The competition hypergraphs of doubly partial orders

Suh-Ryung KIM\textsuperscript{a,1}, Jung Yeun LEE\textsuperscript{b}, Boram PARK\textsuperscript{c,2,*}, Yoshio SANO\textsuperscript{d,3}

\textsuperscript{a}Department of Mathematics Education, Seoul National University, Seoul 151-742, Korea
\textsuperscript{b}National Institute for Mathematical Sciences, Daejeon 305-390, Korea
\textsuperscript{c}DIMACS, Rutgers University, Piscataway, NJ 08854, United States
\textsuperscript{d}National Institute of Informatics, Tokyo 101-8430, Japan

Abstract

Since Cho and Kim \cite{2} had shown that the competition graph of a doubly partial order is an interval graph, it has been actively studied whether or not the same phenomenon occurs for other variants of competition graph and interesting results have been obtained. Continuing in the same spirit, we study the competition hypergraph, an interesting variant of competition graph, of a doubly partial order. Though it turns out that the competition hypergraph of a doubly partial order is not always interval, we completely characterize the competition hypergraphs of doubly partial orders which are interval.

Keywords: Competition hypergraphs, Competition graphs, Doubly partial orders, Interval hypergraphs

2010 MSC: 05C75, 05C20

1. Introduction

Given a digraph $D$, the competition graph $C(D)$ of $D$ is a graph which has the same vertex set as $D$ and has an edge between vertices $u$ and $v$ if and only if there exists a common out-neighbor of $u$ and $v$ in $D$. The notion of competition graph is due to Cohen \cite{4} and has arisen from ecology. Competition graphs also have applications in coding, radio transmission, and modeling of complex economic systems (see \cite{19} and \cite{21}). Since Cohen introduced the notion of competition graph, various variations have been defined and studied by many authors (see the survey articles \cite{9} and \cite{14}).

Cohen \cite{4,5} observed empirically that most competition graphs of acyclic digraphs representing food webs are interval graphs. A graph $G$ is an interval graph if we can assign to each vertex $v$ in $G$ a real interval $J(v) \subseteq \mathbb{R}$ so that there is an edge between two distinct vertices $v$ and $w$ if and only if $J(v) \cap J(w) \neq \emptyset$. Cohen’s observation and the continued preponderance of examples that are interval graphs

*Corresponding author. E-mail address: kawa22@snu.ac.kr; borampark22@gmail.com

\textsuperscript{1}This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (201-20110036).

\textsuperscript{2}This research was supported by National Research Foundation of Korea Grant funded by the Korean Government, the Ministry of Education, Science and Technology (NRF-2011-357-C00004).

\textsuperscript{3}The author was supported by JSPS Research Fellowships for Young Scientists.
led to a large literature devoted to attempts to explain the observation and to study
the properties of competition graphs. Roberts [20] showed that every graph can be
made to the competition graph of an acyclic digraph by adding isolated vertices.
He then asked for a characterization of acyclic digraphs whose competition graphs
are interval. The study of acyclic digraphs whose competition graphs are interval
led to several new problems and applications (see [6, 7, 11, 12, 15, 22]). As one of
consequences, Cho and Kim [2] found an interesting class of acyclic digraphs called
“doubly partial orders” with interval competition graphs. We denote by ≺ a partial
order \( \{(x_1, x_2), (y_1, y_2)\} : x_1 < y_1, x_2 < y_2 \) on \( \mathbb{R}^2 \). A digraph \( D \) is called a doubly
partial order (a DPO for short) if there exist a finite subset \( V \) of \( \mathbb{R}^2 \) and a bijection
\( \phi : V(D) \rightarrow V \) such that \( A(D) = \{(x, y) : \phi(y) \prec \phi(x), x, y \in V(D)\} \). The following
theorem clarifies the relationship between interval graphs and the competition
graphs of doubly partial orders.

**Theorem 1** ([2]). The competition graph of a doubly partial order is an interval
graph, and an interval graph with sufficiently many isolated vertices is the competi-
tion graph of a doubly partial order.

Since then, it has been actively studied whether or not the same phenomenon
occurs for other variants of competition graphs and interesting results have been
obtained.

**Theorem 2** ([8]). The competition-common enemy graph of a doubly partial order
is an interval graph unless it contains a 4-cycle as an induced subgraph. In addition,
an interval graph with sufficiently many isolated vertices is the competition-common
enemy graph of a doubly partial order.

The above result on competition-common enemy graphs was generalized by Lu and
Wu [13] and Wu and Lu [29]. Most recently, the niche graph, the m-step competition
graph, and the phylogeny graph of a doubly partial order were studied.

**Theorem 3** ([10]). The niche graph of a doubly partial order is an interval graph
unless it contains a triangle.

**Theorem 4** ([14]). For any positive integer \( m \), the m-step competition graph of
a doubly partial order is an interval graph, and an interval graph with sufficiently
many isolated vertices is the m-step competition graph of a doubly partial order.

**Theorem 5** ([18]). The phylogeny graph of a doubly partial order is an interval
graph. In addition, for any interval graph \( G \), there exists an interval graph \( \tilde{G} \) such
that \( \tilde{G} \) contains the graph \( G \) as an induced subgraph and that \( \tilde{G} \) is the phylogeny
graph of a doubly partial order.

Continuing in the same spirit, we study the competition hypergraph of a doubly
partial order. The notion of a competition hypergraph which is a variant of a
competition graph was introduced by Sonntag and Teichert [24]. The competition
hypergraph \( C\mathcal{H}(D) \) of a digraph \( D \) is a hypergraph without loops and multiple
hyperedges such that the vertex set is the same as \( D \) and \( e \subset V(D) \) is a hyperedge
if and only if \( e \) contains at least 2 vertices and \( e \) coincides with the in-neighborhood
of some vertex \( v \) in \( D \). As we study the competition hypergraphs of digraphs, we assume that all hypergraphs considered in this paper have no loops and no multiple hyperedges. The notion of a competition hypergraph is considered as one of the important variants of competition graphs and significant results on this topic are being obtained (see [17, 24, 25, 26, 27]). In this paper, we classify doubly partial orders whose competition hypergraphs are interval.

2. Main Results

We will always embed the vertices of a DPO (as well as the vertices of its competition hypergraph) into \( \mathbb{R}^2 \) in natural way. We say two vertices \( u \) and \( v \) are adjacent in a hypergraph \( H \) if there is a hyperedge \( e \) in \( H \) such that \( \{ u, v \} \subset e \). For a positive integer \( r \), a hypergraph \( H \) is called \( r \)-uniform if each hyperedge of the hypergraph \( H \) has the same size \( r \). Obviously, 2-uniform hypergraphs are graphs. A sequence \( v_0v_1 \cdots v_k \) of distinct vertices of a hypergraph \( H \) is called a path if there exist \( k \) distinct hyperedges \( e_1, e_2, \ldots, e_k \) such that \( e_i \) contains \( \{ v_{i-1}, v_i \} \) for each \( 1 \leq i \leq k \). A sequence \( v_0v_1 \cdots v_k \) of distinct vertices of a hypergraph \( H \) is called a cycle if there exist \( k+1 \) distinct hyperedges \( e_1, e_2, \ldots, e_k, e_{k+1} \) such that \( e_i \) contains \( \{ v_{i-1}, v_i \} \) for each \( 1 \leq i \leq k \) and \( e_{k+1} \) contains \( \{ v_0, v_k \} \). A subhypergraph of a hypergraph \( H \) is a hypergraph \( H' \) such that \( V(H') \subseteq V(H) \) and \( E(H') = \{ e \cap V(H') \mid e \in E(H), |e \cap V(H')| \geq 2 \} \). For a vertex \( v \) in a digraph \( D \), we denote by \( N_D(v) \) the in-neighborhood of \( v \), i.e., \( N_D(v) := \{ u \in V(D) \mid (u, v) \in A(D) \} \).

A hypergraph \( H \) is interval if there exists a one-to-one function mapping the vertices of \( V(H) \) to points on the real line such that for each hyperedge \( e \), there exists an interval containing the images of all elements of \( e \), but not the images of any vertices not in \( e \). There is a characterization of interval hypergraphs by forbidden subhypergraphs:

**Theorem 6 ([28]).** A hypergraph \( H \) is an interval hypergraph if and only if \( H \) does not contain any of the hypergraphs in Figure\(^4\) as a subhypergraph.

More precisely, the hypergraphs in Figure\(^4\) are defined as follows: Given a positive integer \( n \geq 3 \), let \( C_n \) be the 2-uniform hypergraph with \( n \) vertices which forms a cycle, and let \( C := \{ C_n \mid n \geq 3 \} \). For a positive integer \( n \), we define hypergraphs \( M_n \) and \( F_n \) with \( n+3 \) vertices by

\[
V(M_n) = V(F_n) = \{ v_1, v_2, \ldots, v_{n+3} \} =: V,
\]

\[
E(M_n) = \{ \{ v_i, v_{i+1} \} \mid 1 \leq i \leq n + 1 \} \cup \{ V \setminus \{ v_1, v_{n+2} \} \},
\]

\[
E(F_n) = \{ \{ v_i, v_{i+1} \} \mid 1 \leq i \leq n + 1 \} \cup \{ V \setminus \{ v_1 \}, V \setminus \{ v_{n+2} \} \}.
\]

Let \( \mathcal{M} := \{ M_n \mid n \geq 1 \} \) and \( \mathcal{F} := \{ F_n \mid n \geq 1 \} \). Let \( O_1 \) be the hypergraph defined by \( V(O_1) = \{ x, x', y, y', z, z' \} \) and \( E(O_1) = \{ \{ x, x' \}, \{ y, y' \}, \{ z, z' \}, \{ x, y, z \} \} \), and let \( O_2 \) be the hypergraph defined by \( V(O_2) = \{ x, y, z, w, v \} \) and \( E(O_2) = \{ \{ x, y \}, \{ z, w \}, \{ x, y, z, w \}, \{ y, z, v \} \} \). Theorem 6 states that a hypergraph \( H \) being an interval hypergraph is equivalent to \( H \) not containing any of the hypergraphs in \( C \cup \mathcal{M} \cup \mathcal{F} \cup \{ O_1, O_2 \} \) as a subhypergraph.
First, we will show that the competition hypergraph of a DPO may not be interval. For a positive integer $n$, we define

$$A_n := \{(i, n-i+1) \in \mathbb{R}^2 \mid i \in \{0, 1, 2, \ldots, n+1\}\},$$

$$B_n := \{(i - \frac{1}{3}, n - i - \frac{1}{3}) \in \mathbb{R}^2 \mid i \in \{0, 1, \ldots, n\}\}.$$ 

In the DPO defined on the set $A_n \cup B_n$, two vertices $(i, n+1-i)$ and $(j, n+1-j)$ of $A_n$ with $i < j$ have a common out-neighbor $(i - \frac{1}{3}, n - i - \frac{1}{3})$ if $j - i = 1$ and have no common out-neighbor if $j - i \geq 2$. Thus, the competition hypergraph of the DPO defined on the set $A_n \cup B_n$ is a path as a 2-uniform hypergraph on the $n+2$ vertices in $A_n$ together with the $n+1$ isolated vertices in $B_n$.

**Lemma 7.** For a positive integer $n$, there exists a doubly partial order whose competition hypergraph contains $M_n$ as a subhypergraph.

**Proof.** For a positive integer $n$, we will define a DPO $D_n$ so that $\mathcal{CH}(D_n)$ contains $M_n$ as a subhypergraph. It is easy to check that, for the DPO $D_1$ defined on the set $A_1 \cup B_1 \cup \{(0,0),(\frac{2}{3},\frac{2}{3})\}$ (see Figure 2), $\mathcal{CH}(D_1)$ contains $M_1$ as a subhypergraph. For a positive integer $n \geq 2$, let $D_n$ be the DPO defined by $V(D_n) = A_n \cup B_n \cup \{(0,0)\}$ (see Figure 3). Then the hyperedges of $\mathcal{CH}(D_n)$ consist of the hyperedges of the 2-uniform path induced by $A_n$ and the hyperedge $N_{D_n}((0,0)) = (A_n \setminus \{(0,n+1),(n+1,0)\}) \cup (B_n \setminus \{(-\frac{1}{3},n-\frac{1}{3}),(n-\frac{1}{3},-\frac{1}{3})\})$. Note that $(\frac{2}{3},n-\frac{4}{3}) \in B_n$ for $n \geq 2$. Thus, it is easy to see that the subhypergraph of $\mathcal{CH}(D_n)$ induced by $A_n \cup \{(\frac{2}{3},n-\frac{4}{3})\}$ is isomorphic to $M_n$. 

![Figure 1: Forbidden hypergraphs for interval hypergraphs](image-url)
Lemma 8. For a positive integer \( n \), there exists a doubly partial order whose competition hypergraph contains \( F_n \) as a subhypergraph.

Proof. For a positive integer \( n \), we will define a DPO \( D_n' \) so that \( CH(D_n') \) contains \( F_n \) as a subhypergraph. It is easy to check that, for the DPO \( D_1' \) defined on the set \( A_1 \cup B_1 \cup \{ (\frac{2}{3}, -\frac{1}{3}), (-1, 0), (0, -1) \} \) (see Figure 4), \( CH(D_1') \) contains \( F_1 \) as a subhypergraph. For \( n \geq 2 \), let \( D_n' \) be the DPO \( D_n' \) defined by \( V(D_n') = A_n \cup B_n \cup \{ (0, -1), (-1, 0) \} \) (see Figure 5). Then the hyperedges of \( CH(D_n') \) consist of the hyperedges of the 2-uniform path induced by \( A_n \) and the hyperedges \( N_{D_n'}((-1, 0)) = (A_n \setminus \{ (n + 1, 0) \}) \cup (B_n \setminus \{ (n - \frac{1}{3}, -\frac{1}{3}) \}) \) and \( N_{D_n'}((0, -1)) = (A_n \setminus \{ (0, n + 1) \}) \cup (B_n \setminus \{ (-\frac{1}{3}, n - \frac{1}{3}) \}) \). Note that \( (\frac{2}{3}, n - \frac{4}{3}) \in B_n \) for \( n \geq 2 \). Thus, it is easy to see that the subhypergraph of \( CH(D_n') \) induced by \( A_n \cup \{ (\frac{2}{3}, n - \frac{4}{3}) \} \) is isomorphic to \( F_n \).

Lemmas 7 and 8 show that a hypergraph isomorphic to an element in \( M \cup F \) is realizable as a subhypergraph of the competition hypergraph of a DPO.

Theorem 9. For each hypergraph \( H \) in \( M \cup F \), there exists a doubly partial order whose competition hypergraph contains \( H \) as a subhypergraph.

Now it is natural to ask if the family \( M \cup F \) contains all the forbidden subhypergraphs for the competition hypergraph of a DPO being interval. The answer is yes as it shall be shown in the rest of this paper.
For the sake of simplicity, we define an irreflexive and transitive relation \( \rightarrow \) on \( \mathbb{R}^2 \) as follows: For \( x, y \in \mathbb{R}^2 \) with \( x = (x_1, x_2) \) and \( y = (y_1, y_2) \),

\[
x \rightarrow y \iff x \neq y, \ x_1 \leq y_1, \text{ and } x_2 \geq y_2.
\]

One can check that, for \( x, y \in \mathbb{R}^2 \) with \( x \neq y \),

\[
x \not\rightarrow y \text{ and } y \not\rightarrow x \text{ if and only if } x \not\prec y \text{ or } y \not\prec x. \tag{\dagger}
\]

The following lemmas are simple but useful:

**Lemma 10.** For vertices \( x \) and \( y \) of a doubly partial order \( D \), if the competition hypergraph of \( D \) has two hyperedges \( e_x \) and \( e_y \) such that \( x \in e_x, \ y \notin e_x, \ y \in e_y, \ x \notin e_y \), then either \( x \not\prec y \) or \( y \not\prec x \).

**Proof.** To reach a contradiction, suppose that \( x \not\prec y \) and \( y \not\prec x \). Then, by (\dagger), either \( x \not\prec y \) or \( y \not\prec x \). If \( x \not\prec y \), then any out-neighbor of \( x \) is also an out-neighbor of \( y \). Therefore, any hyperedge containing \( x \) contains \( y \), which is a contradiction to the existence of the hyperedge \( e_x \). Similarly, we can reach a contradiction if \( y \not\prec x \). Thus \( x \not\prec y \) or \( y \not\prec x \). \( \square \)
Lemma 11. For vertices $x$, $y$, and $z$ of a doubly partial order $D$, if $x \nless y \nless z$ then $y$ and all of its in-neighbors are contained in any hyperedge containing $x$ and $z$ in the competition hypergraph of $D$.

Proof. Let $e$ be a hyperedge containing $x$ and $z$ in $\mathcal{CH}(D)$. Then there exists a vertex $a$ such that $N_D^-(a) = e$. Let $x = (x_1, x_2)$, $y = (y_1, y_2)$, $z = (z_1, z_2)$, and $a = (a_1, a_2)$. Since $a \less x$ and $a \less z$, $a_1 < \min\{x_1, z_1\}$ and $a_2 < \min\{x_2, z_2\}$. Since $x \nless y \nless z$, $x_1 \leq y_1 \leq z_1$ and $x_2 \geq y_2 \geq z_2$, which implies that $\min\{x_1, z_1\} \leq y_1$ and $\min\{x_2, z_2\} \leq y_2$. Therefore $a \nless y$ and thus $y \in e$. Since $y \leq u$ for any in-neighbor $u$ of $y$, $a \less u$ and hence $u \in e$.

Lemma 12. For vertices $x$, $y$, and $z$ of a doubly partial order $D$, if $x \nless y$, $y \less x$, and $z \nless y$, then $z \nless y$.

Proof. Let $x = (x_1, x_2)$, $y = (y_1, y_2)$, and $z = (z_1, z_2)$. Since $x \nless y$, $x_1 \leq y_1$. Since $z \less x$, $z_1 \leq x_1$. Therefore $z_1 \leq y_1$ and so $y \nless z$. Since $y \nless z$ and $z \nless y$, either $z \nless y$ or $y \nless z$ by (1). If $y \nless z$, then $y_1 \leq z_1$ and so $x_1 \leq z_1$, a contradiction to the fact that $z \nless x$. Thus $y \nless z$ and so $z \nless y$.

Lemma 13. For vertices $x$, $y$, and $z$ of a doubly partial order $D$, if $x \nless y$, $y \less x$, and $z \nless x$, then $x \nless z$.

Proof. Let $x = (x_1, x_2)$, $y = (y_1, y_2)$, and $z = (z_1, z_2)$. Since $x \nless y$, $x_2 \geq y_2$. Since $z \less x$, $z_2 \leq x_2$. Therefore $z_2 \leq x_2$. Since $z \nless x$, $z_1 \geq x_1$ or $z_2 \geq x_2$. Since $z_2 < x_2$, $z_1 \geq x_1$. Thus $x \nless z$.

Using Lemmas 10, 11, and 12, we can show that the competition hypergraph of a DPO does not contain any element of $\mathcal{C} \cup \{O_1, O_2\}$ as a subhypergraph.

Lemma 14. The competition hypergraph of a doubly partial order does not contain any hypergraph in $\mathcal{C}$ as a subhypergraph.

Proof. We prove the lemma by contradiction. Suppose that there exists a DPO $D$ such that $\mathcal{CH}(D)$ contains $C_n$ as a subhypergraph for some $n \geq 3$. Let $v_1 v_2 \cdots v_n$ be the vertices of $C_n$ such that $v_i v_{i+1}$ are adjacent for all $1 \leq i \leq n$, where the subscripts are reduced to modulo $n$. Note that for any distinct $i$, $j$ in $\{1, 2, \ldots, n\}$, there exists a hyperedge containing $v_i$ but not $v_j$. Thus, by Lemma 10, $v_i \nless v_j$ or $v_j \nless v_i$ for any distinct $i$, $j$ in $\{1, 2, \ldots, n\}$. Without loss of generality, we may assume that $v_1 \nless v_2$. If $v_3 \nless v_1$, then $v_3 \nless v_1 \nless v_2$ and so, by Lemma 11, $v_1$ and $v_3$ are adjacent, a contradiction. Thus $v_1 \nless v_3$. Suppose that $v_3 \nless v_2$. Then $v_1 \nless v_3 \nless v_2$, and by the argument in the proof of Lemma 11, a common out-neighbor of $v_1$ and $v_2$ is also an out neighbor of $v_3$. This implies that $C_n$ contains a hyperedge containing $\{v_1, v_2, v_3\}$, which contradicts that $C_n$ is 2-uniform. Therefore, $v_2 \nless v_3$. By applying a similar argument, we can claim that $v_3 \nless v_4$. By continuing this argument, we obtain $v_1 \nless v_2 \nless \cdots \nless v_{n-1} \nless v_n \nless v_1$. Since the relation $\nless$ is transitive, we have $v_1 \nless v_1$, which is a contradiction to the irreflexivity of $\nless$. 

Lemma 15. The competition hypergraph of a doubly partial order does not contain $O_1$ or $O_2$ as a subhypergraph.
Proof. Suppose that there exists a DPO $D$ such that $\mathcal{C}(D)$ contains $O_1$ as a subhypergraph. Let $\{x, y, z\}$ be the hyperedge of $O_1$ of size three. By Lemma 10, any two $u, v$ of $\{x, y, z\}$ satisfy $u \not\prec v$ or $v \not\prec u$. Thus $\alpha \not\prec \beta \not\prec \gamma$ for a permutation $(\alpha\beta\gamma)$ on $\{x, y, z\}$. Without loss of generality, we may assume that $x \not\prec y \not\prec z$. Let $y'$ be the vertex of $O_1$ other than $x, y, z$ that is adjacent to $y$. By Lemma 10, either $x \not\prec y'$ or $y' \not\prec x$, and either $z \not\prec y'$ or $y' \not\prec z$. If $y' \not\prec x$, then $y' \not\prec x \not\prec y$ and so, by Lemma 11, $x$ and $y'$ are adjacent, a contradiction. Thus $x \not\prec y'$. If $y' \not\prec z$, then $x \not\prec y' \not\prec z$ and so, by Lemma 11, $x$ and $y'$ are adjacent, a contradiction. Thus $z \not\prec y'$. Then $y \not\prec z \not\prec y'$ and so, by Lemma 11, $y'$ and $z$ are adjacent, a contradiction. Hence, the competition hypergraph of a DPO does not contain $O_1$ as a subhypergraph.

Suppose that there exists a DPO $D$ such that $\mathcal{C}(D)$ contains $O_2$ as a subhypergraph. Let $V(O_2) = \{x, y, z, w, v\}$ and let $\{x, y, z, w\}$ be the hyperedge of $O_2$ of size four, and $\{x, y\}$ and $\{z, w\}$ be hyperedges of size two. The vertices $y$ and $z$ satisfy the condition of Lemma 10 and so $y \not\prec z$ or $z \not\prec y$. Without loss of generality, we may assume that $y \not\prec z$. Now $x$ and $z$ satisfy the condition of Lemma 10 and so $x \not\prec z$ and $z \not\prec x$. If $z \not\prec x$ then $y \not\prec z \not\prec x$ and so $z$ belongs to any hyperedge containing $x$ and $y$ by Lemma 11, a contradiction. Therefore $x \not\prec z$. Similarly we can show that $y \not\prec w$. Since $\{y, z, v\}, \{x, y, z, w\} \in E(O_2)$, there exist $a, b \in V(D)$ such that $\{y, z, v\} = N_D(a) \cap V(O_2)$ and $\{x, y, z, w\} = N_D(b) \cap V(O_2)$. If $a \not\prec b$, then $w \in N_D(a) \cap V(O_2)$, a contradiction. If $w \not\prec a$, then $v \in N_D(b) \cap V(O_2)$, a contradiction. Therefore $a \not\prec w$ and $w \not\prec a$. Thus $y, w, a$ satisfy the condition of Lemma 12 and so $a \not\prec w$. If $a \not\prec x$ then $x \in N_D(a) \cap V(O_2)$, a contradiction. Therefore $a \not\prec x$, and so $x, z, a$ satisfy the condition of Lemma 13 and so $x \not\prec a$. Thus $x \not\prec a \not\prec w$. Since $N_D(b)$ is a hyperedge containing $x$ and $w$, the vertex $v$ which is in-neighbor of $a$ belongs to $N_D(b)$ by Lemma 11. Since $v \in V(O_2), v \in N_D(b) \cap V(O_2)$. However, $\{x, y, z, w\} \in N_D(b) \cap V(O_2)$, a contradiction. Hence the competition hypergraph of a DPO does not contain $O_2$ as a subhypergraph.

By Lemmas 14 and 15, the following holds:

**Theorem 16.** For a doubly partial order $D$, the competition hypergraph of $D$ is interval if and only if it does not contain any hypergraph in $\mathcal{M} \cup \mathcal{F}$ as a subhypergraph.

It follows from Theorems 9 and 16 that $\mathcal{M} \cup \mathcal{F}$ is the family of all the forbidden subhypergraphs for the competition hypergraph of a DPO being interval. A hypergraph is **chordal** if any cycle of length at least 4 has two nonconsecutive vertices that are adjacent. If a hypergraph does not contain any hypergraph in $\mathcal{C}$ as a subhypergraph, then it is chordal. Therefore, by Lemma 14, the following corollary immediately holds.

**Corollary 17.** The competition hypergraph of a doubly partial order is chordal.

Theorem 1 shows that any interval graph can be made into the competition graph of a DPO by adding sufficiently many isolated vertices. In the same context, we may ask whether the following statement is true:

$\ast$ An interval hypergraph can be made into the competition hypergraph of a DPO by adding sufficiently many isolated vertices.
We can show that the answer is no by taking an interval hypergraph $\mathcal{H}$ defined by $V(\mathcal{H}) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and

$$E(\mathcal{H}) = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_5\}, \{v_5, v_6\}, \{v_2, v_3, v_4, v_5\}\}$$

(see Figure 6). To show by contradiction that $\mathcal{H}$ cannot be made into the competition hypergraph of a DPO even if sufficiently many isolated vertices are added, suppose that there is a DPO $D$ such that $\mathcal{C}(D) = \mathcal{H} \cup I$ where $I$ is a set of isolated vertices. One can observe that any two distinct vertices among $v_2, v_3, v_4, v_5$ satisfy the condition in Lemma 10. Thus, for any two distinct vertices $x, y$ in the hyperedge $\{v_2, v_3, v_4, v_5\}$, either $x \nless y$ or $y \nless x$ holds. By applying a similar argument for assuming that $v_1 \nless \ldots \nless v_n$ in the proof of Lemma 14, we may assume without loss of generality that $v_2 \nless v_3 \nless v_4 \nless v_5$. Since $\{v_3, v_4\} \in E(\mathcal{H})$, there exists a vertex $u$ such that $N_D(u) = \{v_3, v_4\}$. If $u = v_i$ for $i \in \{1, 2, 5, 6\}$, then $v_3$ and $v_4$ are contained in any hyperedge containing $v_i$, a contradiction. Thus $u \in I$. Since $v_2 \nless u \nless v_3$, and $u \not\nleq v_2$, it follows from Lemma 13 that $v_2 \nless u$. On the other hand, since $v_4 \nless v_5$, $u \nless v_4$, and $u \not\nleq v_5$, it follows from Lemma 12 that $u \nless v_5$. We have shown that $v_2 \nless u \nless v_5$. Then, by Lemma 11, $u$ is contained in the hyperedge $\{v_2, v_3, v_4, v_5\}$, which is a contradiction to $u \in I$.

The following statement weaker than (*) seems to be worthy of mentioning:

**Theorem 18.** For an interval hypergraph $\mathcal{H}$, there exists a doubly partial order $D$ whose competition hypergraph contains $\mathcal{H}$ as a subhypergraph.

**Proof.** Let $\mathcal{H}$ be an interval hypergraph. Then there exists an ordering $v_1, v_2, \ldots, v_n$ of the vertices of $\mathcal{H}$ such that any hyperedge of $\mathcal{H}$ is consecutive in the ordering. For each $i = 1, 2, \ldots, n$, let $a_i := (i, n + 1 - i) \in \mathbb{R}^2$. For each hyperedge $e$, we denote $\min(e) := \min\{j \mid v_j \in e\}$ and $\max(e) := \max\{j \mid v_j \in e\}$ and let $b_e := (\min(e) - 1, n - \max(e)) \in \mathbb{R}^2$. Let $D$ be a DPO on the set $\{a_i \mid 1 \leq i \leq n\} \cup \{b_e \mid e \in E(\mathcal{H})\}$, and let $\mathcal{H}'$ be the subhypergraph of $\mathcal{C}(D)$ induced by the set $\{a_i \mid 1 \leq i \leq n\}$. We will show that the bijection from $V(\mathcal{H})$ to $V(\mathcal{H}')$ which maps a vertex $v_i$ to the vertex $a_i$ is an isomorphism between hypergraphs $\mathcal{H}$ and $\mathcal{H}'$.

Let $e$ be a hyperedge in $\mathcal{H}$, and let $i := \min(e)$ and $k := \max(e)$. Then $e = \{v_i, v_{i+1}, \ldots, v_k\}$. Since $e$ contains at least two vertices, $i < k$. Then $b_e = (i - 1, n - k) \in \mathbb{R}^2$ is a vertex of $D$. If $i \leq j \leq k$, then $i - 1 < j$ and $n - k < n + 1 - j$. If $1 \leq j < i$, then $i - 1 \geq j$. If $k < j \leq n$, then $n - k \geq n + 1 - j$. Therefore, we have $b_e = (i - 1, n - k) \prec (j, n + 1 - j) = a_j$ if $i \leq j \leq k$ and $b_e = (i - 1, n - k) \not\prec (j, n + 1 - j) = a_j$ if $1 \leq j < i$ or $k < j \leq n$. Since $N_D(b_e)$ is a hyperedge of $\mathcal{C}(D)$, $\{a_i, a_{i+1}, \ldots, a_k\}$ is a hyperedge in $\mathcal{H}'$.

Let $e'$ be a hyperedge in $\mathcal{H}'$. Let $i := \min\{j \mid a_j \in e'\}$ and $k := \max\{j \mid a_j \in e'\}$. Note that $a_j \not\in e'$ for $1 \leq j < i$ and $k < j \leq n$. Since $e'$ contains at least two vertices, $i < k$. Then there exists a vertex $z = (z_1, z_2) \in V(D)$ such that $e' = N_D(z)$. Note
that \( z \prec a_i \) and \( z \prec a_k \). If \( i \leq j \leq k \), then \( z = (z_1, z_2) \prec (j, n + 1 - j) = a_j \) since \( z_1 < i \) and \( z_2 < n + 1 - k \). Therefore, \( e' = \{a_i, a_{i+1}, \ldots, a_k\} \). By the definition of \( D \) and the fact that \( z \not\prec a_j \) for all \( j \in \{1, 2, \ldots, n\} \), \( z = b_e = (\min(e) - 1, n - \max(e)) \) for some hyperedge \( e \) of \( H \). By the consecutive property of the ordering \( v_1, v_2, \ldots, v_n \), the hyperedge \( e \) contains \( v_j \) for each \( j \) such that \( \min(e) \leq j \leq \max(e) \). Now we show that \( \min(e) = i \) and \( \max(e) = k \). Since \( \min(e) - 1 = z_1 < i \) and \( n - \max(e) = z_2 < n + 1 - k \), we obtain \( \min(e) \leq i \) and \( k \leq \max(e) \). Since \( z \not\prec a_i - 1 \), it holds that \( \min(e) - 1 \geq i - 1 \) or \( n - \max(e) \geq n + 1 - (i - 1) \). If the latter happens, then \( \max(e) \leq i - 2 \), which contradicts the choice of \( i \). Thus \( \min(e) \geq i \). Since \( z \not\prec a_k + 1 \), it holds that \( \min(e) - 1 \geq k + 1 \) or \( n - \max(e) \geq n + 1 - (k + 1) \). By the choice of \( k \), the latter holds and so \( \max(e) \leq k \). Thus \( \min(e) = i \) and \( \max(e) = k \).

Hence the theorem holds. \( \square \)

References


