects of the game Alice views that Bob is reasoning about are part of the game she is considering – if she is aware that Bob is aware of some aspect of the game she is also aware of that aspect and find it relevant for the strategic situation (if only for the reason that she find Bob finds it relevant).

The reasoning leading to the behavior described above allows us to define solution concepts for normal form games based on best responses: Alice plays a best response in the game she perceives to a strategy of Bob in the game Alice finds that Bob is considering, etc.

The same approach to defining a game form and solutions can be applied to dynamic settings which allow us to also capture changing perceptions as in our next example. We begin with a story in which Alice – the baker – has just contracted with Bob – the coffee shop owner – to deliver a shipment of cakes from her bakery to his coffee shop. Alice can either ship the cakes on Monday, Tuesday or on Wednesday. The contract states that Alice must ship the cakes on Monday, and that penalties are to be enforced if she ships it late. The contract also states that if unforseen contingencies beyond Alice's control obtain, such as severe weather, she must ship the cakes as soon as possible, with penalties adjusted accordingly. Without any other possible events or actions the game can be written as described in Figure 1.

The payoff to Bob is high if the shipment is sent on Monday and arrives without problems, it is lower if it is sent on Tuesday or Wednesday, but the penalty on late delivery is such that Bob is indifferent between the two. Alice, on the other hand, would rather send the shipment on Monday, however, even though the penalty is higher for shipping on Wednesday rather than Tuesday, it is known that Tuesday is very busy for Alice and she would probably expect to pay overtime to get the shipment out on Tuesday.

Driving to his coffee shop early Monday morning Bob notices that unanticipated road work is scheduled for that day (and that day only). He estimates that if Alice ships the cakes that day then there is a 50% chance it will not be delivered. Hence, his expected payoff if Alice attempts delivery on Monday drops to, say 1.2, somewhat lower than 1.5 since if she attempts to ship on Monday and fails due to the unforseen road work, he will not be compensated for a later delivery according to the contract. But if Alice chooses to ship on Tuesday or Wednesday to begin with, no unforseen contingency can be invoked and payoffs for Bob would remain the same.

The central departure from a standard game in this story is that Bob recognizes the game to include the possibility of road work, but he finds that Alice, much like he was before he drove to his coffee shop, is unaware of the road work. Bob recognizes that it is not that she thinks there will be no road work, it is that the possibility of road work never crossed her mind, Bob thinks that Alice describes the game – models the economic environment – according to the game depicted in Figure 1 just like he did before taking into account road work.

As he drives to work Bob realizes that he can call Alice and let her know that there is road work. In other words, he can strategically *make* her become aware of the road work. If he does this early enough in the morning he expects he can, due to the unforseen road work, agree that she delivers the cakes on Tuesday without being penalized but that if she delivers them of Wednesday he will be compensated for the delay. Hence Bob views the possible actions, including the road work and his ability to call Alice as the game depicted in Figure 2. Bob associates with each of Alice's decision points the game that she is aware of, in particular, if he does not call her, he assumes that she finds the game to be the one depicted in Figure 1 and if he does call her then she will figure out the game is as depicted in Figure 2. Note that if Bob calls Alice and tells her of the road work she will be able to reason that had he not called, she would not have become aware of it.

What Bob is unaware of is that Carol, one of Alice's workers, lives close to Bob's bakery. Furthermore, Carol noticed on Sunday evening the signs indicating scheduled road work on Monday and decided to e-mail Alice to let her know she might be late for work. Hence, Carol's e-mail made Alice aware of the road. Alice realizes the game is actually the one depicted in Figure 3 and that Bob will see the road work Monday morning when driving to work, and that he will model the game to be as in Figure 2 and reason about Alice's view of the game being as in Figure 1 since she would – rightly so – assume he is unaware of Carol's possible actions, and in fact is unaware of Carol's existence. Carol, is simply not modeled as a player in Bob's perception of the game. The situation amounts to Bob thinking that Alice is unaware of the road work and Alice not only aware of it, but also realizing that Bob thinks she is unaware of it.

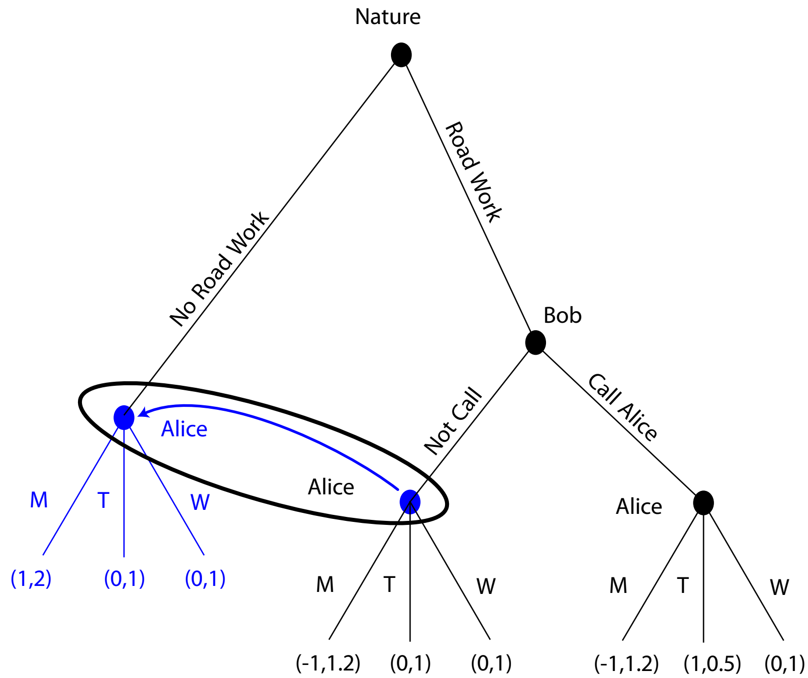

Figure 2: Bob's Calling Decision.

This is the game as seen by Bob, if he does not call Alice he thinks that Alice views the game as in Figure 1 depicted in blue (the arrow indicates that in both decision points of the information set the Alice perceives the same game, in particular she is unaware that there is an information set.)

The payoffs in our story change when we consider Carol, Alice's awareness of road construction, and that she can reason about Bob's restricted view of the game and how Bob views her view of the game as a further restriction of his. First, for simplicity, Carol's payoff are such that e-mailing Alice is a dominant strategy. Alice's payoffs coincide with Bob's view if Carol does not e-mail her, or if Bob decides to call Alice and let her know of the road work. However, if Bob decides not call Monday morning, Alice can claim that she discovered the road work late on Monday[1] and since she was not told in the morning they agree that she would be allowed to deliver on Wednesday without penalty. Obviously, if she does deliver on Wednesday in this case Bob's payoffs are the lowest possible in this game, while Alice's are the highest.

The three games depicted in Figures 1, 2, and 3 as well as how they relate – which decision point is associated with which game – comprise a game with unawareness. They describe the standard extensive form game in Figure 3 as Alice sees it when she is making a decision (it also describes the modelers view of the dynamic game). It describes how Alice views Bob's

[1]To keep this example tractable we have encompassed some possibly strategic actions into the payoffs, e.g. Alice informing Bob of her awareness on Monday afternoon, or even claiming she tried to deliver on Monday.

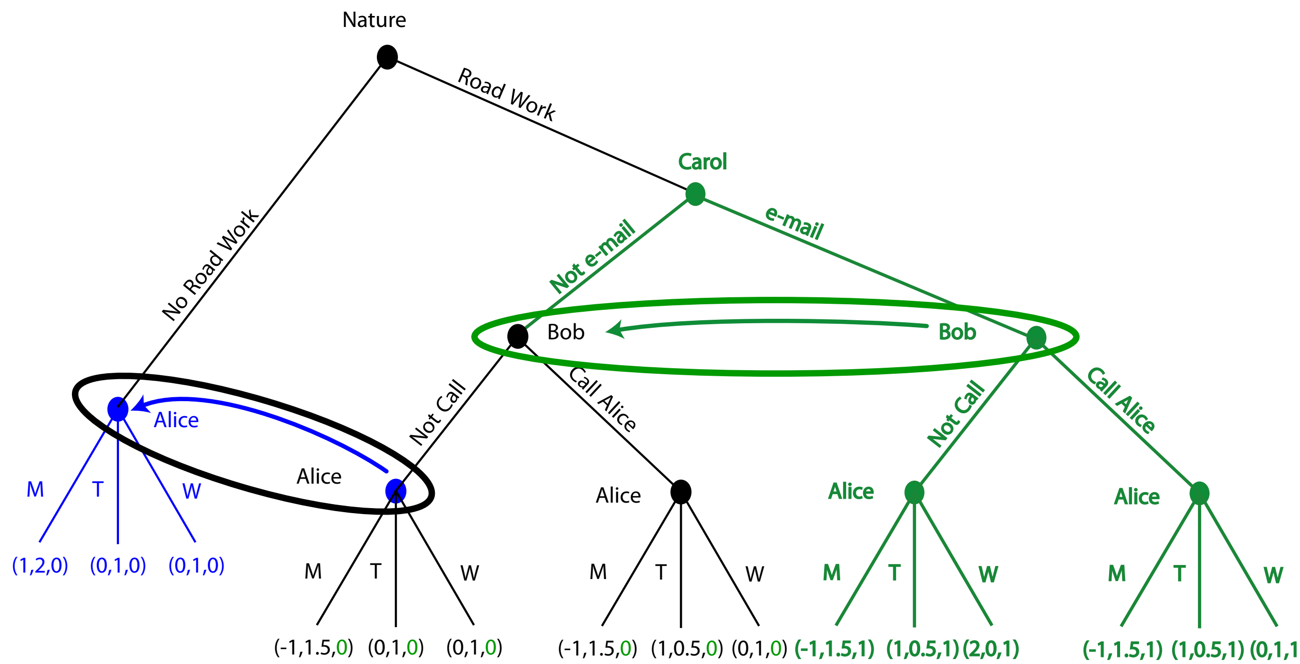

Figure 3: With Carol's E-mail Decision.

This is the game as seen by Alice (once Carol E-mails her, as well as the modeler. We have indicated in green the parts of this game that Bob is unaware of. These include Carol's actions and payoffs – her existence. As well as the fact that Alice may be aware that Bob *is* aware of effect of the road work on Alice's payoff.

viewpoint on what the game is as depicted in Figure 2. This is also the (standard) extensive form game that Bob Actually perceives. Finally, it describes how Bob views Alice's view of the game at *both* her decision points in the game in Figure 2 as corresponding to the game in Figure 1.

The dynamic game with unawareness is a collection of standard extensive form games corresponding to how Alice perceives the game at her decision points, how Alice perceives how Bob perceives the game at his decision point in the game she considers, how she perceives the game he perceives that she perceives in her decision point in the game she perceives he is considering, and so on. Here, Alice's view of the game changes based on the actions of Carol (if she e-mails) and Bob if he were to call Alice. As with the extension of normal form games equilibrium solutions will be based on strategies that are best responses to strategies that others play in the game they are perceived to consider.

1.2 Results and Related Literature

The examples above are meant to illustrate a number of features of our model. First, the definition of games with unawareness for all game forms is based on a collection of standard games, games that describe players' viewpoints, views of other players views of the game and so on. In the case of dynamic games viewpoints correspond to decision points, in incomplete information games they will correspond to types. We can also see the nested features of these games where the actions and players that Alice views Bob is considering become part of the game that Alice is considering. The examples also demonstrate how solutions can come about in these games.

Our structural result shows that the definition of games with unawareness (in all forms) implies that each player's view of the game, view of others, and so on, itself is a game with unawareness satisfying the exact same conditions as the game we, the modelers, are considering. Furthermore, each view of each other player induces a game with unawareness and so on. Next we define a variety of solutions and show that extended Nash equilibria and extended rationalizability coincide with the standard solutions when there is no unawareness. The solutions are extended based on the same reasoning justifying the solutions in standard games. We show existence of solutions. Finally, we present sufficient conditions for a game with unawareness to have a finite representation.

One of the main questions that arises with any new game form is whether it can tell us something that the standard formulations cannot. With restricted perceptions it is particularly important to study whether these situations can be modeled using probability zero rather than unawareness. We show that indeed normal form games with unawareness can be represented as games with incomplete information. Moreover, there is a natural mapping (in the sense that the mapping is independent of the exact payoffs in the game with unawareness) to incomplete information games that fully captures all the relevant views in the game. But while the structure can be mapped between game forms it is the extended Nash equilibrium which does not map to any known solution of Bayesian games. In fact, it seems that generating such a solution in the Bayesian game amounts to reconstructing the normal form game with unawareness. The extended Nash equilibrium turns out to be a refinement of the Bayesian Nash equilibrium of the incomplete information representation since the latter ends coinciding with the extended rationalizable solution for normal form games with unawareness. We note that with other game forms, in particular incomplete information games with unawareness, an extended solution will treat probability zero and awareness in a different manner to begin with, making it at least as challenging as the normal form case to map these games to known game forms.

There are three categories of work that relates to multi-person unawareness in economic theory. The first is a logic, foundational, perspective. Some recent work on modeling multi-person unawareness includes Fagin and Halpern (1998), Halpern (2001), Halpern and Rego (2008, 2009), Feinberg (2004), Heifetz, Meier and Schipper (2006, 2008), Sadzik (2006) Board and Chung (2007, 2009), Li (2009) and Gossner and Tsakas (2010). These provide a variety of models capturing the foundations of interactive unawareness.

The second category of multi-person unawareness studies includes analysis of specific economic situations that involve unawareness and analysis of the role of unawareness in these settings. Some examples include Chen and Zhao (2009), Filiz-Ozbay (2008), Grant, Kline and Quiggin (2012), von Thadden and Zhao (2012), Zhao, X.J. (2008, 2011).

The third category to which the current paper belongs includes models for strategic interaction with unawareness and their solution concepts. Our early work in Feinberg (2004) included a syntactic approach to modeling dynamic games and produced quite a cumbersome definition. In Feinberg (2005) which the current paper subsumes normal form games were defined, these have been modified and generalized to the other game forms. Models for normal form games appear in Copic and Galeotti (2005), Li (2006) and Heifetz, Meier and Schipper (2007). Extensive form games were suggested by Grant and Quiggin (2009), Halpern and Rego (2006), Heifetz, Meier and Schipper (2009) and Rego and Halpern (2012), incomplete information games by Copic and Galeotti (2007) and repeated games by Mengel, Tsakas and Vostroknukov (2009). We point out a number of aspects that distinguish this current work from the literature that followed. First, we note the unified approach to extending all game forms to games with unawareness, all game forms are extended in the same manner following the same conditions for high order unawareness. Second, we use standard games to define games with unawareness – alternative methods introduce new constructs, our game with unawareness is always a set of standard games.

The use of standard games keeps the definition relatively short and simple with four conditions applied to each game form. Furthermore, it allows a relatively clear extension of existing solution concepts to games with unawareness. Using hierarchies of perceptions allows the proof of consistency of the representation in the sense that the modeler, players, and higher order players' view of other players, all have the same form as games with unawareness (we discuss an equivalent "type" space representation below). Finally, our framework allows the comparison between unawareness and probability zero events – we show that games with unawareness can be represented by incomplete information games, however behavior (extended Nash equilibria) under unawareness does not translate to the known solutions of standard incomplete information games.

2 Modeling Games with Unawareness

A game with unawareness is defined by describing the set of players, actions and payoffs in the strategic situation, how each player views these, how they view others' views and so on, while allowing these views to be restricted – exclude players or actions from such perceived games. The main principle for defining these games is that every view point is described by a standard game of the corresponding form.

We extend four game forms to games with unawareness: normal, repeated, incomplete information and dynamic (extensive). We postulate four conditions that are adapted to each game form:

Condition 1 The decision maker Alice views the decision maker Bob to be relevant for the strategic situation if and only if she views Bob to be a player in the game she is considering.

Condition 2 Every action or player that Alice perceives that Bob is modeling in the game, is also part of Alice's perception of the game – Alice can consider the aspects of the strategic situation that she thinks Bob is considering.

Condition 3 Alice's view of Bob's perception of the game coincides with her view of Bob's view of his *own* perception of the game.

Condition 4 Even if Alice is unaware of Bob's participation in the game, the outcomes she considers must agree with the outcomes of the game given one of Bob's actions.

we generalize these conditions to high order reasoning in the formal definitions below. The rest of this section is organized as follows. We begin with a definition of normal form games with unawareness which is followed by a discussion of the hierarchy structure of the game form. The following subsections present the definitions of the other game forms concluding with the proposition stating that our model provides a consistent representation of games with unawareness in the sense that every player's view of the strategic situation is itself a well defined game with unawareness, as is a player's view of the view of others, and so on, and that these forms coincide with our description of the game as modelers.

2.1 Normal Form Games with Unawareness

In standard normal form games the modeler describes the set of players, their possible actions and payoffs for action profiles. The modeler's normal form game is our starting point: $G = (I, \prod_{i\in I} A_i, \{u_i\}_{i\in I})$ where I is a set of players, A_i is the set of actions available for each player and the functions u_i associate the utility for action profiles in $\prod_{j\in I} A_j$. These are the set of players and actions that the modeler is considering whether or not the players are aware of each other or of some of the actions. Each player may have a restricted view of the game. Hence for a player $\mathrm{v} \in I$ we consider a normal form game $G_\mathrm{v} = (I_\mathrm{v}, \prod_{i\in I_\mathrm{v}} (A_i)_\mathrm{v}, \{(u_i)_\mathrm{v}\}_{i\in I_\mathrm{v}})$. Similarly, a player considers how each of the players that appear in her game models the game. In general, a finite sequence of players $v = (i_1, ..., i_n)$ is associated with a normal form game $G_v = (I_v, \prod_{i\in I_v} (A_i)_v, \{(u_i)_v\}_{i\in I_v})$ where I_v is the set of players that i_1 finds that i_2 finds that ... that i_n is considering, and similarly for the sets of actions $(A_i)_v$ and payoffs $(u_i)_v$ defined on the set of action profiles $\prod_{i\in I_v} (A_i)_v$. We call v an *iterated view*, or in short a *view*. Throughout, $v = \emptyset$ corresponds to the modeler's view, i.e. $G_\emptyset = G$. Note that this is the modeler's view of the relevant players and their actions and payoffs not of the unawareness of players. Correspondingly, we say that G_v is the situation as viewed, or perceived at v, that $(A_j)_v$ is the set of j's actions as viewed from v and so on. We denote an action profile in G_v by $(a)_v$. The singletons ($\mathrm{v} = i$) corresponding to a player's view are called *viewpoints* and the set of viewpoints (players, in the case of normal form games) is denoted V with a typical element v.

We denote the concatenation of two views $\bar{v} = (i_1, ..., i_n)$ followed by $\tilde{v} = (j_1, ..., j_m)$ as $v = \bar{v}\hat{\ }\tilde{v} = (i_1, ..., i_n, j_1, ..., j_m)$. The set of all potential views is denoted $\overline{V} = \bigcup_{n=0}^{\infty} (I)^{(n)}$ where $I^{(n)} = \prod_{j=1}^{n} I$ and $I^{(0)} = \emptyset$.

Definition 1 *A collection $\Gamma = \{G_v\}_{v\in\mathcal{V}}$ where G_v are normal form games and $\mathcal{V} \subset \overline{V}$ is a collection of finite sequences of players is called a* normal form game with unawareness *and the collection of views $\mathcal{V}$ is called its* set of relevant views *if the following properties hold:*

C1 For every $v \in \mathcal{V}$ we have

$$v\hat{\ }\mathrm{v} \in \mathcal{V} \text{ if and only if } \mathrm{v} \in I_v \tag{3}$$

The first condition requires that the set of relevant views $\mathcal{V}$ is closed under the set of players considered in the game perceived at a relevant view and that viewpoints of non-players are irrelevant. There would be no impact on our results if players were to consider the views of players not participating in the game, however we find such redundancy unpleasing, as with any scientific modeling. The other direction of this condition is crucial for our setting since if Alice models Bob as one of the players in the game, it is required that Alice should find Bob's view of the game to be relevant.

C2 For every $v\hat{\ }\tilde{v} \in \mathcal{V}$ we have

$$v \in \mathcal{V} \tag{4}$$

$$\emptyset \neq I_{v\hat{}\tilde{v}} \subset I_v \tag{5}$$

$$\text{for all } i \in I_{v\hat{}\tilde{v}} \text{ we have } \emptyset \neq (A_i)_{v\hat{}\tilde{v}} \subset (A_i)_v \tag{6}$$

The first part of this condition states that if a relevant view's perception of another view is relevant, then the first view must itself be relevant, e.g. if it is relevant to consider Alice's view of Bob's view of Carol, then Alice's view of Bob is also relevant. Together with condition **C1** this implies that the set of relevant views is exactly the set of views inductively constructed from considering the players that are perceived to be participating in the game. In particular, if there exists any relevant view at all we also have, fortunately, that $\emptyset \in \mathcal{V}$ – the modeler's view is relevant.

Condition **C2** extends this principle to the set of players and actions: if Alice finds that Bob is considering a player or an action as part of the game, she herself must consider them to be part of the game. In other words, by the mere fact that you find in relevant that others find some aspect of the game to be relevant enough for modeling, you must model that aspect yourself. Much like we, the modelers, do when considering the reasoning of players. Restating this with the notation of awareness the condition states that: what Alice is aware that Bob is aware of, are things that Alice is aware of as well.

C3 If $v\hat{}\mathrm{v}\hat{}\bar{v} \in \mathcal{V}$ we have

$$v\hat{}\mathrm{v}\hat{}\mathrm{v}\hat{}\bar{v} \in \mathcal{V} \tag{7}$$

$$G_{v\hat{}\mathrm{v}\hat{}\bar{v}} = G_{v\hat{}\mathrm{v}\hat{}\mathrm{v}\hat{}\bar{v}} \tag{8}$$

The third condition requires that each player, that is relevant along some view, has a correct perception of their own perceived perception: if Alice perceives that Bob has a certain perception of the game, she also perceives that he perceives to have that perception. With awareness: if Alice is aware that Bob is aware of something, she is also aware that he is aware that he is aware of it. It is important to point out that this does *not* imply that Bob is actually aware of it. We note that when considering a relevant view of the form $v\hat{}(i)$ we have that $v\hat{}(i)\hat{}(i)$ is relevant and by **C1** we have $i \in I_{v\hat{}(i)}$ hence all players are aware of their own participation in the game.

C4 For every action profile $(a)_{v\hat{}\tilde{v}} = \{a_j\}_{j \in I_{v\hat{}\tilde{v}}}$ there exists a completion to an action profile $(a)_v = \{a_j, a_k\}_{j \in I_{v\hat{}\tilde{v}}, k \in I_v \setminus I_{v\hat{}\tilde{v}}}$ such that

$$(u_i)_{v\hat{}\tilde{v}}((a)_{v\hat{}\tilde{v}}) = (u_i)_v((a)_v) \tag{9}$$

Since a view may consider only some of the players considered by another relevant view, or by the modeler, the payoffs may not be uniquely determined by defining the restricted set of players and actions. The fourth condition requires that the payoffs in a restricted game coincide with payoffs in the larger game with more players by fixing some action profile for these players. We note that for different action profiles of the restricted game we can have different completing actions by players that only appear in the less restricted game[2]. In other words, a restricted view of the game cannot introduce new payoffs. If one wishes, this condition can easily be generalized to assume missing players are playing mixed strategies, or even correlated strategies, without impact on our results.

These properties are used in all game forms discussed below. While the objects in the game may change (with the addition of histories, types, or game trees) we obtain analogous properties describing the structure of games with restricted reasoning translating conditions **C1-C4** to the appropriate setting.

2.2 Hierarchies and Types

Before we define the extension to additional game forms, we briefly discuss our choice for describing a game with unawareness as a collection of standard form games – a "hierarchy" of games, rather than a "type space" – a set of type profiles representing each player's perception of the game and perception of the types of other players. First we note that a representation a la Harsanyi's type spaces is possible and actually quite readily obtained and is briefly provided below. Moreover, one can also define a universal type space which corresponds to the incomplete information representation we provide in Section 3.2. Hence, the transition from hierarchies to type spaces and back is quite a standard (if tedious) exercise. However, this brings us to a central reason for choosing hierarchies: in the standard incomplete information framework, the set of fundamentals generating the universal type space is fixed. In particular, all players are implicitly assumed to be reasoning with the same set of fundamentals which, in the Harsanyi's approach, includes the collection of relevant payoff matrices for the fixed and given set of players and their given sets of actions. This makes the universal type space "common knowledge" in the sense that all players would construct the same universal type space if they were modeling the game, see Aumann (1999) for a discussion of this property. In contrast, our starting point is that players may reason using different fundamentals – the building blocks of the game. As such, they would construct different universal type spaces – different from each other and different from the

[2]This condition is weaker than the condition we proposed in Feinberg (2005) where we required that in all the games a certain player is missing, the *same* action of this player will be used to determine the payoffs. We also note that conditions **C1 - C3** coincide with the weak axioms for unawareness we postulated in Feinberg (2005).

one the modeler uses. Moreover, the players would also attribute different universal type spaces to other players. As such, while we can combine all these to an abstract space as sketched below, working with such a space seems much more difficult than the analysis we did in the examples in the introduction via the hierarchy representation. Working with a type space representation that captures reasoning with differing fundamentals requires extra caution since one needs to figure out at each state how each player perceives the type space. While this is possible, it essentially amounts to reconstructing the hierarchies. Hence, we prefer working directly with hierarchies. This is best exemplified when constructing solution concepts. If we want a player type to play a best response to what other players' types are playing we need to consider the types that correspond to this player's type space, which is not the modeler's type space. In addition, this player will need to evaluate the type spaces that each other player is considering, the spaces that each of them associates to other players (more precisely, the spaces that the first player *perceives* the other players think others are considering), and so on – recreating the hierarchies once again. Finally, the hierarchies allow us to obtain an "equal perception" principle. In the sense that the modeler and the players are easily shown to model the situation in the same manner (as well as model others' perceptions and so on). While they may use different fundamentals for reasoning and attribute different fundamentals to others, the rules governing the reasoning, the relationships between these sets of fundamentals and the definition governing the game form all are of the exact same form. Not only are the games the players consider of the same form, the games they perceive other consider are of the same form and all these coincide with the modeler's game form. This is not a minor issue since, at least in economics, since one might consider it desirable that a model used in describing or predicting behavior will not be at odds with its own predictions if the modeled players themselves were to use the exact same modeling approach. When one begins with an abstract modeler type space alone, it is not a priori clear whether this property holds.

We describe a type space representation for normal form games with unawareness with two players that are aware of each other. The generalizations are later described. The type space representation is defined as a product $T_1 \times T_2 \times \lambda$ and mappings τ_1, G_1, τ_2, G_2 such that for $i \in \{1,2\}$ and $j \neq i$ we have: T_i is player i's type space, Λ is a set of normal form games, $\tau_i : T_i \to T_j$ and $G_i : T_i \to \Lambda$. Hence, each $t_i \in T_i$, is mapped to a member $\tau_i(t_i) \in T_j$ and a game $G_i(t_i) \in \Lambda$.

We now add the conditions that correspond to conditions **C1-C4** for the type space representation. Condition **C1** will be represented by the requirement that the game $G_i(t_i)$ is a two person game played by i and j. Condition **C3** will hold immediately from the type space representation. Condition **C4** holds for this specific case where there is no unawareness of players. For condition **C2** we will require that the game $G_j(\tau_i(t_i))$ is obtained from the game $G_i(t_i)$ by (at most) eliminating pure strategies.

We need to show that every game with unawareness corresponds to a hierarchy of some type space and mappings which satisfy the above conditions and that for every type space with mappings as above we have that every pair of types generates a hierarchy that corresponds to a game with unawareness. The first part will be shown when we discuss the canonical representation of games with unawareness later on. The second part follows from noting that for a given type t_i, we can define the following hierarchy of games $G_i(t_i), G_j(\tau_i(t_i)), G_i(\tau_j(\tau_i(t_i))), ...$, which corresponds to the view $i, i\hat{}j, i\hat{}j\hat{}i, ...$ respectively. For every pair of types t_1, t_2 we define $G_\emptyset$ as the game that contains the union of pure strategies of $G_1(t_1)$ and $G_2(t_2)$ (choosing arbitrary payoffs when not determined by G_1 or G_2). We define the games for all views by extending the above inductively so that $G_{v\hat{}\mathrm{v}\hat{}\bar{v}} = G_{v\hat{}\mathrm{v}\hat{}\mathrm{v}\hat{}\bar{v}}$ and we have a game with unawareness satisfying **C1-C4**.

This equivalence extends to more than two players without any change in conditions. For the case where players may be unaware of other players we need to modify τ_i so that it does not necessarily map to a type profile of *all* other players, but only the players that the specific type t_i is aware of. Similarly, $G_i(t_i)$ will include exactly the players that $\tau_i(t_i)$ maps to in addition to player i herself. Finally, we require that for a player j in $G_i(t_i)$ the game $G_j(\tau_i(t_i))$ is obtained from $G_i(t_i)$ by at most eliminating actions and players and preserving the payoff condition as in condition **C4**. Thus, the resulting type space and mappings will represent all normal form games with unawareness. Note that while the representation has a fairly simple space structure, the set of fundamentals considered by different types (even of the same player) can be quite different.

For the most part a compact representation of a game with unawareness can be produced. For example, in the first example in the introduction Alice and Bob are both aware that the game is as in (1), but Alice thinks that Bob is unaware of one of her actions and considers the game as in (2). Bob actually realizes that Alice attributes to him this restricted view. The representation of this game with unawareness can be collapsed to three states $\{x, y, z\}$, two states x, y corresponding to (1) and the state z to (2). Alice is unaware of the actual state x, at state x she considers state y to be the actual state, at state y Bob would consider state z to be the actual state and he would be unaware of state y. Hence, Alice thinks Bob views the game as (2) while Bob realizes this and actually views the game as (1). For the seminal formal treatment of interactive unawareness and in particular the relationship between models (state space formulation) and the syntax of hierarchies see Fagin and Halpern (1988).

2.3 Repeated Games with Unawareness

Repeated games introduce a dynamic environment where the restricted view of the players may change as the games unfolds. Their view may be widened from directly observing behavior they previously did not consider, from reflecting about behavior, or any other means of discovery. A view can also dynamically narrow from forgetfulness, or by deeming some aspect of the game irrelevant at some point.

As with the normal form games, we consider a collection of repeated games corresponding to relevant views – sequence of viewpoints. This time, however, a player's limited view may depend on the period in which the game is played and, most importantly, it may also depend on the history of play that this player observed. Hence, the collection of potential *viewpoints* includes every player at every period conditional on every possible history of actions by a subset of players.

We define repeated games with unawareness as a collection of standard repeated games satisfying the four consistency conditions as above. The first condition accommodates the added dynamic constraint on relevant viewpoints which now refer to specific histories. For example, if Bob cannot reason about a certain action in the first period of the repeated game, he cannot reason in this first period about his, or Alice's, reasoning in the second period conditional on this action being taken. As such, the viewpoint after the action has occurred is not relevant for views that do not consider that action possible. The third condition is modified to allow for imperfect monitoring and forgetfulness. In these cases a player may not be aware of, or remember, the history of play which requires a modification of the self awareness condition. The other two conditions remain intact.

Let $G(T)$ denote the $T \geq 1$ repeated game with a stage game $G = (I, \prod_{i \in I} A_i, \{u_i\}_{i \in I})$ and payoffs determined by the sum or average of the stage game payoffs (similarly we can consider the infinitely repeated δ-discounted games $G(\delta)$). Let $h^t(J) = (\mathrm{a}^1, ..., \mathrm{a}^t)$ denote a history of (restricted) action profiles in the first t periods where for every s we have $\mathrm{a}^s \in \prod_{i \in J} A_i$. Hence a^s is the action profile for the given subset $J \subset I$. We denote by $A_i(h^t(J))$ the actions taken by i along the history $h^t(J)$ assuming $i \in J$, i.e. the set $\{a_i^s | s = 1, ..., t\}$. Let $H = \{h^t(J) | t = 0, 1, ..., T-1 \text{ and } \emptyset \neq J \subset I\}$ be the set of all histories up to length $T - 1$ with $h^0(J) = \emptyset$ for all J.

Each player after each history of play of a subset of players constitutes a possible viewpoint. The set of possible viewpoints is defined as the collection $V = \{(h^t(J), i) | h^t \in H, i \in I, \emptyset \neq J \subset I\}$, i.e. it is each player's view of the game as a function of the history at the time of observation when considering some of the players in the game. As before, a typical viewpoint is denoted by $\mathrm{v} \in V$.

The set of all finite sequences of viewpoints denoted $\overline{V} = \bigcup_{n=0}^{\infty} V^{(n)}$ with $V^{(n)} = \prod_{j=1}^{n} V$

and the convention $V^{(0)} = \emptyset$. As before a finite sequence of viewpoints $v = (\mathrm{v}_1, ..., \mathrm{v}_n)$ is associated with a repeated game $G_v(T) = (I_v, \prod_{i \in I_v} (A_i)_v, \{(u_i)_v\}_{i \in I_v})$ where I_v is the set of players that v_1 finds that v_2 finds that ... that v_n is considering, and similarly for the sets of actions $(A_i)_v$ and payoffs $(u_i)_v$ defined for the stage game with action profiles $\prod_{j \in I_v} (A_j)_v$ and repeated T times.

Definition 2 *A collection* $\Gamma = \{G_v(T)\}_{v \in \mathcal{V}}$ *where* $G_v(T)$ *are repeated games and* $\mathcal{V} \subset \overline{V}$ *is a set of relevant views is called a* repeated game with unawareness *if the following properties hold:*

CR1 For every $v \in \mathcal{V}, \mathrm{v} = (h^t(J), i)$ we have

$$v\hat{\ }\mathrm{v} \in \mathcal{V} \text{ if and only if } i \in I_v, J = I_v, A_j(h^t(J)) \subset (A_j)_v \text{ for all } j \in J \tag{10}$$

The viewpoints that are considered relevant from the view v are exactly those that correspond to players and histories taken from the stage game $G_v(T)$.

CR2 For every $v\hat{\ }\tilde{v} \in \mathcal{V}$ we have

$$v \in \mathcal{V} \tag{11}$$

and

$$\emptyset \neq I_{v\hat{\ }\tilde{v}} \subset I_v \tag{12}$$

as well as

$$\emptyset \neq (A_i)_{v\hat{\ }\tilde{v}} \subset (A_i)_v \tag{13}$$

for every $i \in I_{v\hat{\ }\tilde{v}}$.

CR3 If $v\hat{\ }\mathrm{v}\hat{\ }\bar{v} \in \mathcal{V}$ where $\mathrm{v} = (h^t(J), i)$ then there exists some $\tilde{\mathrm{v}} = (\tilde{h}^t(\tilde{J}), i)$ such that

$$G_{v\hat{\ }\mathrm{v}\hat{\ }\bar{v}}(T) = G_{v\hat{\ }\mathrm{v}\hat{\ }\tilde{\mathrm{v}}\hat{\ }\bar{v}}(T) = ... = G_{v\hat{\ }\mathrm{v}\hat{\ }\tilde{\mathrm{v}}\hat{\ }...\hat{\ }\tilde{\mathrm{v}}\hat{\ }\bar{v}}(T) \tag{14}$$

and $v\hat{\ }\mathrm{v}\hat{\ }\tilde{\mathrm{v}}\hat{\ }...\hat{\ }\tilde{\mathrm{v}}\hat{\ }\bar{v} \in \mathcal{V}$.

This refines condition **C3** by allowing a viewpoint to consider itself with respect to the histories it can reason about. While a view v may consider $\mathrm{v} = (h^t(J), i)$ because the *view* finds the history $h^t(J)$ possible, it will recognize that the decision maker i may not be aware of this history due to imperfect monitoring or forgetfulness. Hence, he might associate himself with other viewpoints such as $\tilde{\mathrm{v}} = (\tilde{h}^t(\tilde{J}), i)$.

CR4 For every action profile $(a)_{v\hat{}\tilde{v}} = \{a_j\}_{j\in I_{v\hat{}\tilde{v}}}$ there exists a completion to an action profile $(a)_v = \{a_j, a_k\}_{j\in I_{v\hat{}\tilde{v}}, k\in I_v\setminus I_{v\hat{}\tilde{v}}}$ such that

$$(u_i)_{v\hat{}\tilde{v}}((a)_{v\hat{}\tilde{v}}) = (u_i)_v((a)_v) \tag{15}$$

The dynamic features of repeated games with unawareness prompt us to consider the properties that relate a player's past and present views. For example, we can consider the assumption that once a player is aware of an action he remembers it and remains aware of it in his future viewpoints. Such a property would suggest a possible explanation for the notion of experience, as more experienced players consider larger, more elaborate, games. To quote Alvin Roth: "One of the most general things that experiments demonstrate is that subjects adjust their behavior as they gain experience and learn about the game they are playing and the behavior of other subjects." (Roth 1995 p.327).

Memory For $v = (i_1(h_{t_1}), ..., i_k(h_{t_k}), ..., i_n(h_{t_n}))$ and $\tilde{v} = (i_1(h_{t_1}), ..., i_k(h_{t_k+l}), ..., i_n(h_{t_n}))$ where h_{t_k+l} is a continuation of h_{t_k} and such that $v, \tilde{v} \in \overline{V}$ we have that $\alpha_v \subset \alpha_{\tilde{v}}$.

Memory assumes that awareness is monotonic, in the sense that what a player (i_k) is aware of after a history (h_{t_k}) he will still be aware of after some continuation of the game. This property states that a player remembers not only actions but also what he was aware of others' awareness, and it also assumes that others are aware the player remembers. Our results hold both with and without this assumption.

To get a better feel for the potential complexity of repeated games with unawareness consider the game depicted in Figure 1 repeated twice. In period 1 Alice and Bob are viewing each others perception at the current period as before: Alice and Bob are aware of the actions $\{a_1, a_2, a_3, b_1, b_2, b_3\}$, and Alice perceives that Bob is unaware of her action a_3, i.e. she views him as viewing the game as depicted in Figure 2 as in our original example. But how Alice views (at either period) the viewpoint of Bob in period 2 will depend on the realization of play in period 1. If Alice played a_3 in period one, she may safely assume that Bob will remember this action and she will attribute to him the awareness he actually already has. Moreover, if Bob plays b_3 in the first period, Alice might deduce that Bob must have been aware of a_3 in the first period (and if he remembers, in the second period as well) even if she did not choose a_3 in the first period. Hence, Alice's perception of Bob's awareness may change not only from the revelation of actions that were considered secretive, but also from behavior that may best be explained by a different scope of awareness Alice should attribute to Bob. In particular, Alice here may realize that her perception was limited (she did not consider Bob considering a_3) and revise her perception. In this case, Alice may actually reason at period 1 about how she would reason in period 2 about Bob's reasoning in period 1 if she observes b_3. The important feature of the game form illustrated here is that the choice of relevant views may depend on how perception changes with observed behavior – a choice we usually associate with a solution manifests here in the game form. For more on this game and how communication impacts strategic interaction with unawareness see Feinberg (2007).

2.4 Incomplete Information Games with Unawareness

Incomplete information games with unawareness allow us to model uncertainty about the awareness of players in addition to uncertainties about the payoffs as well as high order uncertainties about both. They also allows us to model views that do not consider all possible types. As with other game forms we begin with the modeler's view which begins with a standard Bayesian game with a prior $G(B) = (I, \prod_{i \in I} A_i, \Theta_0 \times \prod_{i \in I} \Theta_i, P, \{u_i\}_{i \in I})$ where I is a finite set of players, A_i is player i's finite actions set, $\Theta = \Theta_0 \times \prod_{i \in I} \Theta_i$ is a set of type profiles where Θ_i is the set of player i's types and Θ_0 is the set of states of nature, the u_i's are the players' utilities defined for a realization of Θ and action profiles, and P is a probability distribution over Θ. The distribution P is the probability over the type profiles as seen by the *modeler*. Recall that in a standard incomplete information game each type has a distribution over the other players' type profiles. This will be captured in our setting when we describe the game as viewed by each type which will include a probability distribution corresponding to that type's beliefs.

The set of viewpoints is the set of all possible types $V = \bigcup_{i \in I} \Theta_i$ and a typical viewpoint is denoted by $\mathrm{v} = \theta_i \in V$. The set of all finite sequences of viewpoints is $\overline{V} = \bigcup_{n=0}^{\infty} V^{(n)}$. A finite sequence of viewpoints $v = (\mathrm{v}_1, ..., \mathrm{v}_n)$ is associated with a game $G_v(B) = (I_v, \prod_{i \in I_v} (A_i)_v, (\Theta_0)_v \times \prod_{i \in I_v} (\Theta_i)_v, P_v, \{(u_i)_v\}_{i \in I_v})$ where I_v is the set of players that v_1 finds that v_2 finds that ... that v_n is considering, and similarly for the sets of actions $(A_i)_v$, states of nature $(\Theta_0)_v$, types $(\Theta_i)_v$, with $\Theta_v = (\Theta_0)_v \times \prod_{i \in I_v} (\Theta_i)_v$, the distribution P_v over the viewed type space $\Theta_v = \prod (\Theta_0)_v \prod_{i \in I_v} (\Theta_i)_v$, and payoffs $(u_i)_v$ defined for the viewed states of nature, types and action profiles $(\Theta_0)_v \times \prod_{i \in I_v} (\Theta_i)_v \times \prod_{j \in I_v} (A_j)_v$. The conditions defining the extension to unawareness are almost identical to those used for the normal form games.

Definition 3 *A collection* $\Gamma = \{G_v(B)\}_{v \in \mathcal{V}}$ *where* $G_v(B)$ *are games as above and* $\mathcal{V} \subset \overline{V}$ *is a set of relevant views is called an* Incomplete Information Game with unawareness *if the following properties hold:*

CI1 For every $v \in \mathcal{V}$, $\mathrm{v} = \theta_i \in \Theta_i$ we have

$$v\,\hat{}\,\mathrm{v} \in \mathcal{V} \text{ if and only if } i \in I_v, \theta_i \in (\Theta_i)_v \tag{16}$$

CI2 For every $v\,\hat{}\,\tilde{v} \in \mathcal{V}$ we have

$$v \in \mathcal{V} \tag{17}$$

and

$$\emptyset \neq I_{v\,\hat{}\,\tilde{v}} \subset I_v \tag{18}$$

as well as

$$\emptyset \neq (\Theta_i)_{v\,\hat{}\,\tilde{v}} \subset (\Theta_i)_v \tag{19}$$

$$\emptyset \neq (\Theta_0)_{v\,\hat{}\,\tilde{v}} \subset (\Theta_0)_v \tag{20}$$

$$\emptyset \neq (A_i)_{v\,\hat{}\,\tilde{v}} \subset (A_i)_v \tag{21}$$

for every $i \in I_{v\,\hat{}\,\tilde{v}}$.

CI3 If $v\,\hat{}\,\mathrm{v}\,\hat{}\,\bar{v} \in \mathcal{V}$ we have

$$G_{v\,\hat{}\,\mathrm{v}\,\hat{}\,\bar{v}}(B) = G_{v\,\hat{}\,\mathrm{v}\,\hat{}\,\mathrm{v}\,\hat{}\,\bar{v}}(B) \tag{22}$$

and $v\,\hat{}\,\mathrm{v}\,\hat{}\,\mathrm{v}\,\hat{}\,\bar{v} \in \mathcal{V}$.

CI4 For every state (nature and type profile) and action profile pair $(\theta, a)_{v\,\hat{}\,\tilde{v}} \in (\Theta_0)_{v\,\hat{}\,\tilde{v}} \times \prod_{i \in I_{v\,\hat{}\,\tilde{v}}} (\Theta_i)_{v\,\hat{}\,\tilde{v}} \times \prod_{j \in I_{v\,\hat{}\,\tilde{v}}} (A_j)_{v\,\hat{}\,\tilde{v}}$ there exists a completion to a pair $(\theta, a)_v \in (\Theta)_v \times \prod_{j \in I_v} (A_j)_v$ that agrees with the appropriate coordinates of $(\theta, a)_{v\,\hat{}\,\tilde{v}}$ such that

$$(u_i)_{v\,\hat{}\,\tilde{v}}((\theta, a)_{v\,\hat{}\,\tilde{v}}) = (u_i)_v((\theta, a)_v) \tag{23}$$

Note that the notion of a type in an incomplete information game with unawareness differs from a type in a standard incomplete information game. The difference is that the beliefs a type has over the type space (as he perceives it) need not be held in common knowledge. Moreover, types are allowed to perceive different type spaces. For example, a type θ_i may have a belief P_{θ_i} yet type θ_j may conceive type θ_i's belief to be different, $P_{\theta_j \theta_i} \neq P_{\theta_i}$ and even defined on a different space.

It is natural to ask whether an analog to Harsanyi's consistency condition – the common prior assumption – can be found for these games with unawareness. The obvious prior candidate is the modeler distribution $P = P_{\emptyset}$, however, since a type θ_i may be unaware of the whole space Θ his distribution P_{θ_i} may be defined on a different set of types $(\Theta)_{(\theta_i)}$, and if θ_i is unaware of some players then $(\Theta)_{(\theta_i)}$ may not be a subset of Θ but rather a subset of a projection. Although this implies that P_{θ_i} cannot be merely a conditional of P it indicates that a projection of the prior might do. Letting P_{θ_i} be exactly the conditional probability over $\Theta_{(\theta_i)}$ of the marginal of P with respect to $\Theta_0 \times \prod_{j \in I_{\theta_i}} \Theta_j$, and extending this definition to iterated relevant views provides a candidate for an extended common prior condition.

Definition 4 *We say that* consistency *(in the sense of Harsanyi) holds for an incomplete information game with unawareness if for every relevant $v\hat{\ }\tilde{v} \in \mathcal{V}$ we have*

$$P_{v\hat{\ }\tilde{v}} = \underset{(\Theta_0)_v \times \prod_{j \in I_{v\hat{\ }\tilde{v}}} (\Theta_j)_v}{Marg} P_v \mid (\Theta_0)_{v\hat{\ }\tilde{v}} \times \prod_{j \in I_{v\hat{\ }\tilde{v}}} (\Theta_j)_{v\hat{\ }\tilde{v}}. \tag{24}$$

Copic and Galeotti (2007) have independently modeled incomplete information games with unawareness in a similar manner. The difference is that they consider players and actions as commonly known and modeled unawareness of types and their beliefs. While our model above is more general, we find it reassuring that the structure of the two definitions is essentially the same.

2.5 Dynamic Games with Unawareness

The most general games we consider are dynamic games – extensive form games. This game form captures both uncertainties and dynamics. A dynamic game is composed of a game tree capturing decision points, actions, nature moves, information sets, probabilities for nature moves and payoffs. An extensive form game is denoted $G(D) = ((W, \prec), I, A_0 \times \prod_{i \in I} A_i, \{F_i\}_{i \in I}, P, \{u_i\}_{i \in I})$ where $(W, \prec)$ is a finite tree (infinite dynamic games vary in definition and unawareness can be extended accordingly) with a disjoint union of vertices (a partition) $W = V_0 \cup \bigcup_{i \in I} V_i \cup Z$ where V_0 is the set of natures moves, V_i denotes the set of player i's decision points and Z is the set of terminal vertices and the order $w' \prec w$ denotes that w' occurs before w on the tree.

Denote by $Pred(w)$ and $Succ(w)$ the (immediate) predecessor, and respectively successor, of w – formally it is the maximal vertex smaller than w and the minimal larger than w respectively. A pair $e = (w, Succ(w))$ of a vertex and its successor is called an *edge* and the set of edges emanating from w is denoted $E(w)$. We assume that every two vertices in the tree are connected with a finite path – a finite sequence of edges. Hence, every vertex has a (single) predecessor except for the root that has none and the set of terminal vertices $Z \subset W$ is the set of vertices that have no successor. The set of players is I. An edge $e = (w, w')$ belongs to player i, resp. Nature, when $w \in V_i$, resp. $w \in V_0$. The mappings $A_i(w, w')$ are defined for player i's edges, $w \in V_i, w' = Succ(w)$, ($A_0(w, w')$ when (w, w') is a nature move) and associate an action with each edge in $E(w)$. The partitions F_i of the sets V_i correspond to the information sets of player i and $f_i(w) \in F_i$ denotes the partition member containing a vertex $w \in V_i$. We require that for every w the function $A_i(w, \cdot)$ of the successors of w is one to one, that for every pair $w \neq w'$ the set of values that $A_i(w, \cdot)$ and $A_i(w', \cdot)$ obtain are disjoint unless $w' \in f_i(w) \in F_i$ in which case they are identical. The mapping P associates a probability distribution over the edges following each of nature's vertices. For every $w \in V_0$ we denote by $P(w)$ the probability distributions over $E(w)$. Finally, $u_i : Z \to \mathbb{R}$ are the utilities of players defined for terminal vertices.

The set of viewpoints is the players' set of decision points $V = \bigcup_{i \in I} V_i$ and a typical viewpoint is denoted by v. The set of all finite sequences of viewpoints is $\overline{V} = \bigcup\limits_{n=0}^{\infty} V^{(n)}$. A finite sequence of viewpoints $v = (\mathrm{v}_1, ..., \mathrm{v}_n)$ is associated with a dynamic game $G_v(D) = ((W_v, \prec), I_v, (A_0)_v \times \prod_{i \in I}(A_i)_v, \{(F_i)_v\}_{i \in I_v}, P_v, \{(u_i)_v\}_{i \in I_v})$ where $W_v = (V_0)_v \cup \bigcup_{i \in I_v}(V_i)_v \cup Z_v$ and similarly for all other ingredients of the game in accordance with the description above. We use the same ordering $\prec$ since W_v will be subsets of W.

Definition 5 *A collection* $\Gamma = \{G_v(D)\}_{v \in \mathcal{V}}$ *where* $G_v(D)$ *are games as above and* $\mathcal{V} \subset \overline{V}$ *is a set of relevant views is called a* Dynamic Game with unawareness *if the following properties hold:*

CD1 For every $v \in \mathcal{V}, \mathrm{v} \in V_i$ we have

$$v\hat{\ }\mathrm{v} \in \mathcal{V} \text{ if and only if } i \in I_v, \mathrm{v} \in (V_i)_v \tag{25}$$

CD2 For every $v\hat{\ }\tilde{v} \in \mathcal{V}$ we have

$$v \in \mathcal{V} \tag{26}$$

$$\emptyset \neq W_{v\hat{\ }\tilde{v}} \subset W_v \tag{27}$$

$$\emptyset \neq I_{v\hat{\ }\tilde{v}} \subset I_v \tag{28}$$

as well as for all $i \in I_{v\hat{}\tilde{v}}, w \in (V_i)_{v\hat{}\tilde{v}}$

$$(V_i)_{v\hat{}\tilde{v}} = (V_i)_v \cap (W_{v\hat{}\tilde{v}} \setminus Z_{v\hat{}\tilde{v}}) \tag{29}$$

$$(V_0)_{v\hat{}\tilde{v}} = (V_0)_v \cap (W_{v\hat{}\tilde{v}} \setminus Z_{v\hat{}\tilde{v}}) \tag{30}$$

$$(F_i)_{v\hat{}\tilde{v}} = \{f \cap (W_{v\hat{}\tilde{v}} \setminus Z_{v\hat{}\tilde{v}}) | f \in (F_i)_v\} \tag{31}$$

and

$$(A_i)_{v\hat{}\tilde{v}}(w, w') = (A_i)_v(w, w'') \tag{32}$$

for the unique successor w'' of w in W_v such that $w'' \preceq w'$, where w' is the successor of w in $W_{v\hat{}\tilde{v}}$.

CD3 If $v\hat{}\mathrm{v}\hat{}\bar{v} \in \mathcal{V}$ with $\mathrm{v} \in V_i$ then we have $f_i(\mathrm{v}) \cap (V_i)_{v\hat{}\mathrm{v}} \neq \emptyset$ and for every $\tilde{\mathrm{v}} \in f_i(\mathrm{v}) \cap (V_i)_{v\hat{}\mathrm{v}}$ we have

$$G_{v\hat{}\mathrm{v}\hat{}\bar{v}}(T) = G_{v\hat{}\mathrm{v}\hat{}\tilde{\mathrm{v}}\hat{}\bar{v}}(T) = \ldots = G_{v\hat{}\mathrm{v}\hat{}\tilde{\mathrm{v}}\hat{}\ldots\hat{}\tilde{\mathrm{v}}\hat{}\bar{v}}(T) \tag{33}$$

and $v\hat{}\mathrm{v}\hat{}\tilde{\mathrm{v}}\hat{}\ldots\hat{}\tilde{\mathrm{v}}\hat{}\bar{v} \in \mathcal{V}$.

The third condition for dynamic games first states that a viewpoint must consider its information set to be relevant. Moreover, the viewpoints it considers in its information set are assumed to find themselves relevant and model the game in the same manner. Otherwise, a player at an information set would be able to distinguish decision points based on differing views at the decision points.

CD4 Let $v\hat{}\bar{v} \in \mathcal{V}$. For every terminal vertex $w \in Z_{v\hat{}\bar{v}}$ there exists a vertex $w' \in Z_v$ such that $w \prec w'$ and

$$(u_i)_{v\hat{}\bar{v}}(w) = (u_i)_v(w') \tag{34}$$

As with incomplete information games we did not constrain the subjective probabilities that a viewpoint, or a view, associates with nature moves. If a restricted view of the game omits some nature moves one may still impose an analog for common priors:

Definition 6 *We say that* Harsanyi consistency *holds for a dynamic game with unawareness representing an incomplete information game with unawareness, if for every relevant $v \in \mathcal{V}$ we have at every $w \in (V_0)_v$ that*

$$P_v(w) = P(w)|E_v(w). \tag{35}$$

We note that the definition of condition **CD3** is stronger than the repeated games version **CR3** as it requires that a viewpoint not only see itself as a viewpoint in the same information set and agree with it, but also that it will agree with all its viewpoints in the restricted information set. This definition follows the interpretation of an information set as representing indistinguishable information *and* indistinguishable awareness. We did not specify the monitoring in repeated games with unawareness, or any other notion of information set, allowing for imperfect monitoring and forgetfulness hence the more general allowance for a player reasoning about her information there.

Our main structural result states that games with unawareness are consistent in the sense that from every view the game is seen as a game with unawareness satisfying the exact same conditions as the modeler's game.

Consider a game with unawareness $\Gamma = \{G_v(\cdot)\}_{v \in \mathcal{V}}$ where $G_v(\cdot)$ has one of the four forms: normal, repeated, incomplete information or dynamic. For every relevant $v \in \mathcal{V}$ we define the relevant views as seen from v as: $\mathcal{V}^v = \{\tilde{v} \in \bar{V} | v\hat{\ }\tilde{v} \in \mathcal{V}\}$. For each relevant view $\tilde{v} \in \mathcal{V}^v$ we define the game $G^v_{\tilde{v}}(\cdot) = G_{v\hat{\ }\tilde{v}}(\cdot)$ and the game with unawareness as seen from v is defined as $\Gamma^v = \{G^v_{\tilde{v}}(\cdot)\}_{\tilde{v} \in \mathcal{V}^v}$.

Proposition 1 *For every game with unawareness Γ with a relevant view $v \in \mathcal{V}$ the game Γ^v is a game with unawareness with relevant views $\mathcal{V}^v$.*

The proof of this as well as all other propositions in this paper appears in the Appendix.

We note that this game form allows for imperfect recall, i.e., players may not only discover aspects of the game they were not aware of, they can also forget past actions. If one would rather maintain perfect recall two extra conditions are required. First the actual game with all the players and actions must be a game of perfect recall. Furthermore, at every decision point a player must consider a game that includes the game they considered in the past, i.e., no player may have their view of the game contract over time. Imposing the latter condition on higher order views (everyone views everyone's view ... to include all past views) will allow the conditioning on perfect recall at any reasoning level. Note that this condition still allows a player to revise their view of what other players view of the game is. For example, Alice can observe an action by Bob that may lead her to think that Bob is less aware than she previously thought. This does not imply that she believes Bob forgot, it is a revision of Alice's view of Bob's view at a *given* decision point for Bob. Once she revises this view, she may well assign a restricted view for Bob in *future* decision points. Still, she will assume that Bob has perfect recall as she revises all of his view to be more restrictive.

3 Solutions for Games with Unawareness

In defining solution concepts for games with unawareness we follow the same principle used in constructing the games: Each relevant view considers equilibrium behavior in a manner consistent with the modeler's definition of equilibrium behavior. There are some degrees of freedom when taking this approach. Obviously, one needs to select the solution concept for standard games whose behavior is being generalized, moreover, there may be some flexibility in the extension of behavior to games with unawareness. The first solutions we analyze are Nash equilibrium (NE) and Rationalizability (R) for normal form games with unawareness. The definition and analysis of some solutions for the other game forms follow.

3.1 Rationalizability and Nash Equilibrium in Normal Form Games with Unawareness

In order to define the solutions for games with unawareness we need to associate behavior with each possible view.

Definition 7 *Let* $\Gamma = \{G_v\}_{v \in \mathcal{V}}$ *be a normal form game with unawareness. An* extended strategy profile *ESP in this game is a collection of strategy profiles* $\{(\sigma)_v\}_{v \in \mathcal{V}}$ *where* $(\sigma)_v$ *is a strategy profile in the game* G_v *such that for every* $v\hat{}\mathrm{v}\hat{}\bar{v} \in \mathcal{V}$ *we have*

$$(\sigma_{\mathrm{v}})_v = (\sigma_{\mathrm{v}})_{v\hat{}\mathrm{v}} \tag{36}$$

in the sense that the same pure strategies are assigned the same probabilities in the two games G_v *and* $G_{v\hat{}\mathrm{v}}$*, as well as*

$$(\sigma)_{v\hat{}\mathrm{v}\hat{}\bar{v}} = (\sigma)_{v\hat{}\mathrm{v}\hat{}\mathrm{v}\hat{}\bar{v}} \tag{37}$$

The first condition requires that the strategy that the view v associates with player v in the game G_v is the same strategy that the view v finds the player playing in the game as he sees it, i.e. in the game $G_{v\hat{}\mathrm{v}}$. The interpretation is that whenever a strategy is assigned to a player the player is indeed assumed to be playing the strategy. The second condition follows the same logic behind condition **C3** in stating that the behavior associate by a player to the games he views is identical to the behavior he reasons about himself associating to games, and that this principle holds from every view. In other words, Bob's strategy associated with the game that Alice perceives that Bob perceives is the strategy that Bob is assumed to play in the game as Alice perceives it. In addition Bob's view of his own view of the strategy coincides with his view of the strategy and this is commonly understood at every view.

It is worthwhile noting that the definition of an extended strategy restricts the *behavior* of players to actions they are aware of. In the definition of a game with unawareness we allowed the possibility that a player may have an action he is unaware of, i.e. a view v may perceive a player v as having an available action a, i.e. $a \in (A_{\mathrm{v}})_v$, while at the same time v may perceive that v is unaware of a, i.e. $a \notin (A_{\mathrm{v}})_{v\hat{\ }\mathrm{v}}$. In this case the right-hand side of (36) is defined on a strictly smaller set of actions and we assume that the right-hand side support is in that set, hence the ESP is defined such that $(\sigma_{\mathrm{v}})_v$ assigns 0 probability to the pure strategy a when the player is unaware of it.

We begin by defining rationalizability in games with unawareness. As expected, rationalizability corresponds to playing a best response in the perceived game to perceived strategies that are themselves best responses in how it is perceived the corresponding players view the game, and so on. This extends rationalizability from normal form games to normal form games with unawareness.

Definition 8 *An ESP* $\{(\sigma)_v\}_{v\in\mathcal{V}}$ *in a game with unawareness is called* extended rationalizable *if for every* $v\hat{\ }\mathrm{v} \in \mathcal{V}$ *we have that* $(\sigma_{\mathrm{v}})_v$ *is a best response to* $(\sigma_{-\mathrm{v}})_{v\hat{\ }\mathrm{v}}$ *in the game* $G_{v\hat{\ }\mathrm{v}}$.

The principle governing the extension of NE to games with unawareness follows the epistemic foundation of the solution concept. A Nash equilibrium requires rationalizability – players play a best response to conjectures, and some form of truth – knowledge of conjectures, or agreement on strategies. These correspond in our setting to strategies that are best responses at every view, and to strategies that coincide when the views of the game coincide, respectively. The first property corresponds to rationality in the sense of playing a best response to conjectures. The second property requires that the conjectures, or best responses, are the same when reasoning about the same game. In other words, when players have the same perceptions about the game (with unawareness) they share the same conjectures on behavior – agreement on strategies.

For a game with unawareness two views $v, \bar{v}$ share the same perception of the game if they agree on how all other views consider the game, i.e. they consider the same game *with unawareness* $\Gamma^v = \Gamma^{\bar{v}}$.

Definition 9 *An ESP* $\{(\sigma)_v\}_{v\in\mathcal{V}}$ *in a game with unawareness is called* an extended Nash equilibrium *ENE if it is rationalizable and for all* $v, \bar{v} \in \mathcal{V}$ *such that* $\Gamma^v = \Gamma^{\bar{v}}$ *we have that* $(\sigma)_v = (\sigma)_{\bar{v}}$.

Note that an ENE assigns the same behavior in games corresponding to the concatenation of views once the perceptions of the game coincide, .i.e., $\Gamma^v = \Gamma^{\bar{v}}$ implies that for all $\tilde{v}$ such that $v\hat{\ }\tilde{v} \in \mathcal{V}$ we have $(\sigma)_{v\hat{\ }\tilde{v}} = (\sigma)_{\bar{v}\hat{\ }\tilde{v}}$. This follows from noting that $\Gamma^v = \Gamma^{\bar{v}}$ implies that $(\Gamma^v)^{\tilde{v}} = (\Gamma^{\bar{v}})^{\tilde{v}}$.

We justify the use of the term "extended" with the following result. This result states that when all views see the game in the exact same manner – there is no unawareness – then the extended solutions coincide with their standard counterparts for the normal form game at hand. More generally, at every view such that the game is seen to have no unawareness the extended solution coincides with the standard one.

Proposition 2 *Let G be a normal form game and Γ a normal form game with unawareness such that for some $v \in \mathcal{V}$ we have $G_{v\hat{}\bar{v}} = G$ for every $\bar{v}$ such that $v\hat{}\bar{v} \in \mathcal{V}$. Let σ be a strategy profile in the normal form game G then*

1. *σ is rationalizable for G if and only if $(\sigma)_v = \sigma$ is part of an extended rationalizable profile in Γ.*

2. *σ is a NE for G if and only if $(\sigma)_v = \sigma$ is part of an ENE for Γ and this ENE also satisfies $(\sigma)_v = (\sigma)_{v\hat{}\bar{v}}$.*

While a game with unawareness may correspond to an infinite collection of games, the structure does support the existence of an equilibrium:

Proposition 3 *Every normal form game with unawareness has an ENE. Hence, the weaker extended rationalizability solution is also non-empty.*

Condition **C2** guarantees that every view of how a viewpoint perceives the game is a restriction of the original view of the game. However, a game with unawareness Γ may still incorporate an infinite number of differing views of the game with unawareness, i.e. the set of games with unawareness $\{\Gamma^v\}_{v\in\mathcal{V}}$ could have an infinite number of distinct members:

Example 1 *Consider three players denoted $1, 2, 3$ and let G be a normal form game where player 1 has three actions and players 2 and 3 have a single actions each. Let F be a normal form game obtained from G by removing one of player 1's actions, and let E be a normal form game obtained from F by removing one of the two remaining actions of player 1. We define the game with unawareness Γ as follows.*

*All views are relevant, i.e. $\mathcal{V} = \bar{V}$. Consider a view $v = \mathrm{v}_1, \mathrm{v}_2, ..., \mathrm{v}_n$ such that $\mathrm{v}_i \neq \mathrm{v}_{i+1}$. It suffices to define G_v for such views since $G_{v'}$ for a view v' with consecutive v_i is uniquely derived according to **C3**.*

Define for every prime $p > 2$

$$v'_p = \underbrace{1, 2, 1, 2, ..., 1, 2, 1}_{p}, 3 \tag{38}$$

$$v''_p = v'_p\hat{}v'_p = \underbrace{1, 2, 1, 2, ..., 1, 2, 1}_{p}, 3, \underbrace{1, 2, 1, 2, ..., 1, 2, 1}_{p}, 3 \tag{39}$$

For v with no consecutive identical viewpoints we define:

$$G_v = \left\{ \begin{array}{lc} E & v = v''_p\hat{\ }\tilde{v} \textit{ for some prime } p > 2 \textit{ and some view } \tilde{v} \\ F & \textit{if } G_v \textit{ is not defined above and } v = v'_p\hat{\ }\tilde{v} \textit{ for some prime } p > 2 \textit{ and view } \tilde{v} \\ G & \textit{otherwise.} \end{array} \right) . \tag{40}$$

For v with consecutive identical viewpoints we define G_v to be the same as the view obtained by replacing each string of consecutive identical viewpoints with a single representative.

Claim 4 Γ *is a normal form game with unawareness.*

To prove this claim we need to show that $\Gamma = \{G_v\}_{v \in \bar{V}}$ satisfies the conditions of a game with unawareness. Since for v with some consecutive identical viewpoints we define G_v according to **C3** this condition holds by definition. Since all players participate in all games we have that all views are relevant and condition **C1** holds as well. Similarly, since all players participate in each of the viewed games, the payoffs are well defined and condition **C4** holds as well. Similarly, parts (4) and (5) in condition **C2** hold. It remains to show that (6) holds, or in other words, whenever $G_{v\hat{\ }\bar{v}} = F$ then $G_v \neq E$ and that whenever $G_{v\hat{\ }\bar{v}} = G$ then $G_v = G$. As before, we can assume that $v, v\hat{\ }\tilde{v}$ have no consecutive identical viewpoints. In the first case, if by way of contradiction $G_v = E$ then there is a prime $p > 2$ and some $\tilde{v}$ such that $v = v''_p\hat{\ }\tilde{v}$. In particular, $v\hat{\ }\bar{v} = v''_p\hat{\ }(\tilde{v}\hat{\ }\bar{v})$ which implies that $G_{v\hat{\ }\bar{v}} = E$ – a contradiction. In the second case, we have that if $G_v \neq G$ then $v = v'_p\hat{\ }\tilde{v}$ for some prime $p > 2$ and view $\tilde{v}$ (note that this holds whether G_v equals F or E). Hence $v\hat{\ }\bar{v} = v'_p\hat{\ }\tilde{v}\hat{\ }\bar{v}$ and we must have $G_{v'_p\hat{\ }\tilde{v}} \neq G$ which completes the proof that Γ is a well defined normal form game with unawareness.

Claim 5 *The set of normal form games with unawareness $\{\Gamma^{v'_p}\}_{p>2 \text{ prime}}$ are all different.*

The proof of this claim follows from observing that for every prime $p > 2$ we have that $G_v^{v'_p} = F$ for $v = \emptyset, v = 1, v = 12, ..., v = \underbrace{121...21}_{p}$ and $G_{v'_p}^{v'_p} = G$. Hence for all prime q such that $q > p$ we have that $G_{v'_p}^{v'_q} = F \neq G_{v'_p}^{v'_p}$ which implies that every member of the set differs from all the following members assuring that no two members coincide.

Claim 5 emphasizes that the existence of an ENE follows from the fact that at each view each viewpoint is considered as playing a best response to strategies in some finite game, so although we have an infinite number of games (one for each view) the fixed point conditions are satisfied since payoffs for each view are determined in a finite game. However, we would like to study conditions under which the normal form game with unawareness is finite, in the sense that there is only a finite set of games with unawareness associated with the views in the game. This is particularly important if one wishes to represent the game in a state space approach with a finite set of states. The following results demonstrates such a condition.

Consider a view $v = (\mathrm{v}_1, ..., \mathrm{v}_n) \in \bar{V}$, each view $v' = (\mathrm{v}_{k_1}, ..., \mathrm{v}_{k_m}))$ with $1 \leq k_1 < ... < k_m \leq n$ is called a *sub-word* of v and we denote the order induced by sub-words as $v' \preceq v$. The stronger version of condition **C2** is stated as follows:

S-C2 For every $v \in \mathcal{V}$ we have that for every $v' \preceq v$

$$v' \in \mathcal{V} \tag{41}$$

and

$$\emptyset \neq I_v \subset I_{v'} \tag{42}$$

as well as

$$\emptyset \neq (A_i)_v \subset (A_i)_{v'} \tag{43}$$

for all $i \in I_v$.

We have

Proposition 6 *If a normal form game with unawareness Γ satisfies condition* ***S-C2*** *then the set of games with unawareness $\{\Gamma^v\}_{v \in \mathcal{V}}$ is finite.*

Condition **S-C2** is quite strong as it requires that if Alice models Bob's perception of Carol reasoning about some action then Bob's model of the game *must* indeed include Carol's reasoning about that action. Furthermore, Alice model of *Carol*'s perceptions of the game must assume that Carol is reasoning about this action. The interpretation of this condition is that the players do not "get it wrong", in the sense that when they reason about someone else's perception they cannot attribute to that person an ability that he does not actually posses. While we do not advocate setting condition **S-C2** on the same level as the more intuitive assumption **C2**, we point out that our introductory examples do satisfy this condition. We note that condition **S-C2** also implies that if two views of the game see it as a standard game with no unawareness, then it must be the *same* standard game:

Proposition 7 *For a normal for game with unawareness Γ which satisfies condition* ***S-C2*** *and such that Γ^v and $\Gamma^{v'}$ are both standard games with $v\hat{\ }v', v'\hat{\ }v \in \mathcal{V}$, we have that $\Gamma^v = \Gamma^{v'}$, and in particular we have the same ENE behavior in both.*

We point out that even with condition **S-C2** the length of views at which there is a change of the perception of the game need not be bounded.

Example 2 *Consider three players denoted* $1, 2, 3$ *and let* G *be a normal form game where player* 1 *has two actions and players* 2 *and* 3 *have a single action each. Let* F *be a normal form game obtained from* G *by removing one of player* 1*'s actions. We define the game with unawareness* Γ *as follows.*

All views are relevant, i.e. $\mathcal{V} = \bar{V}$*. For a view* $v = \mathrm{v}_1, \mathrm{v}_2, ..., \mathrm{v}_n$ *such that for all* i *we have* $\mathrm{v}_i \neq 3$ *we define* $G_v = G$ *and otherwise we set* $G_v = F$*.*

We leave it to the reader to check that the example satisfies all required conditions and note that for any view v that does not contain player 3 the consideration of that player changes the game, i.e. $G_v \neq G_{v\hat{\ }3}$.

We conclude this section with the observation that any NE of a standard game G such that all its actions are held in common awareness, corresponds to an ENE of the normal form game with unawareness. The definition of an ENE readily implies:

Claim 8 *Let* Γ *be a normal form game with unawareness with* $G_\emptyset = G$*. Assume that* σ *is a NE of* G *and that every action in the support of* σ *is held in common awareness, i.e., every view finds all the players in* G *and the actions in the support of* σ *to be part of the game, then* $(\sigma)_v = \sigma$ *for all* v *is an ENE.*

3.2 Representing Games with Unawareness as Games with Incomplete Information

The definition of a game with unawareness maps the strategic situation to a collection of standard games – associating one standard game with each possible relevant view of the situation. On the other hand the notion of unawareness or, "leaving some aspect out of the modeled game", begs the comparison to the notion of assigning zero probability to an event. While unawareness has some particular restrictions when it comes to iterated reasoning, it is natural to ask why a novel structure is required, why not represent situations with unawareness in incomplete information games with zero probabilities replacing unawareness. Indeed, such a construction is feasible, moreover there is a canonic mapping of normal form games with unawareness to standard games with incomplete information – canonic in the sense that varying the payoffs in the game, renaming actions, or players, does not alter the state space under this mapping. We provide this mapping below. Furthermore, the extended rationalizable solution exactly coincides with Bayesian Nash equilibria under this mapping. However, it does not preserve is the extended Nash equilibrium. Hence, while normal form games with unawareness can be mapped to corresponding games with incomplete information, Nash equilibrium reasoning in games with unawareness requires some additional structure without which it differs from the solutions of incomplete information games.

With respect to the other forms of games with unawareness the imbedding becomes more complicated, but not impossibly so. For example, incomplete information games with unawareness can be mapped to standard incomplete information games, yet the problem of mapping the solutions of the unawareness form becomes more severe. The reason is that behavior in incomplete information games with unawareness distinguishes between probability zero and unawareness, a distinction that disappears in the representation with games with incomplete information. Since our solution dictates behavior with characteristics unique to unawareness the elimination of this distinction in the standard incomplete information form hinders the reproduction of the solution to games with unawareness. Similar difficulties arise when trying to imbed dynamic and repeated games with unawareness which require the combination of extensive and incomplete information games.

Consider a normal form game with unawareness $\Gamma = \{G_v\}_{v\in\mathcal{V}}$ with a set of relevant views $\mathcal{V}$ with $G_\emptyset = G = (I, \prod_{i\in I} A_i, \{u_i\}_{i\in I})$. We define the following I-player incomplete information game $G_{\text{Bayesian game}} = (I, \prod_{i\in I} A_i, \Theta_0 \times \prod_{i\in I} \Theta_i, \{P_i\}_{i\in I}, \{u_i\}_{i\in I})$ where A_i agree with the game G, $\Theta_0 = \overline{V} = \bigcup\limits_{n=0}^{\infty} I^{(n)}$, for every $i \in I$ the players types are defined by $\Theta_i = \{v\hat{\ }i | v \in \overline{V}\}$, the players beliefs are $P_i(v\hat{\ }i, (v\hat{\ }i\hat{\ }j)_{j\in I} | v\hat{\ }i) = 1$ hence all other $(\tilde{v}, (\bar{v}_j\hat{\ }j)_{j\in I})$ are assigned zero probability by the type $v\hat{\ }i$. Finally, the payoffs u_i are defined as

$$u_i((v, (\tilde{v}_j\hat{\ }j))_{j\in I})((a_j)_{j\in I}) = \begin{cases} (u_i)_v \text{ when } (a_j)_{j\in I} \text{ are an extension where } (u_i)_v \text{ is defined} \\ -\infty \text{ otherwise.} \end{cases} \tag{44}$$

where $(u_i)_v$ are the payoffs to i as defined in G_v. Note that the payoffs are determined by the state of nature in Θ_0 and that, following Harsanyi, we set the payoffs for actions that are not modeled in G_v at $-\infty$. We also emphasize that this mapping is not one to one since a game with unawareness where a player has a single action according to some view v is mapped to the same game as a game with unawareness for which at the same view v the player with the single action is not part of the description of the game and the payoffs are determined according to the single action, yet this minute difference has no impact on behavior.

An equivalent representation of a normal form game with unawareness can be provided via a formulation where the set of states is V and each player possesses an information partition of the state space. For every player i and every $v \in V$ the player's partition member includes two states $v, v\hat{\ }i$ and player i assigns the probabilities 0 and 1 to the two states respectively. This corresponds to the above game in that it adds a partition, but preserves all probabilities. This mapping also provides a graphic representation of games with unawareness which is depicted in Figure 4. The tree representation indicates the relationship dictated by the second condition (for each game form) where the arrow from a view v to

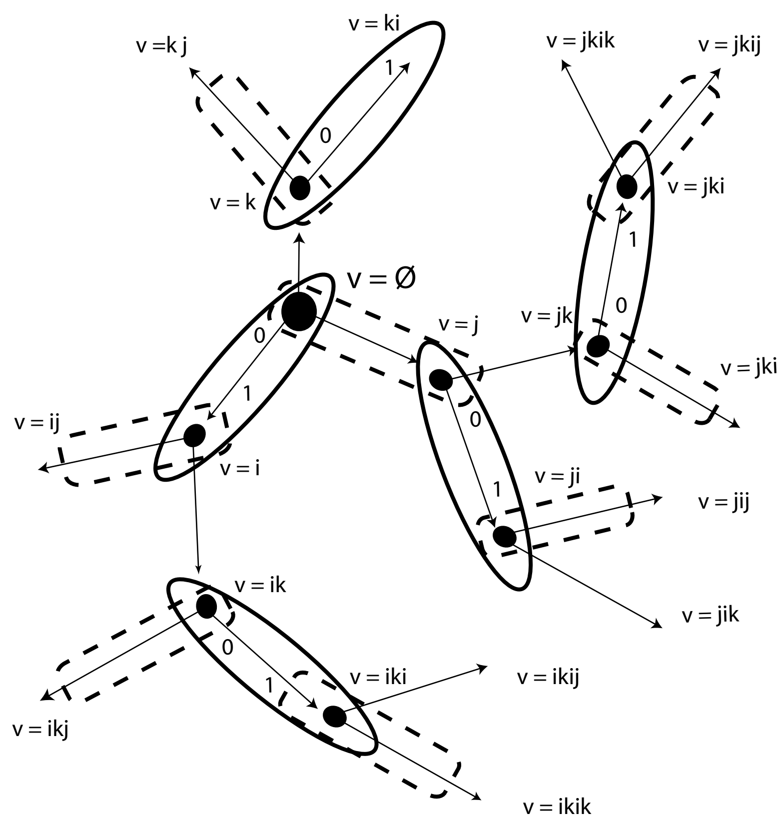

Figure 4: Representing Games with Unawareness.

Each state is denoted by a vertex v and the set of vertices V forms a tree with root $v = \emptyset$ (emphasized in the figure). The pair of states forming player i's partition are circled and the partition of player j is similarly denoted with a dashed line. We also denote the probabilities that player i assigns to each of the two states in the member of the information partition. We omit the notation of player k partition (which exactly corresponds to the edges not circled in the figure). We also removed the probabilities for players j and k which assign 0 to the vertex closer to the root and 1 to further one. The game associated with every state is $G_v(\cdot)$. The edges connecting the views denote how each view views the game as seen from the various viewpoints – the arrows point to the higher level views. We have omitted the views of players' view of themselves as this is redundant based on the third condition.

a view $v\hat{}$v indicates how the viewpoint v perceives the game according to v as a more restricted version of the game $G_v(\cdot)$.

A modification of the tree in Figure 4 can also capture unawareness of players – the relevant views – by trimming the tree at views that are not relevant. Such elimination of views still leaves us with a tree (and well defined partitions) since the second condition implies that if a view is irrelevant so are all the views following it on the tree. We note that all these alternative representations will preserve extended rationalizability as Bayesian

Nash, but will map ENE to a new solution which refines Bayesian Nash.

We note that the representation does not depend on the payoffs of the games, it also generates the same state space and beliefs if the names of the actions, the extent of unawareness of actions and the names of the players are changed. In fact, there will only be a change if the unawareness of players changes and this is only in the last formulation above. The mapping is canonic in the sense that what determines the state space and the beliefs are the set of relevant views, or the set of all views depending on the representation, neither of which constrains payoffs.

We turn now to ENE and compare them the Bayesian representation of the strategic situation when the game with unawareness is mapped to a game with incomplete information. We first note the following:

Claim 9 *Let Γ be a normal form game with unawareness and $G(B)$ be the Bayesian game corresponding to Γ as above. The extended rationalizable strategy profiles of Γ coincide with the Bayesian Nash equilibria (BNE) profiles of $G(b)$ at the state of the world corresponding to $v = \emptyset$.*

The claim follows by observing that at every state the relevant player is playing a best response to the state he assigns probability 1 to, where other players play according to how he views their perception, in turn this applies for iteration of views and as can be seen in Figure 4 the conditions for extended rationalizability coincide with the Bayesian Nash equilibrium conditions. Claim 9 then implies that ENE is mapped to a strict subset of the Bayesian Nash equilibria. In particular, this is a subset where at various states players must play the same strategy if the *games* at the states they assign probability 1 to, they assign probability 1 to others assigning probability 1, and so on, have exactly the same trees of payoffs corresponding to these iterated beliefs. We note that trying to gage these higher order beliefs about payoffs amounts to the construction of the game with unawareness to begin with as a collection of normal form games corresponding to a set of views. Hence, games with unawareness and the ENE provide a novel solution to setting with restricted perceptions.

3.3 Equilibria of Incomplete Information Games with Unawareness

The definition of extended BNE is constructed exactly in the same manner as the normal form case. Every view considers a strategy in the game with incomplete information such that every type plays a best response to the strategies in the game which the type perceives.

In turn, these strategies corresponds to how the type perceives the other players' types play in the game as the first type perceives the other types are considering, and so on. As with ENE we also require that if two views perceive the same incomplete information game with unawareness – have the same perception of the incomplete information game, the same perception of how the types in the game view the incomplete information game, and so on – then they prescribe the same behavior.

Definition 10 *Let* $\Gamma = \{G_v(B)\}_{v \in \mathcal{V}}$ *be an incomplete information game with unawareness. An* extended strategy profile *ESP in this game is a collection of strategy profiles* $\{(\sigma)_v\}_{v \in \mathcal{V}}$ *where* $(\sigma)_v$ *is a strategy profile in the game* $G_v(B)$ *such that for every* $v\hat{\ }\mathrm{v}\hat{\ }\bar{v} \in \mathcal{V}$ *we have*

$$(\sigma_{\mathrm{v}})_v = (\sigma_{\mathrm{v}})_{v\hat{\ }\mathrm{v}} \tag{45}$$

and

$$(\sigma)_{v\hat{\ }\mathrm{v}\hat{\ }\bar{v}} = (\sigma)_{v\hat{\ }\mathrm{v}\hat{\ }\mathrm{v}\hat{\ }\bar{v}} \tag{46}$$

Recall that in this case each viewpoint v corresponds to one type of a player in an incomplete information setting. As with normal form games we extend the solutions:

Definition 11 *An ESP* $\{(\sigma)_v\}_{v \in \mathcal{V}}$ *in an incomplete information game with unawareness is said to be* extended rationalizable *if for every* $v\hat{\ }\mathrm{v} \in \mathcal{V}$ *we have that* $(\sigma_{\mathrm{v}})_v$ *is a best response to* $(\sigma_{-\mathrm{v}})_{v\hat{\ }\mathrm{v}}$ *in the game* $G_{v\hat{\ }\mathrm{v}}$*.*

and

Definition 12 *An ESP* $\{(\sigma)_v\}_{v \in \mathcal{V}}$ *in an incomplete information game with unawareness is called* an extended Bayesian Nash equilibrium *EBNE if it is rationalizable and for all* $v, \bar{v} \in \mathcal{V}$ *such that* $\Gamma^v = \Gamma^{\bar{v}}$ *we have that* $(\sigma)_v = (\sigma)_{\bar{v}}$*.*

We note that, as with ENE for normal form games with unawareness, the EBNE is a new solution concept. In order to capture this solution with standard incomplete information games one would require a fine tailoring of the game that distinguishes the beliefs that stem from uncertainty from those that are generated by unawareness. One would need to choose a particular state space and define the types' beliefs on a case by case basis to capture the EBNE behavior in standard incomplete information games, while in the process essentially mimicking the construction of incomplete information games with unawareness.

3.4 Equilibria of Dynamic Games with Unawareness

Dynamic games provide a host of solution concepts. Many of these consider alternative principles for belief revision. In particular, beliefs and behavior after a deviation from the equilibrium path occurs. The extension of these solutions is no different than the extensions to normal form, or incomplete information, games with unawareness. The extensions require an epistemic foundation for the solution determining the nature of reasoning about beliefs and rationality in a dynamic setting, and will depend on the choice of epistemic characterization. Such characterizations need not be unique as can be seen in the epistemic characterization of sequential equilibria for dynamic games with unawareness in Feinberg (2004).

Definition 13 *Let* $\Gamma = \{G_v(D)\}_{v\in\mathcal{V}}$ *be a dynamic game with unawareness. An* extended strategy profile *ESP in this game is a collection of strategy profiles* $\{(\sigma)_v\}_{v\in\mathcal{V}}$ *where* $(\sigma)_v$ *is a behavior strategy profile in the game* $G_v(D)$ *such that for every* $v\hat{}\mathrm{v}\hat{}\bar{v} \in \mathcal{V}$ *we have*

$$(\sigma_{\mathrm{v}})_v = (\sigma_{\mathrm{v}})_{v\hat{}\mathrm{v}} \tag{47}$$

and for every $\tilde{\mathrm{v}} \in g_i(v) \cap (V_i)_{v\hat{}\mathrm{v}}$ *we have*

$$(\sigma)_{v\hat{}\mathrm{v}\hat{}\bar{v}} = (\sigma)_{v\hat{}\mathrm{v}\hat{}\tilde{\mathrm{v}}\hat{}\bar{v}} \tag{48}$$

Recall that a viewpoint v corresponds to a decision point in the game tree. In particular, this could be one of a number of points in an information set. Furthermore, at that decision point the player may perceive the situation as corresponding to one of the other decision points at the same information set due to beliefs about prior strategic behavior. We need to verify that Using condition (48) agrees with the definition of a dynamic game and with the interpretation of an information set. Indeed, inductively applying (48) we have that

$$(\sigma)_{v\hat{}\mathrm{v}\hat{}\bar{v}} = (\sigma)_{v\hat{}\mathrm{v}\hat{}\tilde{\mathrm{v}}\hat{}\bar{v}} = (\sigma)_{v\hat{}\mathrm{v}\hat{}\tilde{\mathrm{v}}\ldots\tilde{\mathrm{v}}\hat{}\bar{v}} \tag{49}$$

which is a well defined condition according to condition **CD3**.

As with standard dynamic games, rationality in a dynamic setting depends on whether the payoff is calculated ex-ante, or conditional on reaching a decision point. With the latter requiring postulating conditions for belief revision as well as forward, backward and hypothetical rationality analysis. While one can analyze these alternatives as in Feinberg (2004) here we simply extend NE directly without discussion of the epistemic conditions. We do the same for refinements such as sequential equilibria which consider assessments – strategies plus beliefs at information sets.

Definition 14 *An ESP $\{(\sigma)_v\}_{v\in\mathcal{V}}$ in a dynamic game with unawareness is called* an extended Nash equilibrium *ENE if for every $v\hat{\ }\mathrm{v} \in \mathcal{V}$ with $\mathrm{v} \in (V_i)_v$ we have that the behavior strategy $\{(\sigma_{\tilde{\mathrm{v}}})_v \mid$ such that $v\hat{\ }\mathrm{v}\hat{\ }\tilde{\mathrm{v}} \in \mathcal{V}\}$ for player i is a best response to $(\sigma_{-i})_{v\hat{\ }\mathrm{v}}$ in the game $G_{v\hat{\ }\mathrm{v}}(D)$. In addition, for all $v, \bar{v} \in \mathcal{V}$ such that $\Gamma^v = \Gamma^{\bar{v}}$ we have that $(\sigma)_v = (\sigma)_{\bar{v}}$.*

We note that the definition requires that a player's strategy is considered from how a view perceives the perception of a decision point v. Even if the strategy implies that this decision point is not reached and even if the player corresponding to v has a different perception of the game at other decision points. This is not a choice made in defining the solution, but rather a constraint of the definition of games with unawareness since a behavior strategy as defined for player i in the game perceived from v may be beyond the scope of strategies in the game as perceived by $v\hat{\ }\mathrm{v}$.

The definition of solutions based on assessments is more straightforward as they already require reasoning at information sets. However, since these solutions involve beliefs at information sets we must define extended assessments as well.

Definition 15 *Let $\Gamma = \{G_v(D)\}_{v\in\mathcal{V}}$ be a dynamic game with unawareness. An* extended assessment *in this game is a an ESP $\{(\sigma)_v\}_{v\in\mathcal{V}}$ and a collection of belief function $\{(\mu_{\mathrm{v}})_v\}_{v\hat{\ }\mathrm{v}\in\mathcal{V}}$ such that $(\mu_{\mathrm{v}})_v$ is a probability distribution over the information set $g_i(\mathrm{v}) \cap (V_i)_{v\hat{\ }\mathrm{v}}$ such that for every $v\hat{\ }\mathrm{v}\hat{\ }\bar{v} \in \mathcal{V}$ we have*

$$(\mu_{\mathrm{v}})_v = (\mu_{\mathrm{v}})_{v\hat{\ }\mathrm{v}} \tag{50}$$

and for every $\tilde{\mathrm{v}} \in g_i(\mathrm{v}) \cap (V_i)_{v\hat{\ }\mathrm{v}}$ we have

$$(\mu)_{v\hat{\ }\mathrm{v}\hat{\ }\bar{v}} = (\mu)_{v\hat{\ }\mathrm{v}\hat{\ }\tilde{\mathrm{v}}\hat{\ }\bar{v}} \tag{51}$$

We are now set to define the extension of solutions based on assessments and will do so in a general manner.

Definition 16 *An extended assessment $\{(\sigma)_v, (\mu)_v\}_{v\in\mathcal{V}}$ in a dynamic game with unawareness is called* an extended $\star$ equilibrium *if for every $v\hat{\ }\mathrm{v} \in \mathcal{V}$ with $\mathrm{v} \in (V_i)_v$ we have for any $\tilde{\mathrm{v}} \in g_i(\mathrm{v}) \cap (V_i)_{v\hat{\ }\mathrm{v}}$ the mixed strategy $(\sigma_{\tilde{\mathrm{v}}})_{v\hat{\ }\mathrm{v}}$ maximizes the expected payoff to player i at $\tilde{\mathrm{v}}$ in the game $G_{v\hat{\ }\mathrm{v}}(D)$, i.e. the expected payoff to i in $G_{v\hat{\ }\mathrm{v}}(D)$ conditional on the information set $g_i(\mathrm{v}) \cap (V_i)_{v\hat{\ }\mathrm{v}}$ being reached with the probability distribution $(\mu_{\mathrm{v}})_v$, when other players play according to $(\sigma_{-g_i(\mathrm{v})\cap(V_i)_{v\hat{\ }\mathrm{v}}})_{v\hat{\ }\mathrm{v}}$. Where $\star$ stands for sequential or variants thereof, and we require that $(\sigma_{\mathrm{v}})_v$ and $(\mu_{\mathrm{v}})_v$ satisfy the consistency, or other conditions imposed by refinement $\star$ in the game $G_{v\hat{\ }\mathrm{v}}(D)$ with respect to the strategies as perceived from the view $v\hat{\ }\mathrm{v}$.*

To see how the extended refinement can be implemented to various solution concepts, consider sequential equilibria for example. In this case we can require that $(\mu_{\mathrm{v}})_v$ agree with the probability distribution generated by $(\sigma_{-g_i(\mathrm{v})\cap(V_i)_{v\hat{}\mathrm{v}}})_{v\hat{}\mathrm{v}}$ in $G_{v\hat{}\mathrm{v}}(D)$ if the information set is reached, and if not, it is the limit of probabilities assigned by best responses ϵ-completely mixed strategies in the game $G_{v\hat{}\mathrm{v}}(D)$.

We can now state the generalizations of Propositions 3 and 2 which concludes this paper:

Proposition 10 *The set of extended solutions for normal form, incomplete information and dynamic games with unawareness is non-empty. Furthermore, each extended solution coincides with the standard solution when there is no unawareness.*

4 Conclusion

This work provides a uniform framework to model and analyze games with unawareness. The framework follows an identical treatment of various game forms and allows for unawareness of participating players and their actions. The basic approach asks that at every decision point (and given any information the player has) the player has a view of what the game is, a view of how each player in that game views the game and so on. Hence, all these views relate to standard games and the collection of standard games constitutes the game with unawareness. Our four conditions guarantee reasoning consistency in the sense that if Alice reasons about Bob reasoning about a player or an action then she can directly reason about them as well, that Alice's view about Bob's view about his own view coincides with how Alice views Bob and that when there is unawareness of a player the outcomes are consistent with one of the player's action.

We emphasize that these game can usually described much like the examples in the introduction by indicating the relevant view of the game. When higher order views are relevant (as in the gams depicted in (1) and (2) a further state space formulation may be in order (see the discussion in Section 2.2). Games with unawareness can be represented as incomplete information games. Indeed, it has been difficult to envision a situation that cannot be presented as simply a game where players assign zero probabilities instead of being unaware. However, this was not previously made precise. Using the hierarchies approach the mapping to an incomplete information setting allows an explicit formulation of this relationship. Moreover, it also demonstrates why a separate formulation for unawareness may be useful. In particular, it shows that the natural extension of Nash equilibria to games with unawareness does not map to any known solution in the incomplete information formulation.

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5 Appendix

Proof of Proposition 1. Proposition 1 states that from every relevant view, the game with unawareness defined by considering higher order views, i.e., the game as seen from that view, is itself a game with unawareness, i.e., for each relevant view v, the game defined by $\Gamma^v = \{G^v_{\tilde{v}}(\cdot)\}_{\tilde{v}\in\mathcal{V}^v}$ with relevant views $\mathcal{V}^v = \{\tilde{v} \in \bar{V} | v\hat{\ }\tilde{v} \in \mathcal{V}\}$ satisfies the conditions for a game with unawareness.

Recall the notation $G^v_{\tilde{v}}$ denotes the game $G_{v\hat{\ }\tilde{v}}$. Similarly, we denote all the components of the games $G^v_{\tilde{v}}(\cdot)$ accordingly, e.g. the set of players is denoted $I^v_{\tilde{v}}$, player v's actions by $(A_{\mathrm{v}})^v_{\tilde{v}}$ and so on.

Consider a normal form game with unawareness $\Gamma = \{G_v\}_{v\in\mathcal{V}}$ and fix a relevant view $v \in \mathcal{V}$. Recall that $\mathcal{V}^v = \{\tilde{v} \in \bar{V} | v\hat{\ }\tilde{v} \in \mathcal{V}\}$ and $\Gamma^v = \{G^v_{\tilde{v}}\}_{\tilde{v}\in\mathcal{V}^v}$. Since $\tilde{v} \in \mathcal{V}^v$ is equivalent to $v\hat{\ }\tilde{v} \in \mathcal{V}$ we have from applying **C1** to Γ that

$$v\hat{\ }\tilde{v}\hat{\ }\mathrm{v} \in \mathcal{V} \text{ if and only if } \mathrm{v} \in I_{v\hat{\ }\tilde{v}}. \tag{52}$$

Since the left hand side is identical with $\tilde{v}\hat{\ }\mathrm{v} \in \mathcal{V}^v$ and the right hand side coincides with $\mathrm{v} \in I^v_{\tilde{v}}$ we have

$$\tilde{v}\hat{\ }\mathrm{v} \in \mathcal{V}^v \text{ if and only if } \mathrm{v} \in I_{v\hat{\ }\tilde{v}} \tag{53}$$

for all $\tilde{v} \in \mathcal{V}^v$ and the first consistency condition holds for Γ^v.

If $\bar{v}\hat{}\tilde{v} \in \mathcal{V}^v$ we have $v\hat{}\bar{v}\hat{}\tilde{v} \in \mathcal{V}$ which implies $v\hat{}\bar{v} \in \mathcal{V}$ by (4) and is equivalent to $\bar{v} \in \mathcal{V}^v$ proving the first part of condition **C2**. Similarly, we have from (5) and (6) that

$$\emptyset \neq I_{v\hat{}\bar{v}\hat{}\tilde{v}} = I^v_{\bar{v}\hat{}\tilde{v}} \subset I_{v\hat{}\bar{v}} = I^v_{\bar{v}} \tag{54}$$

and

$$\emptyset \neq (A_i)_{v\hat{}\bar{v}\hat{}\tilde{v}} = (A_i)^v_{\bar{v}\hat{}\tilde{v}} \subset (A_i)_{v\hat{}\bar{v}} = (A_i)^v_{\bar{v}} \tag{55}$$

for all $i \in I_{v\hat{}\bar{v}\hat{}\tilde{v}} = I^v_{\bar{v}\hat{}\tilde{v}}$ which proves the second consistency condition holds for Γ^v.

If $\tilde{v}\hat{}\mathrm{v}\hat{}\bar{v} \in \mathcal{V}^v$ we have

$$G_{v\hat{}\tilde{v}\hat{}\mathrm{v}\hat{}\bar{v}} = G_{v\hat{}\tilde{v}\hat{}\mathrm{v}\hat{}\mathrm{v}\hat{}\bar{v}} \tag{56}$$

and $v\hat{}\tilde{v}\hat{}\mathrm{v}\hat{}\mathrm{v}\hat{}\bar{v} \in \mathcal{V}$ which is equivalent to

$$G^v_{\tilde{v}\hat{}\mathrm{v}\hat{}\bar{v}} = G^v_{\tilde{v}\hat{}\mathrm{v}\hat{}\mathrm{v}\hat{}\bar{v}} \tag{57}$$

and $\tilde{v}\hat{}\mathrm{v}\hat{}\mathrm{v}\hat{}\bar{v} \in \mathcal{V}^v$ and the third condition holds.

For every action profile $(a)^v_{\bar{v}\hat{}\tilde{v}} = \{a_j\}_{j \in I^v_{\bar{v}\hat{}\tilde{v}}}$ we have from (9) a completion to an action profile $(a)_{v\hat{}\bar{v}} = \{a_j, a_k\}_{j \in I_{v\hat{}\bar{v}\hat{}\tilde{v}}, k \in I_{v\hat{}\bar{v}} \setminus I_{v\hat{}\bar{v}\hat{}\tilde{v}}}$ hence the existence of an action profile $(a)^v_{\bar{v}} = \{a_j, a_k\}_{j \in I^v_{\bar{v}\hat{}\tilde{v}}, k \in I^v_{\bar{v}} \setminus I^v_{\bar{v}\hat{}\tilde{v}}}$ such that

$$(u_i)_{v\hat{}\bar{v}\hat{}\tilde{v}}((a)_{v\hat{}\bar{v}\hat{}\tilde{v}}) = (u_i)_{v\hat{}\bar{v}}((a)_{v\hat{}\bar{v}}) \tag{58}$$

which is equivalent to

$$(u_i)^v_{\bar{v}\hat{}\tilde{v}}((a)^v_{\bar{v}\hat{}\tilde{v}}) = (u_i)^v_{\bar{v}}((a)^v_{\bar{v}}) \tag{59}$$

and the fourth condition holds for Γ^v as required completing the consistency proof for normal form games with unawareness.

For a repeated game with unawareness $\Gamma = \{G_v(T)\}_{v \in \mathcal{V}}$ fix a relevant view $v \in \mathcal{V}$. For every $\bar{v} \in \mathcal{V}^v, \mathrm{v} = (h^t(J), i) \in V^v$ where the history $h^t(J)$ contains only actions from $(A)^v$ and $i \in I^v, J \subset I^v$ we have from applying (10) to $G_{v\hat{}\bar{v}} = G^v_{\bar{v}}$ that

$$\bar{v}\hat{}\mathrm{v} \in \mathcal{V}^v \text{ if and only if } i \in I^v_{\bar{v}}, J = I^v_{\bar{v}}, A_j(h^t(J)) \subset (A_j)^v_{\bar{v}} \text{ for all } j \in J \tag{60}$$

which is the first consistency condition. The second condition is identical to the normal form games case.

For the third condition let $\tilde{v}\hat{}\mathrm{v}\hat{}\bar{v} \in \mathcal{V}^v$ where $\mathrm{v} = (h^t(J), i)$ and the history and players are taken from G^v. According to condition **CR3** for Γ there exists some $\tilde{\mathrm{v}} = (\tilde{h}^t(\tilde{J}), i)$ such that

$$G_{v\hat{}\bar{v}\hat{}\mathrm{v}\hat{}\bar{v}}(T) = G_{v\hat{}\tilde{v}\hat{}\mathrm{v}\hat{}\tilde{v}\hat{}\bar{v}}(T) = ... = G_{v\hat{}\tilde{v}\hat{}\mathrm{v}\hat{}\tilde{\mathrm{v}}\hat{}...\hat{}\tilde{\mathrm{v}}\hat{}\bar{v}}(T) \tag{61}$$

and $v\hat{}\tilde{v}\hat{}\mathrm{v}\hat{}\tilde{\mathrm{v}}\hat{}...\hat{}\tilde{\mathrm{v}}\hat{}\bar{v} \in \mathcal{V}$. Which is equivalent to

$$G^{v}_{\bar{v}\hat{}\mathrm{v}\hat{}\bar{v}}(T) = G^{v}_{\tilde{v}\hat{}\mathrm{v}\hat{}\tilde{\mathrm{v}}\hat{}\bar{v}}(T) = ... = G^{v}_{\tilde{v}\hat{}\mathrm{v}\hat{}\tilde{\mathrm{v}}\hat{}...\hat{}\tilde{\mathrm{v}}\hat{}\bar{v}}(T) \tag{62}$$

and $\tilde{v}\hat{}\mathrm{v}\hat{}\tilde{\mathrm{v}}\hat{}...\hat{}\tilde{\mathrm{v}}\hat{}\bar{v} \in \mathcal{V}^v$ and the third condition holds. The fourth condition holds exactly as in the normal form case completing the proof of consistency for repeated games with unawareness.

For an incomplete information game with unawareness $\Gamma = \{G_v(B)\}_{v\in\mathcal{V}}$ and a relevant view v we have $G^v_{\tilde{v}}(B) = (I^v_{\tilde{v}}, \prod_{i\in I^v_{\tilde{v}}}(A_i)^v_{\tilde{v}}, (\Theta_0)^v_{\tilde{v}} \times \prod_{i\in I^v_{\tilde{v}}}(\Theta_i)^v_{\tilde{v}}, P^v_{\tilde{v}}, \{(u_i)^v_{\tilde{v}}\}_{i\in I^v_{\tilde{v}}})$. The first three consistency conditions hold in the exact same manner as for normal form games. Note that conditions (19) and (20) are shown to hold in an identical manner as condition (21) which is identical to the normal form case. As for the fourth condition, we note that it too follows a similar proof to the normal form case by replacing partial actions profiles and their completion by partial pairs of state and actions and their completion. More formally, for every state and action profile pair $(\theta, a)^v_{\bar{v}\hat{}\tilde{v}} \in (\Theta_0)^v_{\bar{v}\hat{}\tilde{v}} \times \prod_{i\in I^v_{\bar{v}\hat{}\tilde{v}}}(\Theta_i)^v_{\bar{v}\hat{}\tilde{v}} \times \prod_{j\in I^v_{\bar{v}\hat{}\tilde{v}}}(A_j)^v_{\bar{v}\hat{}\tilde{v}}$ there exists a completion to a pair $(\theta, a)^v_{\bar{v}} \in (\Theta)^v_{\bar{v}} \times \prod_{j\in I^v_{\bar{v}}}(A_j)^v_{\bar{v}}$ that agrees with the appropriate coordinates of $(\theta, a)^v_{\bar{v}\hat{}\tilde{v}}$ such that

$$(u_i)^v_{\bar{v}\hat{}\tilde{v}}((\theta, a)^v_{\bar{v}\hat{}\tilde{v}}) = (u_i)^v_{\bar{v}}((\theta, a)^v_{\bar{v}}) \tag{63}$$

and the proof for incomplete information games with unawareness is complete.

Finally, let $\Gamma = \{G_v(D)\}_{v\in\mathcal{V}}$ be a dynamic game with unawareness and fix a relevant view v and consider the collection $\Gamma^v = \{G^v_{\tilde{v}}(D)\}_{\tilde{v}\in\mathcal{V}^v}$. As before, the components of $G^v_{\tilde{v}}(D)$ are denoted with the superscript v and in particular the game tree is denoted $W^v_{\tilde{v}}$ as are its components (nature moves, players' decision points and terminal nodes) and the collection of information sets are denoted $(F_i)^v_{\tilde{v}}$.

The first condition holds by noting that $\mathrm{v} \in (V_i)^v_{\tilde{v}}$ if and only if $\mathrm{v} \in (V_i)_{v\hat{}\tilde{v}}$. Similarly, the equivalents of (26), (27) and (28) hold as in the normal form case and (29), (30), (31) and (64) follow from noting that the sets of vertices and ordering coincides between $G_{v\hat{}\bar{v}\hat{}\tilde{v}}(D)$ and $G^v_{\bar{v}\hat{}\tilde{v}}(D)$ as do intersections and complements of the sets of vertices and since the order induced from $\preceq$ is preserved, i.e.,

$$(A_i)^v_{\bar{v}\hat{}\tilde{v}}(w, w') = (A_i)^v_{\bar{v}}(w, w'') \tag{64}$$

for the unique successor w'' of w in $W^v_{\bar{v}}$ such that $w'' \preceq w'$, where w' is the successor of w in

$W^v_{\tilde{v}\hat{}\tilde{v}}$.

If $\tilde{v}\hat{}\mathrm{v}\hat{}\bar{v} \in \mathcal{V}^v$ with $\mathrm{v} \in V^v_i$ then we have $f^v_i(\mathrm{v}) \cap (V_i)^v_{\tilde{v}\hat{}\mathrm{v}} \neq \emptyset$ from condition **CD3** applied to Γ as well as for every $\mathrm{v}' \in f^v_i(\mathrm{v}) \cap (V_i)^v_{\tilde{v}\hat{}\mathrm{v}}$ we have

$$G^v_{\tilde{v}\hat{}\mathrm{v}\hat{}\bar{v}}(T) = G^v_{\tilde{v}\hat{}\mathrm{v}\hat{}\mathrm{v}'\hat{}\bar{v}}(T) = ... = G^v_{\tilde{v}\hat{}\mathrm{v}\hat{}\mathrm{v}'\hat{}...\hat{}\tilde{\mathrm{v}}\hat{}\bar{v}}(T) \tag{65}$$

and $\tilde{v}\hat{}\mathrm{v}\hat{}\mathrm{v}'\hat{}...\hat{}\tilde{\mathrm{v}}\hat{}\bar{v} \in \mathcal{V}^v$ simply by applying (33) for Γ with $v\hat{}\tilde{v}$ for v and v' for $\tilde{\mathrm{v}}$.

Similarly, let $\tilde{v}\hat{}\bar{v} \in \mathcal{V}^v$. For every terminal vertex $w \in Z^v_{\tilde{v}\hat{}\bar{v}} = Z_{v\hat{}\tilde{v}\hat{}\bar{v}}$ according to **CD4** for Γ there exists a vertex $w' \in Z^v_{\tilde{v}} = Z_{v\hat{}\tilde{v}}$ such that $w \prec w'$ and

$$(u_i)^v_{\tilde{v}\hat{}\bar{v}}(w) = (u_i)_{v\hat{}\tilde{v}\hat{}\bar{v}}(w) = (u_i)_{v\hat{}\tilde{v}}(w') = (u_i)^v_{\tilde{v}}(w') \tag{66}$$

and the fourth condition holds as well. We have shown that dynamic games with unawareness are also consistent in the sense that how every view is modeled to perceive the game, is itself a game with unawareness and the proof is complete. ■

Proof of Proposition 2. Let $G = (I, \prod_{i\in I} A_i, \{u_i\}_{i\in I})$ be a normal form game and Γ a normal form game with unawareness such that for some $v \in \mathcal{V}$ we have $G_{v\hat{}\bar{v}} = G$ for every $\bar{v}$ such that $v\hat{}\bar{v} \in \mathcal{V}$. We will assume $v = \emptyset$ the general case follows from considering Γ^v instead.

Assume a strategy profile σ is rationalizable for the normal form game G. Then for every player $i \in I$ there is a strategy profile for all other players $\bar{\sigma}_{-i}$ such that σ_i is a best response to $\bar{\sigma}_{-i}$ and such that for every $j \in I \setminus \{i\}$ there is a strategy $\tilde{\sigma}_{-j}$ such that $\bar{\sigma}_j$ is a best response to it, and so on. For every view v we define a strategy profile inductively as follows: Let $(\sigma)_\emptyset = \sigma$, for $\mathrm{v} = i \in I$ let $(\sigma)_\mathrm{v} = (\bar{\sigma}_{-i}, (\sigma_i)_\emptyset)$ i.e., combining player i's strategy σ_i with $\bar{\sigma}_{-i}$ to which it is a best response. Given $(\sigma)_v$ we define $(\sigma)_{v\hat{}\mathrm{v}}$ as the strategy to which $(\sigma_\mathrm{v})_v$ is a best response (augmented by $(\sigma_\mathrm{v})_v$) if the last viewpoint in v does not equal v, and we let $(\sigma)_v = (\sigma)_{v\hat{}\mathrm{v}}$ if the last viewpoint in v coincides with v.

Since the game associated with all view is G and all views are relevant, the above collection of strategies is a well defined ESP. By definition of rationalizability, every view finds the player is playing a best response to the strategies they are considering according to that view which is the definition of extended rationalizability.

Consider now the same game and let $\{(\sigma)_v\}_{v\in\mathcal{V}}$ be an ESP satisfying extended rationalizability. Since at every view the game corresponds to G we have that all views are relevant. In particular, for every sequence of players $i_1, i_2, ..., i_n$ we have a sequence of strategies corresponding to $n+1$ views: $\sigma^0 = (\sigma)_\emptyset$, $\sigma^1 = (\sigma)_{i_1}$, $\sigma^2 = (\sigma)_{(i_1,i_2)}$, ..., $\sigma^n = (\sigma)_{(i_1,i_2,...,i_n)}$. These satisfy by extended rationalizability that $\sigma^k_{i_{k+1}}$ is a best response to $\sigma^{k+1}_{-i_{k+1}}$. Hence, we found that for every player $i \in I$ the strategy $(\sigma_i)_\emptyset$ is a best response to $(\sigma_{-i})_i$ with $(\sigma_j)_i$ each being best responses to $(\sigma_{-j})_{(i,j)}$ and so on making $(\sigma)_\emptyset$ a profile of rationalizable strategies in the normal form game G.

The second part of the proposition follows similarly: Let G be as above and without loss of generality assume the game with unawareness associates G with all views, i.e. we begin with the view $\emptyset$. Let σ be a Nash equilibrium. Then setting $(\sigma)_v = \sigma$ for all views is a well defined ESP. Hence the condition for identical strategies assigned to identical views of the game with unawareness is satisfied. Since σ_i is a best response to σ_{-i} for all $i \in I$ by NE, we have that $(\sigma_i)_v$ is a best response to $(\sigma_{-i})_{v\hat{}i}$ for all v and every $i \in I$ and the ESP satisfies extended rationalizability and is therefore an ENE.

In the other direction, let σ be such that $(\sigma)_v = \sigma$ for all v is an ENE. In particular σ is a strategy profile in G such that $\sigma_i = (\sigma_i)_v$ is a best response to $\sigma_{-i} = (\sigma_{-i})_{v\hat{}i}$ making it a NE. ■

Proof of Proposition 3. The existence of ENE requires the examination of the possibly infinite collection of views of the game assuring that a fixed point exists. Consider a normal form game with unawareness $\Gamma = \{G_v\}_{v\in\mathcal{V}}$ with views $\mathcal{V}$. We define an auxiliary standard normal form game $\mathcal{G}$ as follows:

Let i denote a player in $G_\emptyset$. The set of players in this game is given by

$$\mathcal{N} = \{v \in \mathcal{V} \setminus \{\emptyset\} \mid v = (i_1, ..., i_n) \text{ with } i_k \neq i_{k+1} \text{ for all } k\}. \tag{67}$$

The action set for each player $v = (i_1, ..., i_n) \in \mathcal{N}$, is given by

$$\mathcal{A}_v = (A_{i_n})_v. \tag{68}$$

We define the payoff function for each player $v = (i_1, ..., i_n) \in \mathcal{N}$ in this game by

$$\mathcal{U}_v(\{(a_j)_{\tilde{v}}\}_{\tilde{v}\in\mathcal{N}}) = (u_{i_n})_v(\{(a_j)_v\}_{j\in I_v}) \tag{69}$$

where $(u_{i_n})_v(\{(a_j)_v\}_{j\in I_v})$ is the payoff to i_n in the game G_v when the actions played are $\{(a_j)_v\}_{j\in I_v}$.

The game $\mathcal{G} = (\mathcal{N}, \mathcal{A}, \mathcal{U})$ constitutes a normal form game with a countable number of players. However, the payoff function for each player depends on the actions of only a finite number of other players. Consider the product space of mixed strategies

$$\Sigma = \prod_{v\in\mathcal{N}} \Delta(\mathcal{A}_v). \tag{70}$$

Since the game at each view has a finite action set we have that $\mathcal{A}$ is compact in the product topology. Considering the best response mapping

$$B(\sigma) = \prod_{v \in \mathcal{N}} B_v(\sigma) \tag{71}$$

where

$$B_v(\sigma) = \{\bar{\sigma}_v \in \Delta(\mathcal{A}_v) \mid E(\mathcal{U}_v(\sigma \mid_{\bar{\sigma}_v})) \geq E(\mathcal{U}_v(\sigma \mid_{\tilde{\sigma}_v})) \text{ for all } \tilde{\sigma}_v \in \Delta(\mathcal{A}_v)\} \tag{72}$$

and $E(\mathcal{U}_v(\sigma \mid_{\tilde{\sigma}_v}))$ is the expected payoff when considering the strategy σ modified by having player v play $\tilde{\sigma}_v$. The expected payoff is well defined since for every $v \in \mathcal{N}$ there is only a finite number of players influencing the payoffs.

The set valued functions B_v are non-empty, compact and convex valued since the payoff functions $\mathcal{U}_v$ are linear continuous functions of $\mathcal{A}_v$. Hence, the product map B is a non-empty, compact and convex valued set function on the compact convex set Σ. B is also upper-hemi continuous since for every sequence $\sigma^n \longrightarrow \sigma$ with $\sigma, \sigma^n \in \Sigma$ for $n = 1, 2, ...$, we have that if $\bar{\sigma}^n \in B(\sigma^n)$ for all n and $\bar{\sigma}^n \longrightarrow \bar{\sigma}$ then we must also have $\bar{\sigma} \in B(\sigma)$. This follows from noting that for every $v \in \mathcal{N}$ we have

$$E(\mathcal{U}_v(\sigma^n \mid_{\bar{\sigma}_v^n})) \geq E(\mathcal{U}_v(\sigma^n \mid_{\tilde{\sigma}_v})) \text{ for all } \tilde{\sigma}_v \in \Delta(\mathcal{A}_v) \tag{73}$$

but since $\mathcal{U}_v$ depends only on the actions of $v = (i_1, ..., i_l$ and $v\hat{\ }j$ for $j \in I_v$, $j \neq i_l$, the convergence of σ^n to σ implies convergence in all coordinates and hence the continuity of the expectation of $\mathcal{U}$ yields

$$E(\mathcal{U}_v(\sigma \mid_{\bar{\sigma}_v})) \geq E(\mathcal{U}_v(\sigma \mid_{\tilde{\sigma}_v})) \text{ for all } \tilde{\sigma}_v \in \Delta(\mathcal{A}_v) \tag{74}$$

which holds for all $v \in \mathcal{N}$ and implies $\bar{\sigma} \in B(\sigma)$ as claimed.

The set valued map B satisfies the conditions for the generalized Kakutani fixed point theorem and as shown by Fan (1952) and Glicksberg (1952) there exists a $\sigma \in \Sigma$ such that $\sigma \in B(\sigma)$ hence there exists an equilibrium σ in the auxiliary game $\mathcal{G}$. We now define an ESP for Γ by setting $(\sigma)_v = \sigma_v$ for $v \in \mathcal{N}$ and inductively for $v = (i_1, ..., i_n) \in \mathcal{V}$ for which $(\sigma)_v$ is defined, we set for $\tilde{v} = (i_1, ..., i_k, i_k, i_{k+1}, ..., i_n)$ the strategy $(\sigma)_{\tilde{v}} = (\sigma)_v$ hence we inductively get an ESP. By definition of a NE for the auxiliary game this ESP is an ENE of Γ as required. ■

We note that the NE of the auxiliary normal form game $\mathcal{G}$ defined in the proof of Proposition 3 completely characterizes the set of ENE for the game Γ.

Proof of Proposition 6. Assume by way of contradiction that there is an infinite countable sequence of relevant views $\{v_n\}_{n=1}^{\infty}$ that offer distinct views of the game with unawareness. Formally, for every pair $n \neq m$ there is a $\tilde{v}$ such that

$$(A)_{v_n\hat{}\tilde{v}} \neq (A)_{v_m\hat{}\tilde{v}}. \tag{75}$$

We note that if the games differ in the viewed set of players they will also differ in the set of actions as above.

From Lemma 11 whose proof follows, there exists an infinite countable subsequence $\{v_{n_k}\}_{k=1}^{\infty}$ such that $v_{n_k} \preceq v_{n_{k+1}}$. In particular, for all $\tilde{v}$ we have $v_{n_k}\hat{}\tilde{v} \preceq v_{n_{k+1}}\hat{}\tilde{v}$. From condition **S-C2** we have for every k and $\tilde{v}$ that

$$(A)_{v_{n_k}\hat{}\tilde{v}} \supset (A)_{v_{n_{k+1}}\hat{}\tilde{v}}. \tag{76}$$

From (75) we have for every $k > 1$ there exists some $\tilde{v}_k$ such that

$$(A)_{v_{n_{k-1}}\hat{}\tilde{v}_k} \neq (A)_{v_{n_k}\hat{}\tilde{v}_k} \tag{77}$$

and for this particular $\tilde{v}_k$ we have that

$$(A)_{v_{n_1}\hat{}\tilde{v}_k} \supset ... \supset (A)_{v_{n_{k-1}}\hat{}\tilde{v}_k} \supsetneq (A)_{v_{n_k}\hat{}\tilde{v}_k}. \tag{78}$$

Since (78) holds for every $k > 1$ we have a subsequence $\{v_{n_k}\}_{k=1}^{\infty}$ and a sequence $\{\tilde{v}_k\}_{k=2}^{\infty}$ such that for all $k = 2, 3, ...$ we have

$$(A)_{v_{n_k}\hat{}\tilde{v}_k} \neq (A)_{v_{n_j}\hat{}\tilde{v}_k} \qquad \forall j < k. \tag{79}$$

Using Lemma 11 once more, we can find an infinite subsequence $\{\tilde{v}_{k_l}\}_{l=1}^{\infty}$ such that $\tilde{v}_{k_l} \preceq \tilde{v}_{k_{l+1}}$ and considering the same subset of indices for v_{n_k} we have for all l

$$(A)_{v_{n_{k_l}}\hat{}\tilde{v}_{k_l}} \neq (A)_{v_{n_{k_j}}\hat{}\tilde{v}_{k_l}} \qquad \forall j < l, \tag{80}$$

$$v_{n_{k_l}} \preceq v_{n_{k_{l+1}}}, \tag{81}$$

$$\tilde{v}_{k_l} \preceq \tilde{v}_{k_{l+1}}. \tag{82}$$

From (81) and (82) and since concatenation with the same word (recall that we termed views as "words" comprised of viewpoints as "letters". preserves the order $\preceq$ we have that for all l

$$v_{n_{k_l}}\hat{}\tilde{v}_{k_{l+1}} \preceq v_{n_{k_{l+1}}}\hat{}\tilde{v}_{k_{l+1}} \preceq v_{n_{k_{l+1}}}\hat{}\tilde{v}_{k_{l+2}}. \tag{83}$$

From condition **S-C2** and from (83) we have for every l

$$(A)_{v_{n_{k_l}}\hat{\ }\tilde{v}_{k_{l+1}}} \supset (A)_{v_{n_{k_{l+1}}}\hat{\ }\tilde{v}_{k_{l+1}}} \supset (A)_{v_{n_{k_{l+1}}}\hat{\ }\tilde{v}_{k_{l+2}}}. \tag{84}$$

Since every $(A)_v$ is a subset of the finite set A, we conclude that there exists a t such that

$$(A)_{v_{n_{k_t}}\hat{\ }\tilde{v}_{k_{t+1}}} = (A)_{v_{n_{k_{t+1}}}\hat{\ }\tilde{v}_{k_{t+1}}}. \tag{85}$$

Since (85) contradicts (80) we have reached the desired contradiction and the proof is complete. ■

The proof of the proposition relied on the following lemma due to Higman (1952).

Lemma 11 *For every sequence of views $\{v_k\}_{k=1}^{\infty}$ we can find an infinite countable subsequence $\{v_{n_k}\}_{k=1}^{\infty}$ such that $v_{n_k} \preceq v_{n_{k+1}}$, i.e. each word v_{n_k} can be obtained by deleting some letters of the word $v_{n_{k+1}}$.*

Proof. This Lemma follows immediately from Theorem 4.4 in Higman (1952). Higman shows (as a special case of his finite basis property theorems) that given a finite alphabet I, every set of words X from this alphabet has a finite subset X_0 such that for every word $w \in X$ one can find a word $w_0 \in X_0$ such that the letters of w_0 occur in w in their right order, though not necessarily consecutively. In particular, let $X = \{v_k\}_{k=1}^{\infty}$, from Higman's theorem there exists a finite subset $X_0 \subset X$ such that from each word in $v \in X$ one can obtain at least one of the words in X_0 by eliminating some members in v. Since X_0 is finite and X is countable there exists a word in X_0 denoted v_{n_1} that can be imbedded in an infinite subsequence of words from $X \setminus X_0$. Hence from every countable sequence of words we can find a subsequence such that the first word in the subsequence can be obtained from *every* word that follows by eliminating some letters. We can now consider the subsequence from the second word onwards and find a subsequence such that the second word can be imbedded in all the words that follow. Maintaining the same first element v_{n_1} we now have that the first two words can be imbedded in every word that follows. By induction the required subsequence is derived. ■

The following direct proof of Higman's theorem is due to Michael Ostrovsky:

By induction on k – the number of letters in the alphabet (main induction). For $k = 1$, the claim is obvious. Suppose it is true for k up to n. Let us show that it is also true for $k = n + 1$.

Claim 12 *Any infinite sequence w_i of words (made up of $k = n+1$ different letters) contains two words, w_{i_1} and w_{i_2}, such that $i_1 < i_2$ and $w_{i_1} \preceq w_{i_2}$.*

Proof of claim. By induction on l – the length of the shortest word in the sequence. For $l = 1$ take the one-letter word. Without loss of generality, the letter is A. Eliminate all the words that go before that word from the sequence; we now have $w_1 = A$. If any other word in the remaining sequence contains the letter A, we are done. If not, then the sequence $(w_2, w_3, w_4, \dots)$ is made up of only $n = k - 1$ different letters, and by the assumption of the main induction, this sequence contains an increasing subsequence $(w_{j_1}, w_{j_2}, \dots)$ with any two words, e.g., w_{j_1} and w_{j_2}, satisfying the requirement.

Suppose the claim holds for all l up to m. Let us show that it is also true for $l = m + 1$. Take the shortest word in the sequence. Without the loss of generality, it is the first word in the sequence, and the first letter in this word is A. If there is only a finite number of other words that contain the letter A, then the remaining infinite subsequence is made up of only n different letters and we are done. Otherwise, drop all the words that do not contain the letter A from the sequence. For each remaining word w_i, let L_i be the part of the word that precedes the first occurrence of A in the word, and R_i be the part that follows the first occurrence of A (e.g., if $w_i = BCADCAB$, then $L_i = BC$ and $R_i = DCAB$; if $w_i = ABC$, then L_i is the empty word, and $R_i = BC$). Note that all words in the sequence $(L_1, L_2, \dots, L_i, \dots)$ are made up of only n different letters, and so there exists an increasing subsequence $(L_{i_1}, L_{i_2}, \dots)$ such that for any t, $i_t < i_{t+1}$ and $L_{i_t} \preceq L_{i_{t+1}}$. Note also that since we assumed that w_1 is the shortest word and starts with an A, we can let $i_1 = 1$ – the empty word is smaller than any other word.

Now, consider the corresponding sequence $(R_{i_1}, R_{i_2}, \dots)$. The shortest word in this sequence has length of at most m (because R_1, by construction, has length m), and therefore, by the minor induction assumption there exist u and v such that $u < v$ and $R_{i_u} \preceq R_{i_v}$. But we also know that, by construction, $L_{i_u} \preceq L_{i_v}$, and so $w_{i_u} \preceq w_{i_v}$, and the claim follows. ■

We can now complete the proof of the step of the induction of Higman's theorem. Take any sequence of words made up of $k = n + 1$ different letters. Consider all words w_i in this sequence such that there does not exist $j > i$ such that $w_i \preceq w_j$. There can be at most a finite number of such words, since otherwise the subsequence formed from these words would be a counter-example to the claim. Let w_h be the last one of these words in the sequence, so that for any $j > h$ there exists $k > j$ such that $w_j \preceq w_k$. It is now possible to construct an infinite increasing subsequence, e.g. take the subsequence (w_{i_t}) such that $i_1 = h + 1$ and for all $t > 1$, $i_t = min_{i > i_{t-1}}\{i | w_{i_{t-1}} \preceq w_i\}$ and the proof of Higman's theorem is complete.

Proof of Proposition 7. Let v and v' be such that Γ^v and $\Gamma^{v'}$ are standard games.

Hence, for every $\tilde{v}$ such that $v\hat{}\tilde{v} \in \mathcal{V}$ we have

$$A_{v\hat{}\tilde{v}} = A_v \text{ and } I_{v\hat{}\tilde{v}} = I_v \tag{86}$$

as well as for every $\tilde{v}$ such that $v'\hat{}\tilde{v} \in \mathcal{V}$ we have

$$A_{v'\hat{}\tilde{v}} = A'_v \text{ and } I_{v'\hat{}\tilde{v}} = I'_v. \tag{87}$$

Since $v' \prec v\hat{}v' \in \mathcal{V}$ the condition **S-C2** and (86) imply that $A_v = A_{v\hat{}v'} \subset A'_v$ and using (87) we similarly get $A_{v'} \subset A_v$. Using the same argument we conclude that $I_v = I_{v'}$. Since there is common awareness from these points onward we have that $G_v = G_{v'}$ as well as $G_v = G_{v\hat{}\tilde{v}} = G_{v'\hat{}\tilde{v}} = G_{v'}$ for all $\tilde{v}$ such that $v\hat{}\tilde{v}$ or equivalently $v'\hat{}\tilde{v}$ is relevant. Since ENE for games with unawareness where all views share the same standard game coincides with NE of the standard game, the same behavior is dictated by ENE in both Γ^v and $\Gamma^{v'}$. ∎

Proof of Proposition 10. The proof of existence of the extended solution concepts for repeated, incomplete and dynamic games with unawareness follows quite closely the proof of Proposition 3. Similarly, the proof that the solutions coincide with the standard solution once the views agree follows closely the proof of Proposition 2. We detail the required modifications for the application of these proofs.

The proof for incomplete information games with unawareness is identical to the normal form games proof with the exception that type spaces may be infinite. Hence, there may be an uncountable set of views. Still, the critical condition (74) holds as long as the utility of a type is continuous in other players types strategies, but this amount to continuity in a standard incomplete information game since all we need is for the strategies to be measurable in the type space. Repeated games and dynamic games with unawareness require a modification of the final part of the proof of Proposition 3 where the equilibrium of the auxiliary game is extended to the game with unawareness. The difference here is that a viewpoint for a player may consider itself as a different view point, for example, a viewpoint corresponding to one member of an information set may view herself as corresponding to another member of the (same) information set. This is illustrated in condition **CD3** in (33) (and condition **CR3** for repeated games). Constructing a strategy for the game with unawareness from a strategy for the auxiliary game that we construct in the proof of Proposition 3 requires that we extend the strategies defined for a subset of views to all relevant views. The subset of views is determined as iterating views that can be associated with different strategies in the games with unawareness. For dynamic games the subset of views that the auxiliary game considers will not have two views from the same information set. When mapping these strategies to the game with unawareness we complete the set of strategies, for all relevant views, by identifying views that must be associated with the same strategy. In particular, once the strategy for a view $v\hat{}\mathrm{v}\hat{}\bar{v}$ is defined, we assign the same strategy for the views $v\hat{}\mathrm{v}\hat{}\tilde{\mathrm{v}}\hat{}\bar{v}, ..., v\hat{}\mathrm{v}\hat{}\tilde{\mathrm{v}}\hat{}...\hat{}\tilde{\mathrm{v}}\hat{}\bar{v}$ where $\tilde{\mathrm{v}}$ is as determined by condition **CD3**. The rest of the proof coincides with the proof for the normal form case. ■