Meshless Convection-Diffusion Analysis by Triple-Reciprocity Boundary Element Method

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Abstract. The conventional boundary element method (BEM) needs a domain integral in convection-diffusion analysis. In this paper, we show that the convection-diffusion problem can be solved effectively using triple-reciprocity boundary element method without internal cells. In this method, the convection term is treated in the same manner as the heat generation term, and the values of the term are interpolated using integral equations. In this method, time-dependant fundamental solutions are used. However, the CPU time for calculation does not increase rapidly with increasing number of time steps.

1. Introduction

The convection-diffusion equation is necessary for solving important problems in engineering such as heat conduction and the diffusion of pollutants in fluids. In the calculus of the finite difference method, a gridiron layout is used to solve convection-diffusion equations, but it is difficult to divide an arbitrary region into a gridiron layout in order to analyze the region. Using the finite element method, finite elements can be divided more easily in an arbitrary region. However, the volume of data becomes larger, and the accuracy of the results depends on the mesh of the finite elements. When solving general convection-diffusion problems by the boundary element method (BEM), internal cells are generally required because of the convection term [1, 2]. If the velocity of flow is constant, the convection-diffusion problems can be solved without internal cells by BEM [3, 4].

In this paper, we used the triple-reciprocity boundary element method to solve the convection-diffusion problem with spatially variable convective velocity without internal cells. The conventional boundary element method uses internal cells for a convection term that is nonlinear; however, the use of internal cells in boundary element method removes the advantages of the simple preparation and small volume of data. In this study, on the basis of the triple-reciprocity boundary element method, numerical integration is carried out in such a way that a domain integral with the convection term is converted into a boundary integral. This method involves the uses of arbitrary internal points instead of internal cells. Several examples are given in order to demonstrate the efficiency of this method. In addition, we deal with a heat convection-diffusion problem in this paper.

2. Theory

2.1 Convection-diffusion analysis

Assuming that the convective coefficients are \( A_x(x,y,t) \) and \( A_y(x,y,t) \), the following equation of heat conduction must be solved in order to obtain the temperature \( T \), if there is a heat generation term \( W_1 \),

\[
V_T + \frac{W_1}{\kappa} - \kappa \left( A_x \frac{\partial T}{\partial x} + A_y \frac{\partial T}{\partial y} \right) = \kappa \frac{\partial T}{\partial t},
\]

where \( t \) is the time, \( \kappa \) is the thermal diffusivity, and \( \lambda \) is the heat conductivity. The distance between the observation point \( p \) and the loading point \( q \) is \( r \), the arbitrary time is \( t \), and the initial temperature is \( T_0^q(q,0) \). Then the convective term is given by

\[
W_1 = -\kappa \left( A_x \frac{\partial T}{\partial x} + A_y \frac{\partial T}{\partial y} \right).
\]

The boundary integral equation for the temperature in the unsteady state is given by [3]
\[ cT(P, t) = - \kappa \int \left[ T(Q, r) \frac{\partial T(Q, r,t)}{\partial n} - \frac{\partial T(Q, r,t)}{\partial n} \right] dA \, dt \]
\[ + \frac{c}{2} \int T_t^*(P, q, t) W(q, t) d\Omega \]
\[ + \frac{\kappa}{2} \int \left[ T_t^*(P, q, t) W(q, t) - \frac{\partial T(Q, r,t)}{\partial n} \right] dA \, dt \]
\[ + \left[ T_t^*(P, q, t, 0) + T_t^*(Q, r, 0) \right] d\Omega , \]

where \( c = 0.5 \) on the smooth boundary and \( c = 1 \) inside the domain, respectively. \( \Gamma \) and \( \Omega \) denote boundary and domain, respectively, and \( P \) and \( Q \) are expressed as \( P = \partial Q / \partial n \) on the boundary, respectively. In the two-dimensional case, the time-dependent fundamental solution \( T_t^*(p, q, t, \tau) \) of unsteady heat analysis in Eq. (3) and the differential coefficient with respect to the unit normal \( n \) are given by [6]

\[ T_t^*(p, q, t, \tau) = \frac{1}{4\pi (t - \tau)^3} \exp(-a) , \]

\[ \frac{\partial T_t^*(p, q, t, \tau)}{\partial n} = \frac{-a}{8\pi (t - \tau)^2} \exp(-a) , \]

where

\[ a = \frac{r^3}{4\pi (t - \tau)} . \]

As shown in Eq. (3), the convective term as well as the heat generation term can be dealt with; however, internal cells must be used in the conventional boundary element method.

### 2.2 Interpolation and triple-reciprocity BEM

To avoid the use of internal cells, a new interpolation method is introduced. The method by which we interpolate the convective term \( W_1(q, \tau) \) at the time \( \tau \) using the boundary integral equation is shown below [7-10]. The following equations can be used in the two-dimensional interpolation.

\[ V^2 W_1(q, \tau) = - W_2(q, \tau) \]

\[ V^2 W_2(q, \tau) = - \sum_{m=1}^{M} W_{m}^{p}(q_m, \tau) \delta(q - q_m) \]

Equations (7) and (8) can be rewritten as follows:

\[ V^4 W_1(q, \tau) = - \sum_{m=1}^{M} W_{m}^{p}(q_m, \tau) \delta(q - q_m) . \]

This is the same as the equation used to obtain the deformation of a thin plate with an unknown point load. The deformation is given, and the interpolation is performed by obtaining the unknown point load. The distribution of heat generation and the initial temperature distribution are interpolated using the following equations:

\[ V^2 W_{1N}(q, \tau) = - W_{2N}(q, \tau) , \]

\[ V^2 W_{2N}(q, \tau) = - \sum_{m=1}^{M} W_{mN}(q_m, \tau) \delta(q - q_m) , \]

\[ V^2 T_1^*(q, 0) = - T_2^*(q, 0) , \]

\[ V^2 T_2^*(q, 0) = - \sum_{m=1}^{M} T_{mN}(q_m, 0) \delta(q - q_m) . \]

Next, the function \( T_t^*(p, q, t, \tau) \) given by the following equation is considered.

\[ V^2 T_{r^2}(p, q, t, \tau) = T_t^*(p, q, t, \tau) \]

From Equations (7)-(14), Equation (3) can be rearranged by using Green’s second identity twice.
The two-dimensional polyharmonic function $T_f(P, q, t, \tau)$ in Eq. (14) is determined as

$$T_{f,1s}(P, q, t, \tau) = \int_{\Gamma} T_f(P, q, t, \tau) \, d\Gamma$$

Additionally, the temperature gradient is given by differentiating Equation (15), and expressed as:

$$\frac{\partial T(p, t)}{\partial x_i} = -\kappa \sum_{j=1}^{2} \int \frac{\partial W_j(q, \tau)}{\partial x_j} \frac{\partial T_f(P, q, t, \tau)}{\partial x_i} \, d\Gamma \, dt$$

To interpolate the convective term $W_1$, using the boundary integral equations, the harmonic function $T_1(P, q)$ and the biharmonic function $T_2(P, q)$ in the steady state are used,

$$T_1(P, q) = \frac{1}{2\pi} \ln \frac{1}{r} + B,$$

$$T_2(P, q) = \frac{1}{8\pi} \ln \frac{1}{r} + B + 1,$$

where $B$ is constant and may be regarded as zero. Moreover, the convective terms $W_1$ and $W_2$ are given by Green’s second identity, and Equations (7) and (8) are as follows:

$$cW_1(P, \tau) = \int (T_f(P, \tau) \frac{\partial W_1(q, \tau)}{\partial x_j} - \frac{\partial T_1(P, q)}{\partial x_j} W_1(q, \tau)) \, d\Gamma$$

$$cW_2(P, \tau) = \int (T_f(P, \tau) \frac{\partial W_2(q, \tau)}{\partial x_j} - \frac{\partial T_2(P, q)}{\partial x_j} W_2(q, \tau)) \, d\Gamma$$

In the same manner, the heat generation and the initial temperature terms can be interpolated using the integral equations.

2.3 Polyharmonic function for unsteady state

The two-dimensional polyharmonic function $T_f(P, q, t, \tau)$ is determined as

$$T_{f,1s}(P, q, t, \tau) = \int_{\Gamma} T_f(P, q, t, \tau) \, d\Gamma$$
\( T_j'(P,q,t) \) in the unsteady state and its normal derivative are explicitly given by

\[
T_j'(P,q,t) = \frac{1}{4\pi} E_j(a) + \ln(a) + C
\]

(22)

\[
\frac{\partial T_j'(P,q,t)}{\partial n} = \frac{1}{2\pi} \frac{\partial}{\partial n} \left[ 1 - \exp(-a) \right]
\]

(23)

\[
T_j'(P,q,t) = \frac{r^2}{16\pi} \left[ E_j(a) + \ln(a) + C + \frac{1}{a} \exp(-a) \right]
\]

(24)

\[
\frac{\partial T_j'(P,q,t)}{\partial n} = \frac{r}{8\pi} \frac{\partial}{\partial n} \left[ E_j(a) + \ln(a) + C - 1 + \frac{1}{a} \exp(-a) \right]
\]

(25)

where \( E_j(\cdot) \) is the exponential integral function and \( C \) is Euler's constant [12].

The derivative of the polyharmonic function \( T_j'(P,q,t) \) and the normal derivative with respect to \( x_i \) in Eq.(16) are explicitly given by

\[
\frac{\partial T_j^*}{\partial x_i} = \frac{-r_i}{8\pi} \frac{\partial}{\partial x_i} \exp(-a)
\]

(26)

\[
\frac{\partial^2 T_j^*}{\partial x_i \partial x_j} = \frac{-1}{8\pi} \frac{\partial}{\partial x_i} \left[ E_j(a) + \ln(a) + C - 1 + \frac{1}{a} \exp(-a) \right]
\]

(27)

\[
\frac{\partial^2 T_j}{\partial x^2_i} = \frac{r_i}{2\pi} \left[ 1 - \exp(-a) \right]
\]

(28)

\[
\frac{\partial^3 T_j}{\partial x_i \partial x_j \partial x_k} = \frac{1}{2\pi} \left[ E_j(a) + \ln(a) + C - 1 + \frac{1}{a} \exp(-a) \right]
\]

(29)

\[
\frac{\partial^3 T_j^*}{\partial x_i \partial x_j} = \frac{r_i}{8\pi} \left[ E_j(a) + \ln(a) + C - 1 + \frac{1}{a} \exp(-a) \right]
\]

(30)

\[
\frac{\partial^2 T_j^*}{\partial x_i \partial x_j} = \frac{1}{8\pi} \left[ 1 - \exp(-a) \right]
\]

(31)

where \( r_i = \frac{\partial}{\partial x_i} \).

Numerical solutions are obtained using the interpolation functions for time and space. If constant-time interpolation and the time step \( t_f - t_{f-1} \) are used, the time integral can be treated analytically. The time integrals for \( T_j'(P,q,t) \) and \( \partial T_j'/\partial n \) are given as follows:

\[
\int_{t_f}^{t_j} T_j'(P,q,t) \, dt = \frac{1}{4\pi} E_j(a)
\]

(32)

\[
\int_{t_f}^{t_j} \frac{\partial T_j'}{\partial n} \, dt = \frac{1}{2\pi} \frac{\partial}{\partial n} \exp(-a_f)
\]

(33)

\[
\int_{t_f}^{t_j} T_j'(P,q,t) d\tau = \frac{r^2}{16\pi} \left[ E_j(a_f) + \frac{1}{a_f} \exp(-a_f) \right] + E_j(a_f)
\]

(34)

\[
\int_{t_f}^{t_j} \frac{\partial T_j'}{\partial n} d\tau = \frac{r}{8\pi} \frac{\partial}{\partial n} \left[ 1 - \exp(-a_f) \right]
\]

(35)

\[
\int_{t_f}^{t_j} T_j'(P,q,t) \, dt = \frac{r^3}{256\pi} \left[ 4E_j(a_f) + 4 \ln(a_f) + 4C + 1 - \exp(-a_f) \right]
\]

(36)

\[
\int_{t_f}^{t_j} \frac{\partial T_j'}{\partial n} \, dt = \frac{r^3}{64\pi} \frac{\partial}{\partial n} \left[ 1 - \exp(-a_f) \right]
\]

(37)

where
In the same way, the time integrals for \( \frac{\partial T}{\partial t} \) and \( \frac{\partial^2 T}{\partial x^2} \) in Eq.(16) are given as follows:

\[
\int_T \frac{\partial^2 T}{\partial x^2} (p,q,t) \, dx = \frac{r_a}{2\pi} \exp(-a) ,
\]

\[
\int_T \frac{\partial^2 T}{\partial x^2} (p,q,t) \, dx = \frac{1}{2\pi} \{ a_f E(a_f) + \frac{1}{a_f} [1 - \exp(-a)] \} ,
\]

\[
\int_T \frac{\partial T}{\partial x} (p,q,t) \, dx = \frac{r_a}{8\pi} \frac{\partial x}{a_f} - \frac{1}{a_f} E(a_f) ,
\]

\[
\int_T \frac{\partial T}{\partial x} (p,q,t) \, dx = \frac{1}{8\pi} \{ a_f E(a_f) - \exp(-a) \} ,
\]

\[
\int_T \frac{\partial^2 T}{\partial x^2} (p,q,t) \, dx = \frac{1}{64\pi} \{ a_f E(a_f) + \frac{1}{a_f} [1 - \exp(-a)] \} ,
\]

Assuming that functions \( T(Q,t) \) and \( \frac{\partial T}{\partial n} \) remain constant over time in each time step, Eq.(15) can be written in matrix form. Replacing \( T(Q,t) \) and \( \frac{\partial T}{\partial n} \) with vectors \( T \) and \( Q \), respectively, and discretizing Eq.(15), we obtain

\[
\sum_j B_0 T_j = \sum_j E_0 Q_j + B_0 ,
\]

where \( B \) represents the total effect of the pseudo-initial temperature, heat generation term, and convection term. Adopting a constant time step throughout the analysis, the coefficients of the matrix at several time steps need only computed and stored once. In the conventional boundary element method using the time-dependent fundamental solution, the number of boundary integrals increases rapidly with increasing number of time steps. In this method, there is no such disadvantage, because the integration is considered within several time steps, taking the temperature at the end of the previous time step as the initial (pseudo-initial) value at the current time step. If we use a constant length for the time steps, the integration in the present method is carried out once for the whole time step procedure. However, the iteration process is necessary for the convection term.

3. Numerical examples

We analyzed the rectangular region shown in Fig. 1 to verify the efficiency of this method. The velocities \( A_x \) and \( A_y \) (m/s) are given by

\[
\begin{align*}
A_x &= 40y(1-y) \\
A_y &= 0
\end{align*}
\]

The coefficient of thermal diffusivity \( \kappa \) is 1 m\(^2\)/s. As shown in Fig. 1, one part of the rectangular region is 10°C, the remainder of the region is 0°C and the right side of the region is adiabatic. The internal points are used for interpolation. Figure 2 shows the temperature distributions at \( t=0.025, 0.1 \) and 0.5 s. Figure 3 shows the comparison of BEM result with one by the finite difference method (FDM).
These results using presented method are in good agreement with the values obtained using FDM.

Next, the temperature distribution of L-shaped region with a round corner shown in Fig. 4 is obtained. When the flow is that of an ideal fluid and the flow rates at \( x = 0 \) are \( A_x = 10 \) (m/s) and \( A_y = 0 \), the distributions of the flow rates are obtained using Laplace’s equation. The coefficient of thermal diffusivity \( \kappa \) is 2 m\(^2\)/s\(^{-1}\). As shown in Fig. 4, one part of the region is 10\(^\circ\)C and the remainder is 0\(^\circ\)C. Fig. 5 shows the temperatures distribution at \( t = 0.005 \) and 0.1 s.

We obtained the temperature distributions for the convection-diffusion problem upon heat generation for the region shown in Fig. 6. The homogeneous heat \( \frac{W}{A} = 100 \text{K/m}^2 \) is generated in the hatched region in Fig. 6. The coefficients of convection term \( A_x \) and \( A_y \) (m/s) are given by

\[
\begin{align*}
A_x &= 0, \\
A_y &= 20x(1-x).
\end{align*}
\]  

(47)

The coefficient of thermal diffusivity \( \kappa \) is 1 m\(^2\)/s\(^{-1}\). The temperature at the boundary is 0\(^\circ\)C as shown in Fig. 6. Fig. 7 shows the temperature distributions at \( t = 0.005, 0.1, 0.2 \) and 2 s.

![Rectangular region](image1)

**Fig. 1** Rectangular region

(a) \( t = 0.025 \) s  
(b) \( t = 0.1 \) s  
(c) \( t = 0.5 \) s

**Fig. 2** Temperature distributions

![Temperature distributions](image2)

**Fig. 3** Comparison of BEM an FDM results
Fig. 4  L-shaped region with round corner

Fig. 5  Temperature distributions

Fig. 6  Rectangular region with heat generation
4. Conclusion
In the conventional boundary element method, internal cells are required to solve convection-diffusion problems using time-dependent fundamental solutions. However, domain integrals are converted into boundary integrals by the triple-reciprocity boundary element method, and internal cells thus become unnecessary. Although there is disadvantage of the conventional boundary element method, in which it takes a considerable amount of time to calculate as an arithmetic series with increasing number of time steps when using a time-dependent solution, it has been overcome using the triple-reciprocity boundary element method in which we regard the calculated temperature as the initial temperature. Even if this method uses internal points, the preparation of data is simpler than that for internal cells, and the numerical examples show the efficiency of this method.

References
Modeling of Inhomogeneities and Reinforcements in Elasto-Plastic Problems with the BEM
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Abstract. The Boundary Element Method (BEM) is particular suitable to analyze problems involving infinite and semi-infinite spaces. However, the classical BEM can not deal with heterogeneous ground conditions and inclusions such as rock bolts. In this work a novel approach for the simulation of inclusions in 2- and 3- dimensional problems and in connection with elasto-plastic material behavior is presented. Two kinds of inclusions are treated, common geological inhomogeneities and the reinforcement of the ground by rock bolts. Internal cells are required to consider the effects of the inclusions inside the domain (the domain-force-terms). To avoid the increase of the degrees of freedom in the system of equations, an iterative algorithm is used for the calculation. This iteration procedure is combined with the plasticity-algorithm.

Introduction
In underground engineering problems different kinds of inclusions plays an important role. This includes in particular geological inhomogeneities and the reinforcement of the ground by rock bolts (linear inclusions). Such a rock bolting technique is an often used way of supporting underground openings. In tunneling and underground mining, steel rod inserted in a hole drilled into the excavation surface to provide support to the roof or sides of the cavity.

To analyze an underground structure containing several small inclusions with numerical methods is very time consuming, for the finite element method as well as for the boundary element method. Even for generating a model the effort is huge, in particular for a domain with many rock bolts and in three dimensional analyses.

In this work a BEM approach is presented to generate and analyze elastic or plastic structures containing a large number of inclusions in an efficient way. The inclusions are discretized with cells (volume-cells or line-cells). Only a few numbers of line-cells are necessary to discretize one rock bolt and these cells are independent of the boundary-element mesh (see Fig. 1).

Fig. 1: Discretization of a tunnel involving rock bolts and geological inhomogeneities
By modeling inclusions in this way, the size of the system of equations does not increase with the number of cell nodes. The additional degrees of freedom are treated as an additional right-hand-side and the system is solved within an iterative algorithm. Especially for large scale problems and in elasto-plastic domains this iterative procedure is an efficient alternative. It avoids the increase of the system of equations and the iteration procedure can be combined within the iterations from the plasticity-algorithm.

Solution Procedure

The boundary integral equation for an elastic problem considering initial-stress inside the domain can be expressed as (see also [1], [2], [3], [6], [5]):

$$c(P)u(P) = \int_S U(P, Q)u(Q) - T(P, Q)\frac{\partial u(Q)}{\partial n} dS + \int_E E(P, Q)\sigma_i(Q) dV$$

Where $u$ and $t$ are the displacements and tractions, respectively on the boundary $S$, $c$ is related to the boundary geometry, $U$ and $T$ are the fundamental solutions for the displacements and tractions, respectively. The last integral over the volume $V$ contains the initial-stress $\sigma$ and the strain fundamental solution $\epsilon$. $P$ and $Q$ denote the source point and field point, respectively, for nodes on the boundary and $Q$ are the internal points where the initial-stress is acting on.

In conventional BEM the initial-stress is known and the system of equations (Eq. 1) can be solved directly to obtain the unknown boundary quantities $u$ and $t$. Here the effect of the inclusions is considered by their action-reaction stresses (initial-stresses $\sigma_i$), which are unknown at the beginning.

The problem can either be solved directly or iteratively. In the direct approach the unknown volume integral can be rearranged to the unknowns on the left-hand-side, in this case the system of equations increases with the number of degrees of freedom related to the cell nodes. The method proposed here shows an iterative approach, the unknown volume term is calculated iteratively by a right-hand-side adding to the system of equations. Therefore the system itself has no additional degrees of freedom based on the inclusions; it has only the size of the boundary-element-modell.

In the first step of the calculation the unknown volume-term of Eq. 1 is neglected. An initial analysis is carried out assuming that no inclusions are inside the domain. After solving the system, the displacement- or the stress-field in the domain can be carried out via post-processing using Somigliana’s Identity. The stress on an interior point $P$ is given by:

$$\sigma(P) = \int_D D(P, Q)u(Q) - S(P, Q)\frac{\partial u(Q)}{\partial n} dS + \int_W W(P, Q)\sigma_i(Q) dV$$

Where the kernels $D, S$ and $W$ are the corresponding kernels for the stress computation. The stress in the rock-mass is calculated at all cell-nodes. Because of the different materials of the rock mass and the inclusion (geological inhomogeneities or rock bolts), the stress is different in both. The residual initial-stress increment is computed by:

$$\dot{\sigma}_i = \sigma_{\text{inclusion}} - \sigma_{\text{rock}} = (D_{\text{inclusion}} - D_{\text{rock}})\dot{e} = (D_{\text{inclusion}}D_{\text{rock}}^{-1} - I)\sigma_{\text{rock}}$$

Where $D_{\text{inclusion}}$ and $D_{\text{rock}}$ are the constitutive matrices of the inclusion and of the rock mass, respectively; $I$ is the identity matrix. In the next iteration step the residual initial-stress increment $\dot{\sigma}_i$ is applied to the system as loading in Eq. 1. With this new loading the system is solved again and a new stress-field in the domain can be computed and from this the new residual initial-stress
follows. The iteration proceeds until the residual stress in the incremental step vanish. The sum of the residual stress increments of all iteration-steps gives the complete initial-stress which represents the effect of the inclusions.

To consider elasto-plastic material behavior of the rock mass, a nonlinear algorithm is needed anyway, as discussed for example in [5] and [8]. The inclusion algorithm can be combined with the plasticity algorithm (Fig. 2) and thus produces not much additional expense in comparison to a system without inclusions.

Fig. 2: Flow chart diagram for the iterative solution algorithm
Integral Formulation for Inhomogeneities and Rock Bolts

To calculate the volume-term of the integral equations (Eq. 1, Eq. 2) different methods are possible. Here a cell-integration technique is used: the volume $V$ is subdivided into a number of integration cells $V_i$ and over these cells numerical integration is used.

For geological inhomogeneities general volume-cells are used:

$$\int_{V} E(P, \bar{Q}) \sigma_i(\bar{Q}) dV = \sum_{i=1}^{C} \int_{V_i} E(P, \bar{Q}) \sigma_i(\bar{Q}) dV$$  \hspace{1cm} (4)

To model linear inclusions like rock bolts, line-cells with a predefined cross-section-area are used instead of volume-cells (see also [7], [4], [9]). It is assumed that the rock bolts are in continuous contact with the ground (fully grouted) and that they are only able to carry axial loading. The three dimensional integration over the volume of the bolt is reduced to a one dimensional integration along its length by integrating over the cross-section-area of the bolt $A_{\text{bolts}}$ analytically. The variation of stress across the cross-section is assumed to be constant. The domain integral (Eq. 4) may be replaced by:

$$\int_{V} E(P, \bar{Q}) \sigma_i(\bar{Q}) dV = \sum_{i=1}^{C} \left( \int_{A_{\text{bolts}}} \bar{E}(P, \bar{Q}) dA \right) \bar{\sigma}_i(\bar{Q}) dL = \sum_{i=1}^{C} \left( \int_{r} \bar{E}(P, \bar{Q}) \bar{\sigma}_i(\bar{Q}) dL \right)$$  \hspace{1cm} (5)

Where $\bar{\sigma}_i$ is the initial-stress in axial direction of the bolt (a scalar value) and $\bar{E}$ is a modified fundamental solution for strain which is computed as follows: First the terms of the general fundamental solution $E$ are computed by using the local coordinates of the bolt $x, y, z$ (see Fig. 3).

![Local coordinate system of the bolt in 3 dimensions](image)

The fundamental solution for strain is now given in local coordinates $E$ and the analytical integration over the cross-section-area can easily be done. However, to insert the kernel in the global expressed equations (Eq. 1), the entire terms have to be transformed to the global system, with the geometrical transformation matrix $T_e$. The modified fundamental solution $\bar{E}$ from Eq. 5 is computed by:
\[ \mathbf{E} = \mathbf{T}_s \int_{A_{\infty}} \mathbf{E}(P, \mathbf{Q}) dA \] (6)

Eq. 5 is evaluated numerically over all line-cells by using Gauss Quadrature.

\[ \sum_{v=1}^{N_{\text{cc}}} \left( \mathbf{E}(P, \mathbf{Q}) \mathbf{\bar{E}}_v(\mathbf{Q}) dL \right) = \sum_{v=1}^{N_{\text{cc}}} \mathbf{E}(P, \xi_v) \mathbf{\bar{E}}_v(\xi_v) J W_v \] (7)

The integral (Eq. 5) becomes singular if the field point \( \mathbf{Q} \) coincide with the source point \( P \) on the boundary or with \( \mathbf{P} \) in the domain. In this case the numerical integration using the Gauss Quadrature does not give accurate results. Therefore the integral (Eq. 5) is carried out analytically over the hole bolt volume.

The volume term of the stress equation (Eq. 2) is calculated analog to volume term of Eq. 1: The modified fundamental solution \( \mathbf{W} \) is computed by:

\[ \mathbf{\bar{W}} = \mathbf{T}_s \int_{V_{\infty}} \mathbf{\bar{W}}(P, \mathbf{Q}) dV \] (8)

Where \( \mathbf{W} \) is the fundamental solution calculated in the local coordinate system, and \( \mathbf{T}_s \) is the strain transformation matrix. The numerical integration over the cell length can be done like for Eq. 7 and in the singular case the integral is carried out analytically too.

**Numerical Applications**

The described method is applied in the boundary element program BEFE++ and verification examples for two- and three-dimensional problems are carried out. Here plane strain example is presented.

The example shown here is a circular hole (radius = 10m) in an infinite domain which is reinforced by 24 rock bolts. The boundary of the hole is subjected to an internal tension of 15 MN/m². Figure 3 shows the mesh used for the analysis. It consists of 40 linear boundary elements.
for the hole and two quadratic line-cells for each rock bolt. A finer mesh with 120 linear boundary elements and 4 cells per rock bolt was analyzed, too. For the analysis the following material properties were assumed: \( E_{\text{rock}} = 5000\, \text{MN/m}^2 \), \( v = 0.3 \), \( E_{\text{bolt}} = 400000\, \text{MN/m}^2 \). The Bolts have a cross-section of \( A_{\text{bolt}} = 0.007854\, \text{m}^2 \) and a length of \( l = 10\, \text{m} \). In Fig. 4 the results are compared with a finite element analysis using a very fine mesh. The displacement along the axis of the rock bolt is shown. It can be seen that the solutions compare well, even the coarse mesh gives very accurate results.

Further examples that tread rock bolts in combination with plastic material behavior will be presented at the conference.

Acknowledgements

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Literature References


Modelling of Changing Geometries for the Excavation Process in Tunnelling with the BEM

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Keywords: Tunnelling, Sequential excavation, Boundary Element Method, Corners and Edges, Multiple Regions, Discontinuous Elements

Abstract. The modelling of the sequential excavation process in tunnelling is a complicated task with the Boundary Element Method (BEM), because it depends heavily on changing geometries and changing boundary conditions. One possibility to model the sequential excavation is to use the Multiple Region BEM (MRBEM). The problem of corners and edges has a big influence to the results of this modelling strategy. This will be pointed out and a solution with discontinuous elements will be shown. Another possibility to solve the excavation problem in tunnelling is to use a single region boundary element calculation in combination with internal result computation. Stresses in the interior domain are calculated to obtain the loading for subsequent analysis steps. With this method the problem of corners and edges is largely avoided, but the crucial point is the correct evaluation of the loading for the next excavation step. These methods are shown for tunnelling examples in 2D.

Introduction

The New Austrian Tunnelling Method (NATM) is characterised by a complicated sequence of excavation and installation of support systems. In this context the modelling of the sequential tunnel excavation in an infinite/semi-infinite domain will be discussed. The numerical method which will be applied is the Boundary Element Method (BEM). Fig. 1 shows a typical excavation with the NATM. As a possible example the tunnel cross section is divided into two parts, top heading and bench. The volumes of material are excavated at different time and location. One way to model the sequential tunnel excavation with the BEM is to use multiple regions \cite{1,2,3}. The technique of multiple regions is shown to be applied successfully for the modelling of the sequential excavation process \cite{4,5}.

Another possibility to simulate the sequential excavation is the use of a single region boundary element calculation for each step of excavation \cite{6}. To obtain the loading of each subsequent excavation step stresses in the interior domain are calculated. These two solution strategies are explained in the following sections and an example in 2D is shown.

Multiple Region Boundary Element Method (MRBEM)

As in Fig. 1 shown the model consists of several finite regions which represent the volumes of excavation. These regions are embedded in an infinite region, which represents the infinite extent of the continuum. The excavation process is simulated by deactivating of regions from the model system. The geometry and boundary conditions are changing progressively from excavation step to step. For all regions stiffness matrices are calculated and assembled to a global system of equations, which is solved for the interface displacements.
Fig. 1 – Example for a staged excavation process in 3D (only half of the mesh shown)

The stiffness matrix of a region is determined by using the boundary integral equation shown in Eq. 1

\[
e(P)u(P) = \frac{1}{2} \int_{S} U(P, Q)t(Q)dS - \frac{1}{2} \int_{S} T(P, Q)u(Q)dS
\]

(1)

Where \( u \) and \( t \) are the displacements and tractions on the boundary \( S \), respectively, \( c \) is related to the boundary geometry, \( U \) and \( T \) are the fundamental solutions for displacements and tractions, respectively. In discretized form this equation is equivalent to

\[
\sum_{e=1}^{N} \sum_{i=1}^{n} \Delta T_{ui}^e \cdot u_{ei}^e = \sum_{e=1}^{N} \sum_{i=1}^{n} \Delta U_{ui}^e \cdot t_{ei}^e
\]

(2)

\( \Delta U_{ui}^e \) and \( \Delta T_{ui}^e \) are the integrated kernel shape function products of the element \( e \) for a collocation point \( (P_i) \) at point \( i \). \( u_{ei}^e \) and \( t_{ei}^e \) are vectors containing all displacements and tractions for the element. Collocating at all boundary nodes results in an equation system. In general the boundary conditions are mixed and the known and unknown values are separated to the right and left side of the equation and this gives the following:

\[
B^N \begin{bmatrix} t_{n1}^N \n u_{n1}^N \end{bmatrix} = f_{n1}^N; \quad B^N \begin{bmatrix} t_{nN}^N \n u_{nN}^N \end{bmatrix} = f_{nN}^N \quad \text{for} \quad n = 1, 2, \ldots, N_{dof} \nonumber
\]

(3)

Matrix \( B^N \) is the assembled left hand side for region \( N \), containing the integrated kernel shape function products related to the unknown boundary conditions. \( t_{n1}^N \) is a part of the unknown vector representing the tractions at the interface \( c \) and \( u_{n1}^N \) are the displacements due to the loading (excavation tractions) at the Neumann boundary. The second equation of Eq. 3 obtains the same left hand side as the first but the loading are unit displacements at the interface degrees of freedom. Applying unit displacements at all degrees of freedom at the interface results in the stiffness matrix \( K^N \) of region \( N \). The final solution \( t_{n1}^N \), the tractions at the interface nodes, and \( u_{n1}^N \), the displacements at the Neumann boundary, can be expressed in terms of \( u_{n1}^N \), the displacements at interface nodes:
Using the conditions of equilibrium and compatibility the stiffness matrices $K^N$ and the vectors $t^N$ of all regions can be assembled into a global system of equation, which can be solved for the unknown interface displacements $\mathbf{u}$ as shown in Eq. 5.

$$ t^N \mathbf{u} + K^N \mathbf{u} = 0 $$

$K$ is the assembled stiffness matrix related to the interface nodes only, $t^N$ is the assembled vector of tractions at the interface due to the given boundary conditions at the Neumann boundary. The remaining unknowns, displacements at the Neumann boundary and tractions at the interface can be evaluated returning from system to region level, by using Eq. 4.

The excavation is modelled by deactivating regions, this means removing stiffness matrices from the equation system [5]. This implies that the geometry of the system and the boundary conditions of some regions are changing, for instance from interface conditions to Neumann conditions. For these regions the stiffness matrices have to be updated each calculation step before the assembly process starts.

A problem, which arises is the one of corners and edges [2,5,6,7]. In the following an example (shown in Fig. 2) is shown where this problem is pointed out. This example concerns an excavation of infinite extend out of plane, which is not a real tunnel excavation, but the problem of corners can be shown clearly. The example shown in Fig. 2 consists of 10 regions which are sequentially removed from the system. At the beginning most of the boundary elements belong to interface boundaries. The regions are of rectangular shape and exhibit geometrical corners. If adjacent elements of these corners belong to the interface, Dirichlet boundary conditions are applied for the evaluation of the stiffness matrices. The resultant tractions at these corners are multi-valued in reality, this means different on both sides of the corner node. If continuous elements are used the boundary integral equation is able to deliver only one distinct vector of tractions at the corner node, therefore the tractions are unique. As the resultant tractions at the interface belong to the loading of the subsequent load step erroneous results will be achieved.

In the diagram of Fig. 3 vertical displacements for load case 1 to 5 at the line $\overline{AB}$ (shown in Fig.2) are shown. The dashed lines show the results for calculations with continuous quadratic elements. It can be seen that the results, except for load case 1, are erroneous and these errors accumulate from step to step. With the use of continuous elements the problem of corners and edges is neglected, especially at nodes of the interface where the traction is singular. Doing the same
calculations with discontinuous quadratic elements gives excellent results [2]. These results are validated with single region boundary element calculations as well as with finite element calculations.

Single region boundary element method

The method discussed before requires a predefinition of the complete geometry of the tunnel excavation problem. From beginning the calculation has to deal with all the regions, interfaces, etc. which in subsequent load cases will be part of the excavation process. This means, that the size of the equation system is determined by the number of all degrees of freedom at the entire geometry and it nearly remains the same for every load step of calculation. Now only a single boundary element region is used to represent the actual excavation surface [6]. Stresses are calculated at points inside the domain, provided that the boundary conditions are known. For the respective load step excavation loads are determined with this approach.

The algorithm is explained for the same example as in the previous section. Fig. 4 shows the excavation sequence for 10 load cases. The excavated parts of each step, where the loading is applied, are indicated by hatched areas. With this method it is necessary to discretize the excavated tunnel surface only. A single boundary element region is sufficient for the discretization of the respective load step which can be seen in Fig. 4.

The crucial part is the determination of excavation tractions for the current load step. This is explained next for load case LC4. In Fig. 5 the excavation steps from LC1 until LC4 are shown. The region to be excavated for LC4 (indicated by hatched areas) is shown in the sketches for each load step.
of the previous load cases. The resultant displacement field and stress field for LC4 is the accumulation of incremental results of all previous load cases. This implies that the previous load cases have to be considered for the determination of the loading for LC4.

![Fig. 5 – Excavation of LC4](image)

The excavation tractions are calculated with internal stress evaluation. The stress at an internal point is calculated by the following integral equation:

$$\sigma(P) = \int_R R(P,Q) t(Q) dS - \int_S S(P,Q) u(Q) dS$$

(6)

where $\sigma(P)$ is the stress at an internal Point $P$, $R$ and $S$ are the fundamental solutions for stress, $t$ and $u$ are the boundary traction and displacement values, respectively. The stress is evaluated at the same points for load case 1 to 3. For load case 1 and 2 all points are internal points as shown in Fig. 5. For these load cases there is no difficulty in the evaluation of the stress. For LC 3 some of the points of the excavated volume are boundary points. Because of the sharp corners at point A and B (shown in Fig. 6) the stress is infinite and a calculation directly at these points is not possible. To overcome this problem the stress is evaluated inside the adjacent element very near to the boundary, at an intrinsic coordinate of value $\zeta = -0.90$. The stress is extrapolated to the boundary according to the parabolic shape function of the element. This is shown in Fig. 6.

![Fig. 6 – Treatment of singular point](image)

The traction vector $t$ is calculated by multiplying the stress $\sigma$ with the outward normal vector $n$ to the excavation surface. The resulting traction at the excavation surface for LC4 is the sum of tractions obtained by internal stress calculation for LC1 to LC 3 completed by the tractions due to the virgin stress field. Once the loading tractions are found the solution for the current load step is evaluated by a single region boundary element calculation using Eq. 3. The results of the vertical displacements along the crown of the tunnel are shown in Fig. 7. These results are compared with the reference solution. As can be seen in Fig. 7, it seems that some loading is lost from one load case to the other. The reason for this is the inaccurate evaluation of the tractions near to the singular points A and B, shown in Fig. 6.
Conclusion

The main advantage of the single region approach against the conventional method of domain decomposition (MRBEM) is that the effort for the solution is much less. The simulation starts with a very small BEM mesh and this will extend from excavation step to step. Additional effort has to be spent for the correct evaluation of the stress distribution near to the boundary, where singular values of stress appear. Currently the implementation for 3D into the computer code BEFE++ is an ongoing task. A reasonable comparison of the efficiency to the conventional method of MRBEM only makes sense for a 3D example.

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References

Multi-grain Orthotropic Material Analysis by BEM and its Application

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Key Words : Multi-Grain, Orthotropic, Effective elastic modulus, Effective Poisson’s ratio, BEM

Abstract

Most of the MEMS parts are made of multi-grain silicon wafer, which is the orthotropic material and its material direction is arbitrary. The stress analysis for the multi-grain is important factor in order to apply the MEMS parts to industrial applications. The finite element method (FEM) is commonly used by a stress analysis method but the boundary element method (BEM) is known as the result of the BEM is more accurate than that of the FEM since the fundamental solution are used. In this study, we derived the boundary integration equation for the orthotropic material by applying fundamental solutions with complex variables. The multi-region analysis procedure for the BEM is developed in order to apply the analysis of the multi-grain orthotropic material. The effective elastic modulus and its Poisson’s ratio are calculated by the BEM. The results of the present method are compared with those of the finite element method in order to verify the present procedure.

1. Introduction

Reliability and safety of MEMS parts are an important factor in its application. The material properties are determined by the computational mechanics in order to reduce the testing efforts. Toonder, et al.<sup>(1)</sup> extracted the effective elastic modulus and Poisson’s ratio for an orthotropic material which has 30 grains and has different orientation in each grain with the FEM. Mullen, et al.<sup>(2)</sup> simulated the effective elastic modulus for a multi-grain thin layer by Monte-Carlo method, Yin<sup>(3)</sup> also extracted the elastic modulus by the same method. Chu<sup>(4)</sup> extracted the effective elastic modulus by applying Johnson-Mehl model. Choi<sup>(5)</sup> calculated the effective elastic modulus and Poisson’s ratio by applying the FEM to an simulated multi-grain structure.

For the numerical simulation, the FEM is popular. But the BEM has advantage in numerical accuracy and approximate boundary only and was applied to analysis crack growth analysis by dual boundary element method.<sup>(6)</sup> In this study, we developed the program to simulate the effective elastic modulus and Poisson’s ratio for the multi-grain orthotropic material. For the simulation, we re-derived the fundamental solution for the orthotropic material<sup>(7,8)</sup> and applied the discontinuous element for the corner problem. We applied the developed program to simulate the multi-grain orthotropic material and determined the effective elastic modulus and Poisson’s ratio, which was compared with the results by the FEM.
2. Orthotropic material relation and BE formulation

By Gibson’s composite material principle, the stress – strain relation in orthotropic material is given in Eq. (1). 

\[
\begin{bmatrix}
\varepsilon_{11} & \frac{1}{E_1} & -\frac{v_{12}}{E_1} & -\frac{v_{13}}{E_1} & 0 & 0 & 0 \\
\varepsilon_{22} & \frac{1}{E_2} & \frac{v_{12}}{E_1} & \frac{v_{13}}{E_2} & 0 & 0 & 0 \\
\varepsilon_{33} & \frac{1}{E_3} & \frac{v_{13}}{E_1} & \frac{v_{13}}{E_3} & 0 & 0 & 0 \\
\varepsilon_{12} & 0 & 0 & 0 & \frac{1}{G_{12}} & 0 & 0 \\
\varepsilon_{13} & 0 & 0 & 0 & 0 & \frac{1}{G_{13}} & 0 \\
\varepsilon_{23} & 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{23}} \\
\end{bmatrix}
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{12} \\
\sigma_{13} \\
\sigma_{23} \\
\end{bmatrix}
\]

(1)

Since two-dimensional orthotropic material is \( \tau_{13} = \tau_{31} = 0 \), Eq. (1) can be reduced as following relation. 

\[
\begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
2\varepsilon_{12} \\
\end{bmatrix}
= 
\begin{bmatrix}
S_{11} & S_{12} & 0 \\
S_{12} & S_{22} & 0 \\
0 & 0 & S_{66} \\
\end{bmatrix}
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{12} \\
\end{bmatrix}
\]

(2)

where \( \sigma_{ij} \) and \( \varepsilon_{ij} \) \( (i,j=1,2) \) are stress and strains, respectively, and the coefficients \( S_{ij} \) are the elastic compliances of the material. These compliances can be written in terms of engineering constants as

\[
S_{11} = \frac{1}{E_1}, S_{22} = \frac{1}{E_2}, \quad S_{66} = \frac{1}{G_{12}}
\]

(3)

where \( E_1 \) is Young’s modulus in \( x_1 \) direction, \( G_{12} \) is shear modulus in \( x_1-x_2 \) plane and \( v_{ij} \) is Poisson’s ratio.

The characteristic equation is given as follows:

\[
S_{11} + 2S_{12} + (2S_{12} + S_{66})\mu^2 - 2S_{12} + S_{66} = 0
\]

(4)

The solution of the characteristic equation can be denoted by:

\[
\mu_i = \alpha_i + i\beta_i, \quad \mu_i = \bar{\mu}_i, \quad \mu_i = \bar{\mu}_i.
\]

(5)

Here, \( \alpha, \beta \) are real and \( i \) is imaginary, \( \bar{\mu} \) is the conjugate of the complex variable of \( \mu \).

Since the directions of the grain crystallization of the MEMS parts are determined randomly, the strains can be expressed in terms of stresses in non-principal coordinates of the laminates as:

\[
\begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
2\varepsilon_{12} \\
\end{bmatrix}
= 
\begin{bmatrix}
S_{11} & S_{12} & 0 \\
S_{12} & S_{22} & 0 \\
0 & 0 & S_{66} \\
\end{bmatrix}
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{12} \\
\end{bmatrix}
\]

(6)

where the \( \bar{\sigma}_{ij} \) are the components of the transformed lamina compliance matrix which are defined by:
The invariants \((V_1, V_2, V_3, V_4)\) are given by:

\[
V_1 = \frac{1}{8}(3S_1 + 3S_2 + 2S_3 + S_4)
\]

\[
V_2 = \frac{1}{2}(S_1 - S_2)
\]

\[
V_3 = \frac{1}{8}(S_1 + S_2 - 2S_3 - S_4)
\]

\[
V_4 = \frac{1}{8}(S_1 + S_2 + 6S_3 - S_4).
\]

The boundary integral equation for anisotropic materials may be written as:

\[
C_{ij}(z_i^1, u_i^1(z_i^1)) + \int \frac{U_j}{z_j} (z_i^2, u_j^2(z_i^2)) dT(z_i) = \int \frac{U_j}{z_j} (z_i^2, u_j^2(z_i^2)) dT(z_i).
\]

\[
C_{ij} = \text{determined by the local boundary shape.}
\]

\[
z_i^j = x_i^j + \mu_i z_i^j (\zeta - \zeta^0),
\]

\[
x_i^j = \text{the derivative of } x_i \text{ and } \zeta^0 \text{ is the position of } z_i^0. \text{ The displacement and traction fundamental solution are given as follows:}
\]

\[
U_i = 2 \Re [\epsilon_j A_i \ln(z_i - z_j^0) + r_{ij}A_i \ln(z_i - z_j^0)]
\]

\[
T_i = 2n_j \Re [\epsilon_j A_i (z_i - z_j^0) + \mu_j A_i (z_i - z_j^0)]
\]

\[
T_i = -2n_j \Re [\mu_j A_i (z_i - z_j^0) + \epsilon_j A_i (z_i - z_j^0)]
\]

\[
T_i = -2n_j \Re [\epsilon_j A_i (z_i - z_j^0) + \mu_j A_i (z_i - z_j^0)].
\]

Here \(n_j\) is the normal unit vector in \(x_1-x_2\) coordinate, \(r_{ij}\) is given as:

\[
\begin{align*}
\epsilon_j &= S_1 \mu_j / \mu_i + S_2 / \mu_i - S_3 / \mu_i, \\
\mu_j &= S_1 / \mu_i + S_2 / \mu_i - S_3 / \mu_i.
\end{align*}
\]

And \(A_{ij}\) are complex variables which are determined by the following equation:

\[
\begin{bmatrix}
1 & -1 & 1 & -1 \\
-\mu_i & -S_1 & -S_2 & -S_3 \\
\epsilon_j & \mu_j & \mu_j & \mu_j \\
\mu_j & \mu_j & \mu_j & \mu_j
\end{bmatrix}
\begin{bmatrix}
A_{ij} \\
\delta_{ij}/2\pi \\
0 \\
0
\end{bmatrix}
= \begin{bmatrix}
\delta_{ij}/2\pi \\
0 \\
0 \\
0
\end{bmatrix},
\]

where \(\delta_{ij} = 0, 1, 2\) represents the Dirac delta.
3. Multi-Grain Boundary Element

For the multi-grain boundary element, the matrix equation for the region- in with interface surface denoted by I can be written as follows:

\[ H^I U^I = G^I T^I. \]  

(13)

The relation between the traction \( T^I \) and the displacement \( U^I \) in region- can be expressed as:

\[ K^I U^I = T^I \rightarrow \left\{ \begin{array}{l} K^I U^I_1 = T^I_1 \\ K^I U^I_2 = T^I_2 \end{array} \right\}. \]  

(14)

On the interface surface, the displacement must be the same between two regions and the traction can be given by adding individual value:

\[ U^I_1 = U^I_2 = U_I, \quad T^I_1 = T^I_2 + T^I_2. \]  

(15)

We can write the relation between displacement and traction on the interface surface by combining eq.14 and eq.15 as following:

\[ (K^I + K^I) U_I = T_I. \]  

(16)

4. Discontinuous Boundary Element

The edge and corner problem in the boundary element method occurs at the point, where the normal vector is defined incorrectly. Aliabadi, et al. introduced the discontinuous element in order to remove the singularity in the crack tip. The nodes in this element are shifted on each end nodal point as:

\[ \zeta = \frac{-2}{\pi}, \quad \zeta = 0, \quad \zeta = \frac{2}{\pi} \]  

(17)

The coordinates can be written as:

\[ x_i(\zeta) = N^I_i(\zeta)x_i^*, \quad n = 1, 2, 3, \]  

(18)

where \( N^I_i(\zeta) \) is the discontinuous shape function and \( x_i^* \) is nodal coordinate which is shifted by eq.17. Since this element does not have common nodal points and the shape function in eq.18 is smooth enough within the element, the edge and corner problem does not occur.

For this element, the integral of eq.9 can be represented by:

\[ \int_{\Gamma} [T^I_{ij}(\xi, \eta) w_{ij}(\xi, \eta)] d\Gamma(\xi, \eta) = u^I_j \int_{\Gamma} [N^I_i(\zeta), n(\zeta)] w^I_i(\zeta) n(\zeta) d\zeta = F^I_{ij} u^I_j, \]  

(19)

where \( u^I_j \) denote the nodal displacement component and \( J \) is the Jacobian of the coordinate transformation and is given by \( J = L/2 \), where \( L \) represents the element length. \( F^I_{ij} \) can be given by
\[ f_{ij}(\zeta, \zeta') = \text{Re} \left\{ \frac{1}{|\lambda_n - \lambda_m|} \left[ N_{ij}(\zeta, \zeta') \nabla_{\zeta} \phi_{ij} \right] \right\} \]

Here, \( \phi_{ij} = \begin{bmatrix} \mu_1 & \mu_2 \\ -1 & -1 \end{bmatrix} \).

As the same method, the traction can be given by:

\[ \int_{\Gamma} u_j(\xi) \partial_{ij} \phi_{ij} \partial_{ij} f_{ij}(\xi, \zeta') d\xi = \int_{\Gamma} u_j(\xi) \partial_{ij} \phi_{ij} \partial_{ij} f_{ij}(\xi, \zeta') d\xi = \int_{\Gamma} u_j(\xi) \partial_{ij} \phi_{ij} \partial_{ij} f_{ij}(\xi, \zeta') d\xi \]

Here, \( \lambda_{g} \) is given as follows:

\[ \lambda_{g} = 2 \text{Re} \left[ \phi_{ij} \partial_{ij} \ln(\zeta - \zeta') + \phi_{ij} \partial_{ij} \ln(\zeta' - \zeta) \right] \]

5. Application of multi-grain BEM

5.1 Stress analysis of multi-grain structure

Geometry shapes for the numerical analysis are shown in Fig. 1, which has 8 regions. For the verification of the present method, we apply the FEM method to analysis the same model as shown in Fig. 4(a). We used Altair HyperMesh\(^{11}\) for FEM modeling and ABAQUS V6.6/standard\(^{12}\) for the analysis. For the material properties are given in Table 1 and the element and node numbers are shown in Table 2 for each region, respectively. For the analysis, boundary conditions are given in Fig. 1(b) and the applied force is 10N/mm\(^2\).

Fig. 1 FEM mesh (a) and BEM element (b) and boundary condition for stress analysis

The tractions of two surface A-A’ and B-B’ are compared in Fig. 5 and Fig. 6. These Figures show that the present method is good agreement with the FEM results. Table 3 shows the tractions along the line A-A’ and line B-B’. In the Table 3, the dark row shows that the traction has some gap. At the sharp corner along the line A-A’, the normal direction does not have a unique value. Therefore the traction has some gap on the sharp corner.
Table 1 Material properties of each region

<table>
<thead>
<tr>
<th>Region No.</th>
<th>E1 (MPa)</th>
<th>E2 (MPa)</th>
<th>ν</th>
<th>G (MPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>195973.0</td>
<td>204973.0</td>
<td>0.2956</td>
<td>78449.89</td>
</tr>
<tr>
<td>2</td>
<td>197344.3</td>
<td>205489.8</td>
<td>0.2998</td>
<td>78200.41</td>
</tr>
<tr>
<td>3</td>
<td>209804.3</td>
<td>197017.6</td>
<td>0.3001</td>
<td>77247.72</td>
</tr>
<tr>
<td>4</td>
<td>199086.3</td>
<td>202683.5</td>
<td>0.2962</td>
<td>78641.28</td>
</tr>
<tr>
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<td>197477.1</td>
<td>206849.3</td>
<td>0.2880</td>
<td>77925.13</td>
</tr>
<tr>
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<td>209231.4</td>
<td>197077.1</td>
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</tr>
<tr>
<td>7</td>
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<tr>
<td>8</td>
<td>210000.0</td>
<td>197000.0</td>
<td>0.3000</td>
<td>77200.00</td>
</tr>
</tbody>
</table>

Table 2 Element and node number of each region

<table>
<thead>
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<th>BE model</th>
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</table>

5.2 Extract Effective Elastic Modulus and Poisson’s Ratio and Discussion

Effective elastic modulus and Poisson’s ratio was calculated by Choi(5), who used the FEM with the model as shown in Fig. 4.
Effective Poisson’s ratio and elastic modulus are given by the relation between stress and strain based on the Hooke’s Law. These relations are expressed by a plane stress and plane strain condition respectively:

\[
\nu_{\text{eff}}^x = \frac{\overline{\sigma}_{1x}}{\overline{\sigma}_{2x} - \overline{\sigma}_{1x}} \cdot E_{\text{eff}}^x, \quad \nu_{\text{eff}}^y = \frac{\overline{\sigma}_{2y} - \overline{\sigma}_{1y}}{\overline{\sigma}_{2x} - \overline{\sigma}_{1x}} \cdot E_{\text{eff}}^x
\]

(23)

\[
\nu_{\text{eff}}^x = \frac{\overline{\sigma}_{1x}}{\overline{\sigma}_{1x} + \overline{\sigma}_{2x}} \cdot E_{\text{eff}}^x = \frac{\overline{\sigma}_{2y} - \overline{\sigma}_{1y}}{\overline{\sigma}_{1x} + \overline{\sigma}_{2x}} \cdot E_{\text{eff}}^x \quad \text{(24)}
\]

here, \(\nu_{\text{eff}}^x\), \(E_{\text{eff}}^x\) and \(\nu_{\text{eff}}^y\), \(E_{\text{eff}}^y\) are the effective Poisson’s ratio and elastic modulus in plane stress and plane strain conditions, respectively.

\(\overline{\sigma}_{1x}\) and \(\overline{\sigma}_{2y}\) is the nominal stresses in x and y direction, respectively, and \(\overline{\varepsilon}_{1x}\) is nominal strain in x-direction. The nominal stress and strain in unit thickness are give by:

\[
\overline{\sigma}_{1x} = \frac{F_x}{L_x}, \quad \overline{\sigma}_{2x} = \frac{F_y}{L_y}, \quad \overline{\varepsilon}_{1x} = \frac{\delta}{L_x}
\]

(25)

here, \(F_x\) and \(F_y\) are the reactions in x and y-direction, respectively, and \(\delta\) is the displacement on the applied force surface.

The geometries for the BEM and FEM are shown in Fig. 5, which were generated by the random grain generation process and we assume that the problem is a plane stress condition.

![Fig.5 BEM element(a) and FEM element(b)](image)

The material properties in each region are given in Table 4, in which region number 7 has material properties with 0 rotational angle but the other region has the material properties by changing the rotational angle randomly.
We used Altair HyperMesh for modeling the geometry and AQUAS V6.6/ Standard for analysis. We calculated the nominal reaction in eq.23 on the boundary surface by the FEM and BEM and the effective elastic modulus and Poisson’s rations as show in Table 5. Table 5 shows that the results by the FEM and BEM are almost the same. This means that the present method is very accurate.

6. Conclusion

The tractions analysis of the multi-grain structure shows that the present method is good agreement with the FEM results. The results of the effective elastic modulus and Poisson’s ratio for the multi-grain orthotropic material by the BEM show almost the same as those by the FEM. From these results, the present method is accurate to analysis the multi-grain orthotropic material that we can apply the present method to MEMS structures with multi-grains.

Acknowledgements

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Potential problem for Functionally Graded Materials:
two-Dimensional study

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Keywords: Functionally graded materials, stochastic Hashin-Shtrikman variational principle, Potential problem, Green’s function, Boundary element method.

Abstract. Functionally graded materials (FGMs) are two-phase composites with continuously changing microstructure adapted to performance requirements. Compared to a layered system, an FGM avoids the discontinuity in material properties across the interface. FGMs have already been employed in many important areas, e.g., thermal barrier coatings for aerospace applications and implants for bio-medical applications. As analytical solutions for non-homogeneous materials are rare, there is a need for reliable and efficient numerical methods for solving problems in FGMs. Traditionally, the overall behavior of FGMs has been determined using local averaging techniques or a given smooth variation of material properties. Although these models are computationally efficient, their validity and accuracy remain questionable as their link with the underlying microstructure is not clear. In this paper, we propose a modeling strategy for the 2-D potential problem of FGMs systematically based on a realistic random microstructural model by the BEM. The overall response of FGMs is addressed in the framework of stochastic Hashin-Shtrikman variational principles.

Introduction

Due to the continuously and functionally varying volume composition of constituent particles, functionally graded materials (FGMs) provide superior thermo-mechanical performance under given loading circumstances compared with classical laminated composite materials. In order to achieve the best performance, computing the temperature and overall stress/strain distributions in FGMs needs an appropriate estimate for properties of the graded layer, such as the thermal conductivity, coefficient of thermal expansion, the Young modulus, Poisson’s ratio and so on. A detailed description of the geometry of actual graded composite microstructure is usually not available, except for information on volume fraction distribution and approximate shape of the dispersed phase or phases. Therefore, the evaluation of thermo-mechanical response and local stresses in graded materials must rely on analysis of micromechanical models with idealized geometries. There are significant differences between the analytical models of macroscopically homogeneous composites and FGMs. It is well known that the response of macroscopically homogeneous systems can be described in terms of certain thermo-elastic moduli that are evaluated for a selected representative volume element, subjected to uniform overall thermo-mechanical fields. However, such representative volumes are not easily defined for systems with variable phase volume fractions, subjected to non-uniform overall fields.

Heat conduction problems can be efficiently solved with the boundary element method. In this work a two-dimensional FGM plate subjected to steady state heat conduction is solved. The model used here is systematically derived from a fully penetrable sphere microstructural model introduced by [1]. The statistics of local fields then follow from re-formulation of the Hashin-Shtrikman (H-S) variational principles introduced, e.g., in [2,3] and summarized in the current context together with
the Galerkin scheme allowing to treat general bodies. The application of the Boundary Element Method (BEM) is covered in the spirit of [4], [5] and [6].

**Microstructural model**

As already indicated in the introductory part, the morphological description adopted in this work is the one-dimensional case of a microstructural model studied by Quintanilla and Torquato [1]. A particular realization can be depicted as a collection on N fully penetrable spheres (fibres in 2D) of radius R randomly distributed within a plate (matrix material) of size $H_1 \times H_2$, whose particle density obeys any specified variation in volume fraction, see Figure 1. The position of the i-th sphere is specified by the $x$ and $y$ coordinates of its reference point which in our case coincides with the centre of the sphere. As individual spheres are allowed to freely overlap each other, the family of microstructures attainable by the model is very versatile.

**Hashin-Shtrikman variational principle**

**Problem statement.** Laplace’s equation of heat transfer $\nabla^2 u = 0$, is considered in a 2-D rectangular domain depicted in Figure 1 with a basic example of discretization of the boundary into elements and nodes.

![Figure 1: Two-dimensional potential problem associated with realization $\alpha$.](image)

**Hashin-Shtrikman decomposition** Following the seminal ideas of [7,8], the solution of the stochastic problem is sought as a superposition of two auxiliary problems, each characterized by constant coefficient of thermal conductivity $k_0$.

By introducing a comparison medium, with constant thermal conductivity coefficient and defining “polarization flux” $\tau$ through the relation

$$q(x; \alpha) = (k_0 \cdot \nabla u(x; \alpha) - \tau(x; \alpha)) \quad (1)$$

and by introducing it into the steady state condition we obtain:

$$\nabla \left( k_0 \cdot \nabla u(x; \alpha) - \tau(x; \alpha) \right) = 0 \quad (2)$$
Multiplying the above equation by the test function \( v(x) \) for the temperature, integrating it over the domain and using the Gauss divergence theorem, the following variational boundary value problem (BVP) is obtained (weak formulation of the Laplace’s equation):

\[
\int_{\Omega} \left( k^0 \cdot \nabla u(x;\alpha) \cdot \nabla v(x) - \tau(x;\alpha) \cdot \nabla v(x) \right) = 0
\] (3)

The unknown polarization flux is now a new variable to be determined from the critical point of the two-field Hashin-Shtrikman-Willis functional

\[
\arg\min_{\nu(x;\alpha) \in \mathcal{T}(\alpha)} U(\nu(x), \tau(x;\alpha); \alpha)
\] (4)

and, hence the functional for this problem yields

\[
U(\nu(x), \tau(x;\alpha); \alpha) = \int_{\Omega} \left( \frac{1}{2} k^0 \cdot \nabla u(x;\alpha) \cdot \nabla v(x) - \tau(x;\alpha) \cdot \nabla v(x) \right) d\Omega
\] (5)

where \( \tau \) denotes an admissible polarization flux from the realization-dependent set \( \mathcal{T}(\alpha) \). In order to determine the distribution minimizers for a given probability distribution \( p(\alpha) \), we introduce the average energy functional:

\[
\Pi(\nu(x); \alpha) = \int \Pi(\nu(x); \alpha) \cdot p(\alpha) d\alpha,
\] (6)

The minimization of the functional with respect to \( \nu \) can be performed using Green’s function technique. Therefore the decomposition of the temperature field \( u(x;\alpha) \) is introduced.

\[
u(x;\alpha) = u^0(x) + u^1(x;\alpha),
\] (7)

where \( u^0 \) is the solution of the reference problem and \( u^1 \) denotes the temperature field related to a structure loaded by test polarization flux \( \tau \). Determination of \( u^0 \) is a standard task, which can be solved by a suitable numerical technique. Furthermore \( u^1 \) can be solved by introducing the Green’s function of the polarization problem satisfying

\[
k^0(x) \nabla G^\alpha(x,y) + \delta(y-x) = 0,
\] (8)

When the body is loaded by polarization flux \( \tau \), we get

\[
\nabla (k^0 \cdot \nabla u(x;\alpha) - \tau(x;\alpha)) = 0,
\] (9)

Solving the above equation will result in the temperature field

\[
u^1(x;\alpha) = -\int \frac{\partial G^\alpha(x,y)}{\partial y} \delta(y;\alpha) dy - \int \Lambda^\alpha(x,y) \theta(y;\alpha) dy,
\]

\[
\nabla u^1(x;\alpha) = -\int \frac{\partial^2 G^\alpha(x,y)}{\partial x \partial y} \delta(y;\alpha) dy - \int \Gamma^\alpha(x,y) \theta(y;\alpha) dy.
\] (10)

By exploiting the optimality properties of the minimizing temperature field \( u(x;\alpha) \), after some steps described in e.g. [9,10], the Hashin-Shtrikman functional in Eq. (5) is restored solely in terms of the polarization flux

\[
\tau(x;\alpha) = \arg\min_{\nu(x;\alpha) \in \mathcal{T}(\alpha)} H(\tau(x;\alpha); \alpha)
\] (11)

where \( H(\tau(x;\alpha); \alpha) \) is the “condensed” energy functional and is defined as
\[
H \left( \theta(\mathbf{x}; \alpha) \alpha \right) = \min_{\nu(\mathbf{x}; \alpha)} \mathcal{U}(\nu(\mathbf{x}), \nu(\mathbf{x}; \alpha); \alpha) = \Pi^0 \left( \nu^0(\mathbf{x}) \right)
\]

\[
- \int d^2 y \Gamma(\mathbf{x}, y) \frac{1}{2} \theta^2(\mathbf{x}, \alpha) \left[ k_n(\mathbf{x}) - k_p \right]^{-1} \frac{\nu^0(\mathbf{x})}{dx}
\]

with \( \Pi^0 \) denoting the thermal potential energy of the reference structure. The stochastic problem then can be solved by repeating the previous equation in probabilistic framework. Taking the ensemble average of the Eq. (7) and Eq. (10) yields to

\[
\langle \psi \rangle(\mathbf{x}) = \psi^0(\mathbf{x}) - \int dy \left[ \Delta^0(\mathbf{x}, y) \langle \tau \rangle(\mathbf{y}) \right]
\]

where the expectation \( \langle \tau \rangle \) is a solution of the stochastic variational problem

\[
\arg \min \mathcal{H}(\nu(\mathbf{x}; \alpha); \alpha)p(\alpha)d\alpha.
\]

From the above equations, it is now possible to link the unknown temperature field and flux to the test polarization flux \( T \), which becomes the primary variable to be solved. Due to limited knowledge of detailed statistical characterization of phase distribution, the previous variational principle can only be solved approximately. The following form of polarization flux is assumed:

\[
\tau(\mathbf{x}; \alpha) = \sum_r \tau_r(\mathbf{x}) \chi_r(\mathbf{x}; \alpha),
\]

\[
\theta(\mathbf{x}; \alpha) = \sum_r \theta_r(\mathbf{x}) \chi_r(\mathbf{x}; \alpha).
\]

where \( \tau_r \) and \( \theta_r \) denote a realization-dependent polarization stresses related to the \( r \)-th phase and \( \chi_r \) is the characteristic function which specifies the material distribution in case of binary heterogeneous materials. Replacing the stochastic polarization flux into the "condensed" Hashin-Shtrikman functional Eq. (14) and after some manipulations detailed in e.g. [11,10], leads to the following variational principle

\[
\langle \tau \rangle(\mathbf{x}) = \arg \min \mathcal{H}(\nu(\mathbf{x}; \alpha); \alpha)p(\alpha)d\alpha.
\]

Discretization. The condition in Eq. (16) presents an infinite system to be fulfilled. Two steps need to be taken for converting the equation to a finite-dimensional system: (i) to represent the reference flux field and the Green’s function related quantities and (ii) discretization of the phase polarization flux. Next by using the standard Galerkin procedure the Eq. (16) is reduced to a finite-dimensional format. To that end, the following discretization of the phase polarization flux is introduced

\[
\tau_r(\mathbf{x}) \approx N_r(\mathbf{x}) d_r,
\]

\[
\theta_r(\mathbf{x}) \approx N_r(\mathbf{x}) d_r^p
\]

where \( N_r \) is the matrix of (possibly continuous) shape functions; \( d_r \) and \( d_r^p \) denote the degrees-of-freedom (DOFs) of trial and true polarization flux, the latter related to the discrete Green’s function. Introducing the approximation Eq. (17) into the variational statement Eq. (16) and using the arbitrariness of \( d_r^p \) leads to the system of linear equations

\[
K_r d_r + \sum_{s=1}^2 K_{rs} d_s = 0
\]

with the individual terms given by \( (r, s = 1, 2) \).
Finally, once the approximate values of phase polarization flux are available, the elementary statistics of the temperature field follows from the discretized form of Eq. (13)

\[ (u')_0(x) = (u')_0 - \sum_{j=0}^N \left( \int_D (u^j(x,y)N'(x,y))^2 \right) \delta x \delta y \]  

Note that additional information such as conditional statistics or higher-order moments can be extracted from the polarization fields in post-processing steps similar to Eq. (20); see [12], [13] and [14] for more details.

BEM approximation of reference problem

Starting point for the boundary element formulation is the weighted residual (or weak) statement of the differential equation. For Laplace’s equation \( \nabla^2 u = 0 \), is given by

\[ \int_D \nabla^2 u(x) \varphi(x) \, d\Omega = 0 \]  

with the Green’s function as the test function \( \varphi(x) \) and boundary condition

\[ u(x) = \varphi(x) \quad \text{on} \quad \Gamma_u, \]
\[ q(x) = \varphi(x) \quad \text{on} \quad \Gamma_q. \]  

Next step is to transform the differential operator to the boundary terms. This transformation is done by applying Green’s theorem twice to the weighted residual statement. Substituting for the Green’s function i.e. the fundamental solution \( u^* \) as the test function and integrating it twice by parts, taking the load point \( \xi \) to the boundary and accounting for the jump of the left-hand side integral as is described in [15], yields the boundary integral equation

\[ c(\xi)u^0(\xi) = \int_{\Gamma_u} \frac{\partial \varphi(x)}{\partial n} G^{0,\infty}(\xi, x) \, d\Gamma_u(x) + \int_{\Gamma_q} \frac{\partial \varphi(x)}{\partial n} G^{0,\infty}(\xi, x) \, d\Gamma_q(x) \]
\[ - \int_{\Gamma_u} \frac{\partial G^{0,\infty}(\xi, x)}{\partial n} u(x) \, d\Gamma_u(x) + \int_{\Gamma_q} \frac{\partial G^{0,\infty}(\xi, x)}{\partial n} u(x) \, d\Gamma_q(x) \]  

and \( q^0(x) = \frac{\partial u^0(x)}{\partial n} \).

Since one component of the pair \( (\varphi, \varphi') \) is always specified on \( \Gamma_u \) and \( \Gamma^q \neq 0 \), the above equation determines the unknown boundary data (i.e. \( u \) on \( \Gamma_u \) and \( q \) on \( \Gamma_q \)).

We introduce a decomposition of the Green’s function into the discretization-independent infinite-body part and the discretization-dependent boundary contribution:

\[ G^{0} = G^{0,\infty}(\xi, x) + G^{0,\delta}(\xi, x) \]

where \( G^{0,\infty} \) refers to the two-point GF for the corresponding homogeneous material and the second term \( G^{0,\delta} \) is bounded and refers to the additional term due to property gradation;

Expression of \( \delta \) is derived following an analogous procedure. We exploit the infinite-body-boundary split

\[ \Delta^0 (\xi, x) = \Delta^{0,\infty}(\xi, x) + \Delta^{0,\delta}(\xi, x) \]
The first part $\Delta^\lambda (\xi, x)$ is directly obtained from equation (10) for the reference problem. The BEM-based approach is completed by approximating $\Gamma^\lambda$ function, in particular we get

$$
\Gamma^\lambda (\xi, x) = \Gamma^{\alpha, \alpha} (\xi, x) + \Gamma^{\alpha, \beta} (\xi, x),
$$

$$
\Gamma^{\alpha, \alpha} (\xi, x) = \frac{\partial \Delta^{\alpha, \alpha} (\xi, x)}{\partial x},
$$

$$
\Gamma^{\alpha, \beta} (\xi, x) = \frac{\partial \Delta^{\alpha, \beta} (\xi, x)}{\partial x}.
$$

Some examples of these functions are illustrated in Figure 2 illustrated below.

Finally note that the previous procedure can be directly translated to multi-dimensional and/or vectorial case; see [6] for more details.

![Figure 2: (a)-(d) Examples of $\Gamma^\lambda$ function](image)

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**References**


Scalar Wave Equation by the Domain Boundary Element Method with Non-homogeneous Initial Conditions

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Keywords: scalar wave equation, initial conditions, D-BEM

Abstract. This work is concerned with the development of a D-BEM approach to solve 2D scalar wave propagation problems. The process of time-marching is accomplished with the use of the Houbolt method, as usual. Special attention is devoted to the development of a procedure that allows for the computation of the initial conditions contributions. Two examples are presented and the D-BEM results are compared with the corresponding analytical solutions.

Introduction

The development of BEM formulations for solving time-dependent problems is a very attractive area of research. For this reason, a great number of formulations appeared during the last years, enriching the BEM literature concerning this matter. The reader is referred to Beskos [1] for a very complete discussion concerning dynamic analysis by the BEM.

Bearing in mind that the integral equations can be obtained by means of a weighted residuals statement, the solution of time-dependent problems can be accomplished with the use of time-dependent fundamental solutions: in this case, BEM formulations are denominated TD-BEM (TD meaning time-domain), e.g. Mansur [2], Dominguez [3].

Alternatively, one can use static fundamental solutions instead of time-dependent fundamental solutions. In this case, the BEM basic integral equation presents a domain integral with the kernel constituted by the fundamental solution multiplied by the second order time derivative of the potential, thus generating the so-called D-BEM formulations, D means domain, e.g. Carrer and Mansur [4], Hatzigeorgiou and Beskos [5]. It is important to note that the time variable does not appear explicitly in the integral equations. As a consequence, in order to perform the time-marching, the choice of an adequate approximation to the acceleration plays an important role: this approximation seems to be, as long as the authors know, the Houbolt method [6]. Note that unless responses only at early times are required, D-BEM formulation is not appropriate for infinite domain analyses. In spite of this drawback, D-BEM formulations provide reliable results, justifying the interest in its development.

In this work a D-BEM formulation for the solution of 2-D scalar wave equation problems with non-homogeneous initial conditions is presented. Two examples are included at the end of the article, with the aim of validating the proposed formulation.

D-BEM Formulation

The starting D-BEM equation is written as follows:

\[
c(\xi) u(\xi,t) = \int_{\Gamma} u^* (\xi,X) p(X,t) \, d\Gamma(X) - \int_{\Gamma} p^* (\xi,X) u(X,t) \, d\Gamma(X) - \frac{1}{c} \int_{\Omega} u^* (\xi,X) \ddot{u}(X,t) \, d\Omega(X) \tag{1}\]

In eq(1), \(u(X,t)\), that is generically referred to as potential function, can represent the transversal displacements of a membrane, and \(p(X,t)\) represents its normal derivative (support reaction in a membrane). Besides, \(c\) is the wave propagation velocity, \(t\) is the time, \(\Gamma\) is the boundary and \(\Omega\) is the domain of the
problem. As usual in BEM formulations, \( X \) and \( \xi \) represent the field and source points, respectively. The fundamental solution, \( u^*(\xi;X) \), is given by:

\[
u^*(\xi;X) = \frac{1}{2\pi} \ln\left(\frac{1}{r}\right)
\]

(2)

where \( r \) is the distance between \( X \) and \( \xi \), and \( p^*(\xi;X) \) is the normal derivative of the fundamental solution, computed according to:

\[
p^*(\xi;X) = \frac{du^*(\xi;X)}{dn} = \frac{du^*(\xi;X)}{dr} = \frac{du^*(\xi;X)}{dn}
\]

(3)

In order to solve eq(1), boundary and domain discretization must be carried out and an approximation to the acceleration must be adopted. In the present work, the discretization of the boundary is accomplished by linear boundary elements; the discretization of the domain, by triangular linear cells. The reader is referred to Mansur [2] for further details concerning this matter. Once the spatial discretization has been carried out, the resulting matrices can be assembled thus generating an enlarged system of equations, written below:

\[
\begin{bmatrix}
H^{bb} & 0 \\
H^{bd} & 1
\end{bmatrix}
\begin{bmatrix}
u^b_{n+1} \\
u^d_{n+1}
\end{bmatrix}
= \begin{bmatrix}
G^{bb} & \frac{1}{c^2}M^{bd}
\end{bmatrix}
\begin{bmatrix}
u^b_{n+1} \\
\phi^b_{n+1}
\end{bmatrix}
\]

(4)

In eq(4), in order to simplify the notation, the subscript \((n + 1)\) represents the time \( t_{n+1} = (n + 1)\Delta t \), where \( \Delta t \) is the selected time interval. The superscripts \( b \) and \( d \) correspond to the boundary and to the domain (internal points), respectively. In the sub-matrices, the first superscript corresponds to the position of the source point and the second superscript, to the position of the field point. The identity matrix is related to the coefficients \( c(\xi) = 1 \) of the internal points. From eq(4), the boundary unknowns are the values of \( u \) and \( p \) at \( \xi^* \) (as usual in BEM formulations) and the values of \( u \) at \( \Gamma \). It is important to mention that the assemblage of such an enlarged system of equations is necessary because the domain integral relates boundary values to domain values. This remark is confirmed by matrix \( M^{bd} \) in eq(4).

Time-marching Scheme. The Houbolt method [6] is obtained by cubic Lagrange interpolation of \( u = u(t) \) from time \( t_{n-2} = (n-2)\Delta t \) to time \( t_{n+1} = (n + 1)\Delta t \). Exact differentiation with respect to time gives the approximations to the velocity and acceleration below:

\[
\begin{align*}
\dot{u}^b_{n+1} &= \frac{1}{6\Delta t}\left[11u^b_{n+1} - 18u^b_n + 9u^b_{n-1} - 2u^b_{n-2}\right] \\
\ddot{u}^b_{n+1} &= \frac{1}{\Delta t^2}\left[2u^b_{n+1} - 5u^b_n + 4u^b_{n-1} - u^b_{n-2}\right]
\end{align*}
\]

(5)

(6)

After substituting (6) in (4), the time-marching scheme can start:

\[
\begin{bmatrix}
(c\Delta t^2)H^{bb} + 2M^{bd} \\
(c\Delta t^2)H^{bd} + 2M^{dd}
\end{bmatrix}
\begin{bmatrix}
\dot{u}^b_{n+1} \\
\dot{u}^d_{n+1}
\end{bmatrix}
= \begin{bmatrix}
(c\Delta t^2)G^{bb} \\
(c\Delta t^2)G^{bd}
\end{bmatrix}
\begin{bmatrix}
u^b_{n+1} \\
\phi^b_{n+1}
\end{bmatrix}
\]

(7)

Eq(7) can be represented in a concise manner as:
\[ H \mathbf{u}_{n+1} = G \mathbf{p}_{n+1} + \mathbf{g}_n \]  \hspace{1cm} (8)

The contributions of the previous instants of time are stored in vector \( \mathbf{g}_n \). After the boundary conditions are imposed, a final system of equations, that arises from eq(8), can be solved.

It is important to point out that an adequate choice of the time-step plays a fundamental role in the analysis. A dimensionless variable, say \( \beta_0 \), was adopted in order to provide a measure of the time-step \( \Delta t \), see Mansur \[2\] and Carrer and Mansur \[4\]:

\[ \beta_0 = \frac{c \Delta t}{t} \]  \hspace{1cm} (9)

where \( t \) is the length of the smallest element used in the boundary discretization.

**Initial Conditions Contributions.** In the Houbolt method, the computation of the velocity and accelerations at time \( t_{n+1} = (n + 1)\Delta t \) requires the knowledge of the values of \( \mathbf{u} \) from time \( t_{n-2} = (n - 2)\Delta t \) to time \( t_{n-1} = (n + 1)\Delta t \). At the beginning of the analysis, i. e., at the beginning of the time-marching process, \( n = 0 \) and, consequently, the values \( \mathbf{u}_{-2} \) and \( \mathbf{u}_{-1} \) must be computed appropriately in order to provide a good start of the analysis.

For the determination of \( \mathbf{u}_{-1} \), \( \mathbf{u}_0 \) is computed by employing the forward and the backward finite difference formulae at \( t = 0 \) and assuming that the finite difference expressions are equal, that is:

\[ \frac{u_0 - u_1}{\Delta t} = \frac{u_2 - u_1}{\Delta t} \]  \hspace{1cm} (10)

Solving eq(10) for \( u_1 \), one has:

\[ u_1 = 2u_0 - u_{-1} \]  \hspace{1cm} (11)

One can also assume that \( \mathbf{u}_0 \) can be computed by employing a central finite difference formula, which gives:

\[ \frac{u_1 - u_{-1}}{2\Delta t} \]  \hspace{1cm} (12)

Solving eq(12) for \( u_1 \):

\[ u_1 = 2\Delta t \frac{u_0 - u_{-1}}{\Delta t} + u_{-1} \]  \hspace{1cm} (13)

From (11) and (13) one has \( u_{-1} \) as a function of \( \mathbf{u}_0 \) and \( \mathbf{u}_0' \):

\[ u_{-1} = u_0 - \Delta t \mathbf{u}_0' \]  \hspace{1cm} (14)

Now, for the determination of \( \mathbf{u}_{-2} \), initially \( \mathbf{u}_{-1} \) is computed by employing the forward and the backward finite difference formulae at \( t = -\Delta t \) and assuming that the finite difference expressions are equal, that is:

\[ \frac{u_{-2} - u_{-1}}{\Delta t} = \frac{u_{-1} - u_{-2}}{\Delta t} \]  \hspace{1cm} (15)
Solving eq(15) for $u_2$:

$$u_2 = 2u_1 - u_0$$  \hspace{1cm} (16)

Substituting (14) in (16) one obtains the value of $u_2$ as a function of $\dot{u}_0$ and $u_0$:

$$u_2 = u_0 + 2\Delta t \dot{u}_0$$ \hspace{1cm} (17)

**Examples**

In the following examples, the BEM results are always compared with the corresponding analytical solution, computed following the procedures described by Kreyszig [7].

**Square Membrane under Prescribed Initial Displacement over the Entire Domain.** This example deals with a square membrane defined over the domain $0 \leq x \leq a$, $0 \leq y \leq a$, subjected to the initial conditions given by eq(18), see Fig.1:

$$u_0(x,y) = U x(x-a)y(y-a); \hspace{0.5cm} v_0(x,y) = 0$$  \hspace{1cm} (18)

Boundary discretization employed 80 elements and the domain, 800 cells, see Fig. 2.

The analytical solution to this problem, for the general case of a rectangular membrane with dimensions $a$ and $b$, according to Kreyszig [8], is:

$$u(x,y,t) = \frac{64Ua^2b^2}{m^4n^4\pi^4} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin \left(\frac{mnx}{a}\right) \sin \left(\frac{mny}{b}\right) \cos \left(\lambda mn^2\pi^2t\right), \hspace{0.5cm} \text{with} \hspace{0.5cm} \lambda_{mn} = \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$$  \hspace{1cm} (19)

The results corresponding to the displacement at point $A(a/2,a/2)$ and to the support reaction at node $B(a,a/2)$ are shown in Fig. 3 and Fig. 4, respectively. The analysis was carried out with the time interval computed from $t_\Delta = 3/10$. Good agreement between D-BEM and analytical results was already expected, as in this example time jumps do not appear.
Square Membrane under Prescribed Initial Velocity over Part of the Domain. The square membrane shown in Fig. 5, with the initial condition $v_0 = V$ prescribed over $\Omega_a$, is analysed here.

The analytical solution to this problem is given by, see Mansur [2] and Kreyszig [7]:

$$u(x,y,t) = \frac{2V}{\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn\lambda_{mn}} \sin \left( \frac{mx}{a} \right) \sin \left( \frac{ny}{a} \right) G_{mn}$$

(20)

where:

$$\lambda_{mn} = \frac{c}{2} \sqrt{\frac{m^2}{a^2} + \frac{n^2}{a^2}}$$ and $$G_{mn} = \left( \cos \left( \frac{3mx}{5} \right) - \cos \left( \frac{2mx}{5} \right) \right) \left( \cos \left( \frac{3ny}{5} \right) - \cos \left( \frac{2ny}{5} \right) \right) \sin(2\pi \lambda_{mn} t)$$

(21)

This example was analysed previously with the use of the TD-BEM formulation, e.g. Mansur [2]. In the D-BEM formulation presented here, care should be taken when considering the initial conditions contributions. Because linear interpolation is adopted to the initial conditions in the internal cells, the use of double-nodes in the boundary of $\Omega_a$ became necessary in order to avoid the spreading of the initial conditions to the cells in the neighbouring of $\Omega_a$. Aiming at providing a good representation of the time jumps in the support reaction response, a more refined mesh, constituted...
of 160 elements and 3200 cells, not shown here, was employed. The results corresponding to the
displacement at point \( A(\frac{a}{2}, \frac{a}{2}) \) and to the support reaction at node \( B(a, \frac{a}{2}) \), are depicted in Fig. 6 and Fig. 7, respectively. These results were obtained by adopting \( \beta_0 = \frac{3}{10} \). The D-BEM responses are in good agreement with the analytical ones, although it must be pointed out that the TD-BEM formulation provides a better representation of the time jumps. Nevertheless, the good responses furnished by the D-BEM formulation demonstrate its applicability to the solution of this kind of problems.

![Figure 6](image1.png)

**Figure 6.** Square membrane under prescribed initial velocity over part of the domain: displacement at point \( A(\frac{a}{2}, \frac{a}{2}) \).

![Figure 7](image2.png)

**Figure 7.** Square membrane under prescribed initial velocity over part of the domain: support reaction at node \( B(a, \frac{a}{2}) \).

**Conclusions**

The D-BEM formulation is a very promising approach and, consequently, it is expected that some research work concerning its development will be done in the next years. Aiming at contributing with the development of the D-BEM formulation, this work is concerned with the solution of 2-D scalar wave equation problems with non-homogeneous initial conditions; in other words, once the Houbolt method was adopted to approximate the acceleration, this work is concerned with finding general expressions to the terms \( u_2 \) and \( u_1 \) that appear at the beginning of the time-marching. The results presented here demonstrate the validity of the expressions found, and also encourage the extension of the present work to elastodynamics, as well as to the DR-BEM formulation.

**References**


Shear Deformation Effect in Plates Stiffened by Parallel Beams by BEM

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**Key words:** Stiffened plate, reinforced plate with beams, bending, nonuniform torsion, warping, ribbed plate, slab-and-beam structure, shear deformation, Reissner's theory

**Abstract.** In this paper a general solution for the analysis of shear deformable stiffened plates subjected to an arbitrary loading is presented. The analysis of the plate is based on Reissner's theory, while the analysis of the beams is performed employing the linearized second order theory taking into account shear deformation effect. Six boundary value problems are formulated and solved using the Analog Equation Method (AEM), a BEM based method. The solution of the aforementioned plate and beam problems, which are nonlinearly coupled, is achieved using iterative numerical methods. The adopted model permits the evaluation of the shear forces at the interfaces in both directions, the knowledge of which is very important in the design of prefabricated ribbed plates.

**1. Introduction**

Structural plate systems stiffened by beams are widely used in buildings, bridges, ships, aircrafts and machines. Stiffening of the plate is used to increase its load carrying capacity and to prevent buckling especially in case of in-plane loading. Moreover, for cases wherein the plate or the beams are not very “thin” or the stiffeners are closely spaced, the error incurred from the ignorance of the effect of shear deformation may be substantial, while the accuracy of a classical analysis decreases and the truthfulness of the results is lost with growing plate or beam thickness. The extensive use of the aforementioned plate structures necessitates a rigorous analysis.

In this paper a general solution for the static analysis of plates stiffened by arbitrarily placed parallel beams of arbitrary doubly symmetric cross section subjected to an arbitrary loading is presented taking into account shear deformation effect in both the plate and the beams. The employed structural model is an improved one of that presented by Sapountzakis and Mokos in [1], so that a nonuniform distribution of the interface transverse shear force and the nonuniform torsional response of the beams are taken into account. According to this model, the stiffening beams are isolated again from the plate by sections in the lower outer surface of the plate, taking into account the arising tractions in all directions at the fictitious interfaces. These tractions are integrated with respect to each half of the interface width resulting two interface lines, along which the loading of the beams as well as the additional loading of the plate is defined. The utilization of two interface lines for each beam enables the nonuniform torsional response of the beams to be taken into account as the angle of twist is indirectly equated with the corresponding plate slope. The unknown distribution of the aforementioned integrated tractions is established by applying continuity conditions in all directions at the two interface lines. The analysis of both the plate and the beams is accomplished on their deformed shape taking into account second-order effects. The analysis of the plate is based on Reissner's theory, which may be considered as the standard thick plate theory with which all others are compared, while the analysis of the beams is performed employing the linearized second order theory taking into account shear deformation effect. Six boundary value problems are formulated and solved using the Analog Equation Method (AEM) [2], a BEM based method. The effectiveness, the range of applications of the proposed method and the influence of shear deformation effect are illustrated by working out numerical examples with great practical interest.

**2. Statement of the problem**

Consider a plate of homogeneous, isotropic and linearly elastic material with modulus of elasticity $E$, shear modulus $G$ and Poisson ratio $\mu$, having constant thickness $h_p$ and occupying the two dimensional multiply connected region $\Omega$ of the $x,y$ plane bounded by the boundary $\Gamma$. The plate is stiffened by a set
of \( i = 1, 2, \ldots, I \) arbitrarily placed parallel beams of arbitrary doubly symmetric cross section of homogeneous, isotropic and linearly elastic material with modulus of elasticity \( E^b_i \), shear modulus \( G^b_i \) and Poisson ratio \( \mu^b_i \), which may have either internal or boundary point supports. For the sake of convenience the \( x \) axis is taken parallel to the beams. The stiffened plate is subjected to the lateral load \( g = g(x, y) \). For the analysis of the aforementioned problem a global coordinate system \( Oxy \) for the analysis of the plate and local coordinate ones \( O^i x^i y^i \) corresponding to the centroid axes of each beam are employed.

The solution of the problem at hand is approached by an improved model of that proposed in [1]. According to this model, the stiffening beams are isolated again from the plate by sections in its lower outer surface, taking into account the arising tractions at the fictitious interfaces (Fig.1). Integration of these tractions along each half of the width of the \( i \)-th beam results in line forces per unit length in all directions in two interface lines, which are denoted by \( q^i_x, q^i_y \) and \( q^i_z \) \((j = 1, 2)\) encountered in this way the nonuniform distribution of the interface transverse shear forces \( q^i_q \), which in the aforementioned model in [1] was ignored. The aforementioned integrated tractions result in the loading of the \( i \)-th beam as well as the additional loading of the plate. Their distribution is unknown and can be established by imposing displacement continuity conditions in all directions along the two interface lines, enabling in this way the nonuniform torsional response of the beams to be taken into account, which in the aforementioned model in [1] was also ignored.

**a. For the plate.**

The plate undergoes transverse deflection and inplane deformation. Thus, for the transverse deflection, according to Reissner’s theory, if \( \psi(x, y) \) is a stress function satisfying the equation

\[
k \psi - \nabla^2 \psi = 0, \quad k = 10 / h_p^2 \quad \text{in } \Omega
\]

the average rotations \( \phi_{px}, \phi_{py} \) of the plate with respect to the axes \( y, x \), respectively, taking into account the effect of shear deformation can be written as

\[
\phi_{px} = -\frac{\partial \psi}{\partial x} + \frac{\sigma}{5Gh_p} \frac{Q_{px}}{Q_{px}} \quad \phi_{py} = -\frac{\partial \psi}{\partial y} + \frac{\sigma}{5Gh_p} \frac{Q_{py}}{Q_{py}}
\]

where the second term in the right hand side of these equations represents the additional angle of rotation due to shear deformation. Moreover, the stress resultants are given as
and the equation of equilibrium employing the linearized second order theory can be written as

\[ D \frac{\partial^2 w_p}{\partial x^2} - \left( \frac{N_x}{E} + 2N_{xy} + \frac{N_y}{E} \right) \frac{\partial^2 w_p}{\partial x \partial y} + \frac{N_{xy}}{E} \frac{\partial^2 w_p}{\partial y^2} = \frac{h_p^2}{10(1-\mu)} \frac{\partial^2 w_p}{\partial y^2} - \frac{g}{k} \left( \frac{\partial Q_{px}}{\partial x} + \frac{\partial Q_{py}}{\partial y} \right) \]

in \( \Omega \) (4)

where \( w_p = w_p(x,y) \) is the transverse deflection of the plate; \( D = \frac{Eh_p^3}{12(1-\nu^2)} \) is its flexural rigidity; \( N_x = N_x(x,y) \), \( N_y = N_y(x,y) \), \( N_{xy} = N_{xy}(x,y) \) are the membrane forces per unit length of the plate cross section and \( \delta(y-y') \) is the Dirac’s delta function in the y direction. The governing differential equations (1), (4) are also subjected to the pertinent boundary conditions of the problem, which are given as

\[ \alpha_{pl} w_p + \beta_{pl} \psi_{pl} + \gamma_{pl} M_{pl} = \gamma_{pl} \]

on \( \Gamma \) (5a,b,c)

where \( \alpha_{pl}, \beta_{pl}, \gamma_{pl} \) (1 = 1, 2, 3) are given functions specified on the boundary \( \Gamma \), \( Q_{pl}, M_{pl} \) are the shear force, the bending moment and the twisting moment along the boundary, respectively and \( \psi_{pl}, \phi_{pl} \) are the average rotations of the plate with respect to the axes \( i, n \), respectively.

Since linearized plate bending theory is considered, the components of the membrane forces \( N_x, N_y, N_{xy} \) are given as

\[ N_x = C \left( \frac{\partial u_p}{\partial x} + \mu \frac{\partial v_p}{\partial y} \right) \]
\[ N_y = C \left( \frac{\partial u_p}{\partial y} - \mu \frac{\partial v_p}{\partial x} \right) \]
\[ N_{xy} = C \frac{1-\mu}{2} \left( \frac{\partial u_p}{\partial x} + \frac{\partial u_p}{\partial y} \right) \]

(6a,b,c)

where \( C = \frac{Eh_p}{1-\mu^2} \); \( u_p = u_p(x,y), v_p = v_p(x,y) \) are the displacement components of the middle surface of the plate arising from the line body forces \( \psi_{ij} \) (1 = 1, 2, ..., I, 3 = I, 2). These displacement components are established by solving independently the plane stress problem, which is described by the following boundary value problem (Navier’s equations of equilibrium)
\[ \nabla^2 u_p + \frac{1 + \mu}{1 - \mu} \left( \frac{\partial^2 u_p}{\partial x^2} + \frac{\partial^2 u_p}{\partial y^2} \right) - \frac{1}{Gh_p} \sum_{j=1}^{2} \frac{q_j'(y-y_j)}{c^j} (y-y_j) = 0 \quad (7a) \]

\[ \nabla^2 v_p + \frac{1 + \mu}{1 - \mu} \left( \frac{\partial^2 v_p}{\partial x^2} + \frac{\partial^2 v_p}{\partial y^2} \right) - \frac{1}{Gh_p} \sum_{j=1}^{2} \frac{q_j'(y-y_j)}{c^j} (y-y_j) = 0 \quad \text{in } \Omega \quad (7b) \]

\[ \delta_p u_{p1} + \delta_p u_{p2} N_q = \delta_{p3} \quad \epsilon_p u_{p1} + \epsilon_p u_{p2} N_q = \epsilon_{p3} \quad \text{on } \Gamma \quad (8a,b) \]

In which \( G = E/2(1+\nu) \) is the shear modulus of the plate; \( N_q, \ N_j \) and \( u_{p1}, \ u_{p2} \) are the boundary membrane forces and displacements in the normal and tangential directions to the boundary, respectively; \( \delta_{pl} \), \( \epsilon_{pl} \) \( (l=1,2,3) \) are functions specified on the boundary \( \Gamma \).

b. For each \((i\text{-th})\) beam.

Each beam undergoes transverse deflection with respect to \( z_i \) and \( y_i \) axes, axial deformation along \( x_i \) axis and nonuniform angle of twist along \( x_i \) axis. Thus, for the transverse deflection with respect to \( z_i \) axis the equation of equilibrium employing the linearized second order theory and taking into account shear deformation effect can be written as [27]

\[ E_i^I \frac{d}{dx} \left( I + \frac{N_i}{G_i A_i} \right) \frac{d^2 w_i}{dx^2} = \sum_{j=1}^{2} \left( q_{ij}^l - q_{ij}^l \right) \frac{\partial w_i}{\partial x} + N_i \frac{\partial^2 w_i}{\partial x^2} \frac{\partial m_i}{\partial x} \quad \text{in } L_i, \ i=1,2,...,I \quad (9) \]

subjected to the following boundary conditions

\[ a_i^{ij} w_i + b_i^{ij} M_i = b_i^{ij} \quad \text{at the beam ends } x_i = 0, L_i \quad (10a,b) \]

where \( w_i = w_i(x_i) \) is the transverse deflection of the \( i\text{-th} \) beam with respect to \( z_i \) axis; \( I_i \) is its moment of inertia with respect to \( y_i \) axis; \( N_i = N_i^l(x_i) \) are the axial forces at the \( x_i \) centroid axis arising from the line body forces \( q_{ij}^l \); \( a_i^{ij} \), \( b_i^{ij} \) \( (l=1,2,3) \) are coefficients specified at the boundary of the \( i\text{-th} \) beam; \( \theta_{bi} \), \( R_{bi} \), \( M_{bi} \) are the slope, the reaction and the bending moment at the \( i\text{-th} \) beam ends, respectively given as

\[ \theta_{bi} = \frac{E_i^I I_i}{G_i A_i} \sum_{j=1}^{2} \left( \frac{N_i}{G_i A_i} \right) \frac{\partial^2 w_i}{\partial x^2} \frac{\partial w_i}{\partial x} \quad R_{bi} = -\frac{E_i^I I_i}{G_i A_i} \sum_{j=1}^{2} \left( \frac{N_i}{G_i A_i} \right) \frac{\partial^2 w_i}{\partial x^2} \frac{\partial^2 w_i}{\partial x^2} \quad M_{bi} = -\frac{E_i^I I_i}{G_i A_i} \sum_{j=1}^{2} \left( \frac{N_i}{G_i A_i} \right) \frac{\partial^2 w_i}{\partial x^2} \quad (11a,b,c) \]

Similarly, the \( v_i = v_i(x_i) \) transverse deflection with respect to \( y_i \) axis must satisfy the following boundary value problem
where $I_{bz}$ is the moment of inertia of the $i$-th beam with respect to $y$ axis; $a_{1}^{y}$, $a_{2}^{y}$, $M_{1}$ are the slope, the reaction and the bending moment at the $i$-th beam ends, respectively given as

$$
\theta_{bc}=-\frac{E_{b}J_{bc}}{I_{bc}} \sum_{j=1}^{2} \left( 1 + N_{by} \right) \frac{d^{2}v_{b}}{d\chi^{2}} \frac{\partial v_{b}}{\partial \chi} \quad R_{by}=-\frac{E_{b}J_{bc}}{I_{bc}} \sum_{j=1}^{2} \left( 1 + N_{by} \right) \frac{d^{2}v_{b}}{d\chi^{2}} \frac{\partial v_{b}}{\partial \chi}
$$

(13a,b,c)

In eqns (10), (12)-(14), (15), (17)-(19) $E_{b}, A_{y}, A_{z}$ are the shear rigidities of the Timoshenko’s beam theory, where

$$A_{y}=\kappa_{y} A'=\frac{1}{a_{y}}, \quad A_{z}=\kappa_{z} A'=\frac{1}{a_{z}}
$$

(15a,b)

are the shear areas with respect to $y$, $z$ axes, respectively with $\kappa_{y}$, $\kappa_{z}$ the shear correction factors, $a_{y}$, $a_{z}$ the shear deformation coefficients and $A'$ the cross section area of the $i$-th stiffening beam.

Since linearized beam bending theory is considered the axial deformation $u_{b}^{x}$ of the beam arising from the arbitrarily distributed axial forces $q_{ij}^{x}$ ($i=1,2,...,I$, $j=1,2$) is described by solving independently the boundary value problem

$$
E_{b}A_{b} \frac{\partial^{2} u_{b}^{x}}{\partial \chi^{2}} = \sum_{j=1}^{2} q_{ij}^{x} \quad \text{in} \quad L', \ i = 1,2,...,I
$$

(16)

$$
\gamma_{1}^{y} u_{b}^{x} + \gamma_{2}^{y} N_{b}^{y} = \gamma_{1}^{y} \quad \text{at the beam ends} \quad \chi'=0, L'
$$

(17)

where $N_{b}^{y}$ is the axial reaction at the $i$-th beam ends given as

$$
N_{b}^{y} = \sum_{j=1}^{2} N_{by}^{xy} = E_{b}A_{b} \frac{\partial v_{b}^{y}}{\partial \chi_{i}}
$$

(18)

Finally, the nonuniform angle of twist with respect to shear center axis has to satisfy the following boundary value problem
\[
E_{ij}'l_{ai} \frac{\partial^4 \theta_i^x}{\partial x^4} - G_{ij}'l_{ai} \frac{\partial^2 \theta_i^x}{\partial x^2} = \sum_{j=1}^{n} m_{aij} \quad \text{in } l_i', i = 1, 2, \ldots, l
\]  
\[
a_i^y \theta_i^y + a_i^z M_{bx} = \alpha_i^z \quad \beta_i^y \frac{\partial \theta_i^x}{\partial x} + \beta_i^z M_{bx} = \beta_i^z \quad \text{at the beam ends } x_j = 0, L_i
\]

where \(\theta_i^x = \theta_i^x(x')\) is the variable angle of twist of the \(i\)-th beam along the \(x'\) shear center axis;
\(G_{bx}^0 = G_{bx}^0/2(1+\nu_b)\) is its shear modulus; \(l_{ai}'\), \(l_{bi}'\) are the warping and torsion constants of the \(i\)-th beam cross section, respectively given as
\[
l_{ai}' = \int_{x'} \left( \phi^2 \right) \, dx'^4 \quad l_{bi}' = \int_{x'} \left( \phi^2 \right)^{1/2} \left( \frac{\partial \phi^y}{\partial x'} - z' \frac{\partial \phi^z}{\partial x'} \right) \, dx'^4
\]
with \(\phi^y(x', z')\) the primary warping function with respect to the shear center \(S\) of the \(A'\) beam cross section; \(a_i^y, \beta_i^y (i = 1, 2, 3)\) are coefficients specified at the boundary of the \(i\)-th beam; \(\frac{\partial \theta_i^x}{\partial x'}\) denotes the rate of change of the angle of twist and it can be regarded as the torsional curvature; \(M_{bx}^i\) is the twisting moment and \(M_{bx}^i\) is the warping moment due to the torsional curvature at the boundary of the \(i\)-th beam.

Eqs. (1), (4), (7a), (7b), (9), (12), (16), (19) constitute a set of eight coupled partial differential equations including fourteen unknowns, namely \(w, w_p, w_q, u_p, v_p, v_q, \theta^x, \psi_{1f}, \psi_{2f}, \psi_{3f}, \psi_{1b}, \psi_{2b}, \psi_{3b}\).

Six additional equations are required, which result from the displacement continuity conditions in the directions of \(z', x'\) and \(y'\) local axes along the two interface lines of each (\(i\)-th) plate – beam interface. These conditions can be expressed as

**In the direction of \(z'\) local axis:**
\[
w_{p1} - w_b = -\frac{b_j}{4} \theta_{i1}\quad \text{along interface line 1 (} f_{j1}^1 \text{)} \quad (22a)
\]
\[
w_{p2} - w_b = -\frac{b_j}{4} \theta_{i2}\quad \text{along interface line 2 (} f_{j2}^2 \text{)} \quad (22b)
\]

**In the direction of \(x'\) local axis:**
\[
u_{p1} - u_b = \frac{b_p}{2} \frac{\partial w_{p1}}{\partial x'} - \frac{b_y}{2} \frac{\partial w_{b1}}{\partial x'} + \frac{b_y}{4} \frac{\partial \phi}{\partial x'} \quad \frac{\partial \phi}{\partial x'} \quad \text{along interface line 1 (} f_{j1}^1 \text{)} \quad (22c)
\]
\[
u_{p2} - u_b = \frac{b_p}{2} \frac{\partial w_{p2}}{\partial x'} - \frac{b_y}{2} \frac{\partial w_{b2}}{\partial x'} + \frac{b_y}{4} \frac{\partial \phi}{\partial x'} \quad \text{along interface line 2 (} f_{j2}^2 \text{)} \quad (22d)
\]

**In the direction of \(y'\) local axis:**
\[
v_{p1} - v_b = \frac{b_p}{2} \frac{\partial w_{p1}}{\partial y'} - \frac{b_y}{2} \frac{\partial w_{b1}}{\partial y'} \quad \text{along interface line 1 (} f_{j1}^1 \text{)} \quad (22e)
\]
\[
v_{p2} - v_b = \frac{b_p}{2} \frac{\partial w_{p2}}{\partial y'} - \frac{b_y}{2} \frac{\partial w_{b2}}{\partial y'} \quad \text{along interface line 2 (} f_{j2}^2 \text{)} \quad (22f)
\]

where
while
\[
\theta_{ij} = \frac{\partial^2 \psi_{ij}}{\partial x^2} \frac{E_b I_f}{G_b A_f^2} \left( -E_b I_f \frac{\partial^2 \psi_{ij}}{\partial x^2} + E_b I_f \frac{\partial^4 \psi_{ij}}{\partial x^4} + N_{b,ij} \frac{\partial^2 \psi_{ij}}{\partial x^2} - 2 \mu_{ij} \frac{\partial^4 \psi_{ij}}{\partial x^2 \partial y^2} + \mu_{ij} \frac{\partial^4 \psi_{ij}}{\partial y^2} \right) + m_{ij}
\]
(23a, b)

and \((\theta^p_{ij})_{yj}\) is the value of the primary warping function with respect to the shear center \(S\) of the beam cross section at the point of the \(j\)-th interface line of the \(i\)-th plate–beam interface \(f\). It is worth here noting that the coupling of the aforementioned equations is nonlinear due to the terms including the unknown \(q_{ij}^p\) and \(q_{ij}^p\) interface forces.

3. Numerical Solution

The numerical solution of the boundary value problems described by eqns (4-5a,b,c), (7a,b-8a,b), (9-10a,b), (12-13a,b), (16-17) and (19-20a,b) will be accomplished employing the Analog Equation Method [2]. This method is applied for the aforementioned problems as this is presented in [1].

4. Numerical example

A concrete \(C20\) \((E = 2.9 \times 10^7 \text{ kPa}, \mu = 0.0)\) rectangular plate with dimensions \(a \times b = 18.0 \times 9.0 \text{ m}\) subjected to eccentric uniform loading \(g = 20 \text{ kN/m}^2\) and stiffened by five concrete \(C35/40\) \((E_b = 3.35 \times 10^7 \text{ kPa}, \mu_b = 0.2)\) I-section beams symmetrically placed (Fig. 2) forming a bridge deck has been studied.

The plate along its small edges is simply supported according to its transverse boundary conditions, clamped according to its inplane ones, while the other two edges are free according to both its transverse and inplane boundary conditions. The beams at their edges are also simply supported according to their bending.
boundary conditions and clamped according to their axial and torsional ones. In Table 1 the obtained
deflections of the stiffened plate at its center and at the middle of the free edges A and C (Fig. 2) and in Fig.3
the obtained axial forces $N_i$ of each stiffening beam are presented taking into account or ignoring shear
deformation effect.

<table>
<thead>
<tr>
<th></th>
<th>Including Inplane Forces</th>
<th>Ignoring Inplane Forces</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>with s.d.</td>
<td>without s.d.</td>
</tr>
<tr>
<td>Center</td>
<td>1.1094E+00</td>
<td>8.9581E-01</td>
</tr>
<tr>
<td>Middle of the free edge A</td>
<td>-9.3156E-02</td>
<td>-6.6141E-02</td>
</tr>
<tr>
<td>Middle of the free edge C</td>
<td>9.6674E-01</td>
<td>7.1705E-01</td>
</tr>
</tbody>
</table>

Fig. 3 Axial forces of the stiffening beams taking into account (a) or ignoring (b) shear deformation effect.

5. Concluding remarks
a. The proposed model permits the study of a stiffened plate subjected to an arbitrary loading, while both
the number and the placement of the nonintersecting stiffening beams are also arbitrary.
b. The proposed model permits the evaluation of both the longitudinal and the transverse inplane shear
forces at the interfaces between the plate and the beams (estimation of shear connectors).
c. The evaluated lateral deflections of the plate - beams system are found to exhibit considerable
discrepancy from those of other models, which neglect inplane and axial forces and deformations.
d. In some cases, the influence of the shear deformation effect is remarkable and should not be neglected.

References
Slow viscous migration of a solid particle in a rigid and motionless cavity: boundary-element techniques against a new and boundary-integral-equation-free approach

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Abstract. The slow viscous rigid-body migration of a solid particle immersed in a Newtonian liquid bounded by a solid and motionless cavity is examined. For this purpose four techniques are successively proposed, tested and compared. The first two methods resort to boundary-integral equations obtained by putting an unknown Stokeslet density on both the particle and cavity surfaces or solely on the particle's boundary once a cavity Green tensor, by essence vanishing on the cavity, is available. Two additional and new procedures, free from any boundary-integral equation, are furthermore advocated. These methods rest on the key reciprocal identity and appeal either to the free-space or cavity Green tensor. The merits (easy implementation, accuracy,...) but also the drawbacks of each approach are discussed by achieving benchmark tests when both the particle and the cavity admit a spherical geometry. Indeed, for such specific shapes the cavity Green tensor is known in closed form and very accurate results, recently obtained elsewhere by using the quite different approach of bipolar coordinates, are also available.

Introduction

In many applications (sedimentation, electrophoresis,...) the migration of small solid particles suspended in a Newtonian liquid is encountered. For a dilute suspension particle-particle interactions are negligible but as soon as the liquid is confined it is not always possible to also discard particle-boundary interactions. In the literature many papers therefore handle the case of one solid particle immersed in a liquid weakly bounded by one (see [1-5]) or two [see [6-9] parallel solid plane(s) or strongly bounded by a close cavity (see [10-12]). Although appealing to different techniques (semi-analytical approaches using the bipolar coordinates [1,2,10,11]; the multipole images [5]; the collocation technique [3,4,6,7] and the Green tensor [8,9]) all the previously-quoted works are unfortunately restricted to the case of a spherical particle. Therefore, it is worth proposing other approaches valid for arbitrarily-shaped particles. This is achieved in the present work for the challenging and tricky case of a strongly bounded liquid, i.e. for a particle moving in a liquid bounded by a solid cavity. More precisely, four techniques are successively given for arbitrary particle and cavity. These procedures are moreover tested against accurate semi-analytical solutions [12] when both the particle and the cavity exhibit a spherical shape.

Governing assumptions and equations. Basic issues

Let us consider, as illustrated in Fig. 1, a solid particle $P$ immersed in a Newtonian liquid with uniform viscosity $\mu$ occupying a domain $\Omega$ bounded by a solid and motionless cavity $\Sigma$. When $P$ translates and/or rotates with respect to a Cartesian framework $(O,x_1,x_2,x_3)$ attached to $\Sigma$ at the translational velocity $\mathbf{U}$ (the velocity of its points $O'$) and angular velocity $\Omega$ it induces a liquid flow with velocity $\mathbf{u}$, pressure $p$ and stress tensor $\sigma$ and also experiences net hydrodynamic force $\mathbf{F}$ and torque $\mathbf{T}$ (with respect to $O'$) such that

$$
\mathbf{F} = \int_S \sigma \cdot \mathbf{n} \, dS, \quad \mathbf{T} = \int_S \mathbf{O'M} \wedge \sigma \cdot \mathbf{n} \, dS
$$

with $\mathbf{n}$ the unit outward normal on the particle's surface $S$. 

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Figure 1. A solid particle $\mathcal{P}$ immersed in a Newtonian liquid strongly bounded by a solid and motionless cavity $\Sigma$. The point $O$ is attached to the particle.

If $\mathcal{P}$ has length scale $a$ and $u$ has typical magnitude $V$, for most encountered applications the particle is small enough so that $Re = \rho Va/\mu \ll 1$. Accordingly, inertial effects are negligible and the flow $(u, p)$ thus obeys the following quasi-static Stokes equations

$$\mu \nabla^2 u = \nabla p \quad \text{and} \quad \nabla \cdot u = 0 \quad \text{in} \quad \Omega, \quad (2)$$

$$u = 0 \quad \text{on} \quad \Sigma \quad \text{and} \quad u = U + \Omega \wedge \mathbf{OM} \quad \text{on} \quad S. \quad (3)$$

If $(U, \Omega)$ is provided one thus obtains $(u, p)$ and the resulting net force $\mathbf{F}$ and torque $\Gamma$ it exerts on the particle by solving (2)-(3) and using the relations (1). How such a basic task can be numerically achieved for arbitrary particle and cavity shapes is addressed in the next sections by proposing and testing four different techniques.

Two boundary-integral-equation approaches

This section briefly introduces two methods to compute the velocity field $u$ and the required net force $\mathbf{F}$ and torque $\Gamma$ which reduce to the treatment of a boundary-integral equation.

A basic velocity integral representation

One fruitful way of solving (2)-(3) is to introduce Green tensors. More precisely, for any so-called pole $y$ in $\mathcal{D} = \Omega \cup \mathcal{P}$ and indices $j = 1, 2, 3$ we introduce the Stokes flows $(v^{(j)}, p^{(j)})$, with stress tensor $\sigma^{(j)}$, such that

$$\mu \nabla^2 v^{(j)} = \nabla p^{(j)} - \delta^{(j)}(x - y)\epsilon_j, \quad \nabla \cdot v^{(j)} = 0 \quad \text{in} \quad \mathcal{D} \quad (4)$$

where $\delta^{(j)}(x - y) = \delta^{(j)}(x_1 - y_1)\delta^{(j)}(x_2 - y_2)\delta^{(j)}(x_3 - y_3), y_j = y_j \epsilon_j$, and $\delta^{(j)}$ designates the Dirac pseudo-function. We note that both the velocity fields $v^{(j)}(x, y)$ and the associated Green tensor $G$ with Cartesian components $G_{kj}(x, y) = v^{(j)}(x, y)\epsilon_k$ are not unique because (4) is free from any boundary condition on the surface $\Sigma \cup \mathcal{D}$. In addition [13] for $(u, p)$ fulfilling (2) and adopting the usual tensor summation convention and setting $f = \sigma \cdot n = f_k \epsilon_k$ on the boundary $\Sigma \cup \mathcal{D}$ one gets the following key integral representation

$$u_j(x) = \int_{\Sigma \cup \mathcal{D}} u_j(y) \sigma^{(j)}(x, y) \cdot n(y) dS(y) - \int_{\Sigma \cup \mathcal{D}} f_k(y) G_{kj}(y, x) dS(y) \quad \text{for} \quad x \in \Omega, \quad j = 1, 2, 3. \quad (5)$$

One should mention that, if needed, a similar integral representation also holds [13] for the pressure field $p$ with involves this time a pressure Green vector associated with the selected Green tensor $G$.

Now, exploiting the boundary condition (3) makes it possible to cast (5) into the simple form

$$u_j(x) = -\int_{\Sigma \cup \mathcal{D}} f_k(y) G_{kj}(y, x) dS(y) \quad \text{for} \quad x \in \Omega, \quad j = 1, 2, 3. \quad (6)$$

Inspecting (6) thus reveals that it is sufficient to compute the traction $f$ on the boundaries $\mathcal{S}$ and $\Sigma$ to obtain the velocity field $u$ in the entire liquid domain $\Omega$.

First boundary-integral technique
A well-known and simple Green tensor $G$ governed by (4) is the free-space Green-Burgers tensor $G_{\infty}$ with Cartesian coordinates $G_{ij}$ such that [14]

$$s \mu G_{ij}^{\infty}(x, y) = \frac{\delta_{ij}}{|x - y|} + \frac{[(x - y) \cdot e_j][(x - y) \cdot e_i]}{|x - y|^3}$$

(7)

where $\delta$ denotes the Kronecker delta symbol. This tensor becomes weakly singular as $x$ tends to $y$ and therefore letting $r = \min(|x|, |y|)$ in (6) the point $x$ tends to the boundary shows that the required surface traction $f$ obeys the widely-employed Fredholm boundary-integral equation of the first kind

$$u_j(x) = - \int_{S \cup \Sigma} f_k(y) G_{kj}^{\infty}(x, y) dS(y), \text{ for } x \in S; j = 1, 2, 3.$$  

(8)

Inverting (8) actually provides the vector $f$ on both surfaces $S$ and $\Sigma$ up to a constant multiple of $n$.

### Second boundary-integral technique

As previously outlined, the Green tensor is not unique and supplementing (4) with boundary conditions on $S \cup \Sigma$ permits one to select a Green tensor. In practice it is convenient to use a so-called cavity Green tensor $G^c$ further denoted by $G^c$ and vanishing on the cavity boundary $\Sigma$. More precisely, one has $G_{ij}^{\infty}(x, y) = 0$ for $x$ on $\Sigma$. For such a specific boundary condition it is possible to prove [13] that $G^c_{ij}(x, y) = G_{ij}^{\infty}(y, x)$. Consequently, one also has $G^c_{ij}(x, y) = G^{\infty}_{ij}(y, x) = 0$ as soon as $y$ lies on $\Sigma$. Exploiting this latter property, the integral representation (6) this time yields

$$u_j(x) = - \int_{S \cup \Sigma} f_k(y) G_{kj}^{\infty}(x, y) dS(y), \text{ for } x \in \Omega; j = 1, 2, 3.$$  

(9)

Moreover, from (4) the cavity Green tensor $G^c$ admits Cartesian components $G_{ijk}^c(x, y)$ such that

$$G_{ij}^c(x, y) = G_{ij}^{\infty}(x, y) + \Sigma_{kij}$$

(10)

with velocity fields $\Sigma_{ijk}(x, y)e_k$ smooth and bounded in the domain $\Omega$ for $j = 1, 2, 3$. Accordingly, the tensors $G^c$ and $G_{\infty}$ exhibit the same weakly singular behaviour as $x$ tends to $y$ and (9) thus yields, as $x$ tends onto $S$, the following simple boundary-integral equation for the surface traction $f$ on the particle’s boundary $S$

$$u_j(x) = - \int_{S \cup \Sigma} f_k(y) G_{kj}^c(x, y) dS(y), \text{ for } x \in S; j = 1, 2, 3.$$  

(11)

The second boundary-integral approach consists in inverting (11). By contrast to the first one, which reduces to the treatment of (8), this second boundary-integral approach thus (only) provides the requested traction $f$ on the smaller surface $S$ (i.e. not on the entire surface $S \cup \Sigma$ which may become very large as the cavity increases in volume).

### Two new approaches free from any boundary-integral-equation

The key integral representations (6) and (9) admit a straightforward and easy numerical implementation because the occurring kernels $G_{\infty}^c$ and $G^c$ are regular for $x$ in $\Omega$. Due to the previously-mentioned weak singularity at $x = y$ arising for those kernels the accurate treatment of the boundary-integral equations (9) and (11) by contrast requires a special attention and much more efforts to ascertain a good accuracy. Hence, it would be nice to obtain the surface traction $f$ without resorting to a boundary-integral equation. This is achieved by two new methods presented in this section.

A key relation: the reciprocal identity

Remind that the flows $(u, p)$ obeys (2). For any arbitrary additional flow $(u', p')$ with stress $\sigma'$ and subject to the Stokes equations (2) the well-known reciprocal identity [14] holds and reads

$$\int_{S \cup \Sigma} u' \cdot \sigma' dS = \int_{S \cup \Sigma} u \cdot \sigma' dS.$$  

(12)
Exploiting the boundary conditions (3) and setting \( x = \mathbf{O} \mathbf{O}' + x' \), then provides the key identity

\[
\int_{S \cup \Sigma} [u'(x) \cdot f(x)] dS(x) = \mathbf{U} \int_{S} [\sigma'(x) \cdot n(x)] dS(x) + \mathbf{N} \left\{ \int_{S} x' \wedge [\sigma'(x) \cdot n(x)] dS(x) \right\}.
\]

(13)

The relation (13) suggests to determine the required surface traction \( f \) on \( S \cup \Sigma \) by adequately selecting

\[
\text{Stokes flows} \ (u', p')
\]

for which it is possible to accurately evaluate the right-hand side of (13) and the associated velocity \( u' \) on the surface \( S \cup \Sigma \). As in the previous section, two different types of auxiliary Stokes flows \( u' \) are successively selected below.

**Auxiliary Stokes flows induced by the free-space Green tensor**

Let us consider a pole \( y \) located outside the liquid domain \( \Omega \), i.e., either inside the particle \( P \) or in \( R^3 \setminus (\Omega \cup P) \). For any prescribed value of \( j = 1, 2, 3 \) the flow \( u'(x; y, j) = G^*_j(x, y) e_j \) thus obeys

(2) and one can apply (13). The value of such a flow is easy to calculate on the surface \( S \cup \Sigma \) by employing the definition (7) and one moreover obtains in that specific case the key relations [7]

\[
\int_{S} \sigma'(x; y, j) \cdot n(x) dS(x) = \int_{S} x' \wedge [\sigma'(x; y, j) \cdot n(x)] dS(x) = 0 \quad \text{if} \quad y \in R^3 \setminus (\Omega \cup P),
\]

(14)

\[
\int_{S} \sigma'(x; y, j) \cdot n(x) dS(x) = -e_j \int_{S} x' \wedge [\sigma'(x) \cdot n(x)] dS(x) = [\mathbf{O} \mathbf{O}' - y] \wedge e_j \quad \text{if} \quad y \in \mathcal{P}.
\]

(15)

In summary, using these specific auxiliary Stokes flows the relation (13) admits the following form

\[
\sum_{j=1}^{3} \int_{S \cup \Sigma} G^*_j(x, y) f_j(x) dS(x) = 0 \quad \text{for} \quad j = 1, 2, 3 \quad \text{if} \quad y \in R^3 \setminus (\Omega \cup \mathcal{P}),
\]

(16)

\[
\sum_{j=1}^{3} \int_{S \cup \Sigma} G^*_j(x, y) f_j(x) dS(x) = -e_j \left[ \mathbf{U} + \Omega \wedge [y - \mathbf{O} \mathbf{O}'] \right] \quad \text{for} \quad j = 1, 2, 3 \quad \text{if} \quad y \in \mathcal{P}.
\]

(17)

The equations (16)-(17) permit one to obtain the traction \( f \) on the entire boundary \( S \cup \Sigma \) by using as many poles \( y \) as the number of collocation points employed when discretizing the surface \( S \cup \Sigma \). More precisely, if one puts \( N \) and \( M \) nodes on \( S \) and \( \Sigma \), respectively, then one will employ \( N + M \) poles \( y \). In practice it appears more efficient to spread the poles on closed surfaces \( S' \) and \( \Sigma' \) drawn in \( \mathcal{P} \) and outside the cavity, respectively. One furthermore can use \( N \) and \( M \) poles on \( S' \) and \( \Sigma' \), respectively.

**Auxiliary Stokes flows induced by the cavity Green tensor**

When the cavity Green tensor is available it is also fruitful to select the auxiliary Stokes flows \( u' \) originating from a given pole \( y \) located inside the particle \( \mathcal{P} \) as \( u'(x; y, j) = G_j(x, y) e_j \). In such circumstances, (16)-(17) still hold provided that one replaces \( G^*_j \) with \( G_j \) and \( S \cup \Sigma \) with \( S \). It is then possible to numerically obtain the traction \( f \) on the particle’s surface \( S \).

**Case of a spherical cavity**

There is no restriction regarding the shapes of the particle and of the cavity for the previously-proposed procedures. In order to test those approaches against results obtained in another fashion the case of spherical particle cavity has been examined. This section gives the associated cavity tensor and also briefly introduces the selected tests whereas, for the sake of conciseness, the oral presentation will present and discuss the comparisons for a spherical particle and time-permitting also for a few ellipsoidal particles.

**Cavity Green tensor for a spherical cavity**

To the author’s very best knowledge the cavity Green tensor \( G^* \) is available in the literature for a spherical cavity only. This permits one to efficiently work out the previous approaches employing this Green tensor. The tensor \( G^* \) has been obtained for a spherical cavity with center \( O \) and radius \( R \) in closed form by Green [7]. Introducing the notations

\[
y' = \frac{R^2 y}{|y'|}, \quad r' = |x - y'|, \quad r = |x|, \quad \cos \theta = \frac{|x' y'|}{|x| |y'|},
\]

(18)
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This author obtained

$$G_{jk}(x, y) = G_{jk}^B(x, y) - \frac{\partial G_{jk}}{\partial r} \left\{ \frac{R}{|y|} \frac{[r^' \cdot e_j] [r^' \cdot e_k]}{r^2} \right\}$$

$$- \frac{|y|^2 - R^2}{2|y|^2} \left\{ \frac{y^' \cdot e_j}{R^2 r^2} \right\} + \frac{3|x| + [r^' - |y^'|] \cos \theta}{R|x| |y|^2 \sin^2 \theta} \left[ R^2 x - (x \cdot y') y \right].$$

(19)

with $v = \eta \cdot e_j$ the vector

$$v = \frac{R}{r} \left( 2[r' \cdot y'] y' \right) + \frac{3[x + [r^' - |y^'|] \cos \theta]}{R|x| |y|^2 \sin^2 \theta} \left[ R^2 x - (x \cdot y') y \right].$$

(20)

### Numerical implementation and proposed benchmark tests

Even for a spherical cavity only a numerical treatment is possible. This is achieved by putting boundary elements [16] on the surfaces $S$ and, if needed, $\Sigma$. The adopted numerical implementation and the treatment of the encountered boundary-integral equations will be detailed at the oral presentation.

Accurate results are available for a spherical particle with radius $a$ and center $O'$ and a spherical cavity with radius $R > a$ and center $O$. Two different cases have been selected to test the proposed techniques:

A) Case of a translating sphere with $O' = O$

The sphere located at the center of the spherical cavity translates (without rotating) at the velocity $v$. Such circumstances have been analytically treated [14]: the sphere experiences no couple and a net force $F = -6\pi \mu a^2 \beta \epsilon$ with $\beta = a/R$ and the following coefficient $c$ by the exact formula

$$c(\beta) = \frac{1 - \frac{\beta^5}{1 + \frac{3\beta^5}{1 - \frac{9\beta^5}{1 + \beta^6}}} \beta = a/R < 1. \quad(21)$$

B) Arbitrarily-located sphere

We set $OO' = d_3$ with $0 \leq d < R - a$. For symmetry reasons it is sufficient to consider four different rigid-body motion for which we give the net force $F$ and torque $\Gamma$ exerted on the sphere. The selected cases are:

(i) A sphere translating at the velocity $e_1: F = -6\pi \mu a^2 \epsilon e_1$ and $\Gamma = 8\pi \mu a^2 \epsilon e_2$,

(ii) A sphere translating at the velocity $e_2: F = -6\pi \mu a^2 \epsilon e_2$ and $\Gamma = 0$,

(iii) A sphere rotating at the velocity $e_1: F = -8\pi \mu a^2 \epsilon e_1$ and $\Gamma = 8\pi \mu a^2 \epsilon e_1$,

(iv) A sphere rotating at the velocity $e_2: F = 0$ and $\Gamma = -8\pi \mu a^2 \epsilon e_2$

where the arising and dimensionless friction coefficients $c_1, c_2, s, e_1$ and $e_3$ not only depend upon $(d/a, R/a)$ but have been also accurately obtained by using the bipolar coordinates [12]. Such results will be compared with our computations at the oral presentation.

### Conclusions

Four different approaches have been proposed to compute the net force and torque experienced by a solid particle migrating in a Newtonian liquid bounded by a closed, solid and motionless cavity. This is achieved by determining the surface traction arising on the particle’s boundary which also permits one, exploiting a well-know integral representation, to calculate the liquid flow in the entire cavity. Contrary to previous works in the field, the advocated methods hold in essence for arbitrarily-shaped particle and cavity. Two techniques require to solve a boundary-integral equation whereas the other ones are new procedures free from any boundary-integral equation and thereby of easier and faster numerical implementation. In order to compare the methods in terms of accuracy the special case of spherical particle and cavity, for which very accurate results obtained by the method of bipolar coordinates are available [12], is handled. The oral presentation will successively report and discuss comparisons for such circumstances and also give new results for a solid ellipsoid moving in a spherical cavity.
References


High-Order Spectral Elements for the Integral Equations of Time-Harmonic Maxwell Problems

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Abstract. We present a novel, high-order, Method-of-Moment (MoM) with interpolatory vector functions, on quadrilateral patches. The main advantage of this method is that the H\textsubscript{div} conforming property is enforced, meanwhile it can be interpreted as a point-based scheme. Our approach consists in a specific choice of Degrees Of Freedom (DOF), made in order to fulfill a fast integral evaluation. We apply this method to Field Integral Equations (FIE) to solve time-harmonic electromagnetic scattering problems. We also discuss the hybrid Integral Equation - Finite Element Method (IE-FEM), for simulations performed on coated scatterers.

Introduction
Field integral representation methods allow an electromagnetic field to be expressed in terms of its Cauchy data: the electric and magnetic current densities. The discretization is then performed on the boundary of the scatterer. A drawback of this method is that one has to solve a full matrix system. Even if fast evaluation techniques can be used to speed up an iterative solver, reducing the number of unknowns is a challenge which still remains.

It is well known that high-order schemes provide a significant reduction of the number of unknowns and yield more accurate solutions. As the Method-of-Moment (MoM) with Galerkin testing results in optimal convergence for scattering problems, one should try to expand MoM low-order basis functions into their higher-order form ([1]). Unfortunately, this choice increases dramatically the matrix fill time, thus rendering classical high-order MoM useless for large scatterers. The problem lies in an overpriced evaluation of the double integral, for every term of the impedance matrix.

By contrast, high-order point-based discretizations, such as the Nyström method ([2][3]), excel with their low precomputation time. However, these schemes are commonly not H\textsubscript{div} conforming, and line charges may appear on patch boundaries with the Maxwell problem.

Various authors have developed high-order MoM which can be interpreted as point-based methods too ([4][5][6][7]). The main idea consists in choosing specific Degrees Of Freedom (DOF), defined by Lagrange polynomials whose roots are some quadrature points (spectral element method). It results in fast integral evaluation since both testing and basis functions cancel on these. Nevertheless, the frequently used Gauss quadrature rule does not lead to edge functions, thus enforcing the H\textsubscript{div} property becomes problematical.

The purpose of our talk is to present such a Galerkin/point-based scheme, with H\textsubscript{div} conforming elements. The key component, introduced in [8][9][10] for volumic methods, lies in the use of Gauss-Lobatto quadrature rules which lead to edge functions. Therefore, the H\textsubscript{div} conforming property is easily enforced.

In the following sections, we first adapt the spectral element method to the first family of Nédélec H\textsubscript{div} conforming elements ([11]). Next, we compare this formulation to other spectral schemes ($L^2$ functions, Gauss quadrature rules) to state on the advantages/drawbacks of our method. We then describe slight modifications which lead to a robust discretization scheme, with fast matrix fill time and accurate solutions, even on scatterers with sharp edges. Finally, we expand our approach to the...
hybrid Integral Equation - Finite Element Method (IE-FEM), with high-order anisotropic hexahedral elements, for simulations performed on coated scatterers.

Hdiv Conforming Spectral Elements

In this section, we adapt the spectral element method to the first Nédélec family of Hdiv conforming elements, on quadrilateral patches. We start with a brief description of the variational formulations we use. Then, we present our DOF and their associated quadrature rules, which yield to a spectral formulation.

Let \((E_{\text{inc}}, H_{\text{inc}})\) be the incident field which illuminates an arbitrarily-shaped object \(\Omega\) of boundary \(\Gamma\) (in \(\mathbb{R}^3\)), and \(\Gamma_h\) a mesh of \(\Gamma\). This field induces electric and magnetic current densities on \(\Gamma\), which radiate and generate a scattered field \((E_{\text{scat}}, H_{\text{scat}})\) in \(\mathbb{R}^3 \setminus \Omega\). For simplicity, we restrict the study to perfectly conducting (PEC) scatterers so that the magnetic current density vanishes. Hence, our unknown is the electric current density \(J\). Let \(J_h\) be the approximation of \(J\) by a linear combination of \(N\) basis functions \(\Phi_n\):

\[
J_h(r) = \sum_{n=1}^{N} I_n \Phi_n(r) \quad \forall r \in \Gamma_h, \quad \text{where} \quad I_n, n = 1, \ldots, N \text{ are the unknown coefficients. (1)}
\]

Variational formulations of Magnetic and Electric Field Integral Equations (MFIE (2) and EFIE (3), respectively) are given by:

\[
\begin{align*}
\frac{1}{2} \int_{\Gamma_h} \Phi_m(r) \cdot \Phi_n(r) d\Gamma_h + \int_{\Gamma_h} \Phi_m(r) \cdot n \times \left( \int_{\Gamma_h} \nabla' \bar{G}(r, r') \times \Phi_n(r') d\Gamma' \right) d\Gamma_h &= \int_{\Gamma_h} \Phi_m(r) \cdot n \times H_{\text{inc}} d\Gamma_h, \quad (2) \\
\int_{\Gamma_h} \int_{\Gamma_h} \bar{G}(r, r') \Phi_m(r) \cdot \Phi_n(r') d\Gamma' d\Gamma_h &= -jk \int_{\Gamma_h} \Phi_m(r) \cdot E_{\text{inc}} d\Gamma_h, \quad (3)
\end{align*}
\]

where \(\bar{G}(r, r') = \frac{e^{-jk|r-r'|}}{4\pi||r-r'||} \cdot \bar{G}(r, r') = [1\delta^2 + \nabla \nabla'] G(r, r'), k\) is the wave number, and \(n\) is the exterior normal to \(\Gamma\). The Dyadic Green’s function \(\bar{G}\) yields a hypersingular integral. Applying Stokes’ theorem, the use of Hdiv conforming functions allows us to handle the double gradient operator, and results in a weakly singular integral. We make use of this process when \(r\) and \(r'\) are close to each other (near interactions). Therefore, we deal with divergence terms of basis and testing functions.

Both EFIE and MFIE have some resonance frequencies. A common way to avoid this problem is to solve the Combined Field Integral Equation (CFIE). The CFIE is obtained by a linear combination of the two previous ones, and doesn’t present any resonance frequencies (for real frequencies):

\[
\text{CFIE} = \alpha \text{EFIE} + (1 - \alpha) \text{MFIE}, \quad \text{with} \quad 0 < \alpha < 1.
\]

We set \(\Gamma_h = \bigcup_{i=1}^{N} \{K_i\}\), where \(K_i\) are quadrilaterals (with straight or curved edges). Let \(\hat{K}\) be the unit square and \(F_i\), the conforming mapping of \(\mathbb{R}^2\), such that \(F_i(\hat{K}) = K_i\). \(DF_i\) is the Jacobian matrix of \(F_i\) and \(J_i = \text{det}(DF_i)\). \((u_1, u_2)\) denotes the \(2D\)-coordinates on \(\hat{K}\), and \((e_1, e_2)\) the canonical basis of \(\mathbb{R}^2\). We have

\[
\forall r \in \Gamma_h, \exists K_i \subset \Gamma_h, \exists (u_1, u_2) \in \hat{K}, \quad \text{s.t.} \quad r = F_i(u_1, u_2). \quad (4)
\]

Let \(\Phi \in \{\Phi_n\}_{n=1}^{N}\) be defined on \(K\) (for simplicity, indexes \(i\) are suppressed in following equations).
As we use Lagrange polynomials to expand our vector function $\Phi$, it reads:

$$\Phi_{\mathcal{K}}(r) = \frac{1}{|J(u_1, u_2)|} \mathbf{DF}(u_1, u_2) \cdot \mathbf{\hat{D}}(u_1, u_2) = L_1(u_1) L_2(u_2) \mathbf{e}_\mu, \quad \mu = 1, 2$$

(5)

where $L_{1,2}$ are the Lagrange polynomials, and $\frac{1}{|J|} \mathbf{DF}$ is the local $H_{\text{div}}$ conforming isomorphism.

We introduce two sets of Lagrange polynomials complete to $p$th order ($1D$), and a space on $\mathbf{\hat{K}}$:

- the set of Lagrange polynomials whose roots are the $p+1$ Gauss quadrature points: $G^p = \{g^p_i\}_{i=1}^{p+1}$,
- the set of Lagrange polynomials whose roots are the $p+1$ Gauss-Lobatto quadrature points: $GL^p = \{g^{\text{GL}}_i\}_{i=1}^{p+1}$,
- $\mathbf{\hat{U}}^p = (GL^{p+1} \otimes G^p) \mathbf{e}_1 \oplus (G^p \otimes GL^{p+1}) \mathbf{e}_2$.

Our approximate space for the first Nédélec family of $H_{\text{div}}$ conforming elements is:

$$U^p = \left\{ \Phi \in H_{\text{div}}(\Gamma); \forall \mathcal{K} \in \Gamma_h, \Phi_{\mathcal{K}} \in \frac{1}{|J|} \mathbf{DF} \mathbf{\hat{U}}^p \right\}.$$  

(6)

The Gauss-Lobatto quadrature rule gives us DOF on the edges, shared with the neighbour elements, so that the $H_{\text{div}}$ conforming property is easily implemented (notice that functions in $U^0$ correspond to Rooftop functions).

Now let's define our specific quadrature rules: $\{(\xi, \omega)^{p+1}_{\text{GL}}\}$ denotes the set of points and weights for the $p$th order Gauss-Lobatto quadrature rule ($\{(\xi, \omega)^{p}_{\text{G}}\}$ for the Gauss one). We have:

$$g^{\text{GL}}_i(\xi^p_{\text{GL}}) = \delta^p_{i, q}, \quad g^p_i(\xi^p_{\text{G}}) = \delta^p_{i, q}, \quad 1 \leq i, q \leq p+1.$$  

(7)

Thus, choosing $\{(\xi, \omega)^{p+1}_{\text{GL}}\} \otimes \{(\xi, \omega)^{p}_{\text{G}}\}$ within $GL^{p+1} \otimes G^p$ polynomials, and $\{(\xi, \omega)^{p+1}_{\text{GL}}\} \otimes \{(\xi, \omega)^{p}_{\text{G}}\}$ within $G^p \otimes GL^{p+1}$, leads to a fast evaluation of the impedance matrix: each double integral estimation requires only one pair of quadrature points (far interactions), instead of $(p+2)^4$ with classical $p$-order MoM.

**Establishing a Robust Discretization Scheme with Spectral Elements**

Enforcing the $H_{\text{div}}$ conforming property, with spectral elements, involves the use of Gauss-Lobatto quadrature rules in direction of current’s flow. However, these rules are not as accurate as Gauss ones: the Gauss quadrature rule with $p+1$ points is exact for polynomials of degree $2p+1$, whereas the Gauss-Lobatto one is exact for $2p-1$ (1D). Hence, replacing $(p+2)^2$ Gauss points (classical $p$-order MoM) with $(p+2) \otimes (p+1)$ Gauss-Lobatto and Gauss points, respectively, provides an increase of the error estimate with small values of $p$ ($p = 0, 1, 2$). When simulations are performed on scatterers with sharp edges, or on coarse meshes (less than 5 points per wavelength), higher-order approximations ($p = 3, 4, 5,...$) also suffer from the error increase.

In order to relieve the constraint on quadrature choice, one should make use of $L^2$ functions with DOF associated to Gauss points. This set is characterized by (8) and (9). Nevertheless, line
charges may appear on patch boundaries, thus yielding accurate solutions on scatterers with sharp edges becomes problematic too.

\[
\hat{V}_p = (G^{p+1} \otimes G^p) e_1 \oplus (G^p \otimes G^{p+1}) e_2
\]

\[
V_p = \left\{ \Phi \in L^2(\Gamma) : \forall K \in \Gamma_h, \Phi|_K = \frac{1}{|J|} \text{DP} \hat{V}_p \right\},
\]

\[
\{(\xi, \omega)^{p+1}_q \otimes \{(\xi, \omega)^p_q\} \text{ and } \{(\xi, \omega)^p_q \otimes \{(\xi, \omega)^{p+1}_q\} \text{ quadrature points.}
\] (8)

(9)

To improve precision, we have to incorporate the quadrature set (9) with the Hdiv conforming set of functions (6). This combination is called semi-spectral, since roots of polynomials in $GL^{p+1}$ do not match their associated quadrature points anymore. Of course, it involves a slower pre-computation time: $(p+2)^2$ points are required to estimate each term instead of one.

Obviously, there is a trade-off between speed and accuracy in the choice of quadrature rules, with Hdiv conforming functions. We choose to combine these rules, with regards to the geometry discretization:

- the semi-spectral form is restricted to functions on sharp edges, or on coarse elements,
- the spectral form is employed with remaining functions.

As a result, matrix fill time is reduced and accuracy is ensured. Consequently, the number of unknowns can be minimized (4-5 points per wavelength), as illustrate in Fig.1.

Figure 1: Cobra cavity, 5GHz, polar Phi ($\theta_{inc} = 110, \phi_{inc} = 20$): $|\text{Re}(J_h)|$ (top) and bistatic Radar Cross Section (bottom).
The Hybrid Integral Equation - Finite Element Method
A few years ago, M. Duruflé and G. Cohen have developed spectral methods for FEM, applied to the time-harmonic Maxwell problem ([9][10]). We’ve employed the same procedure to define our set of Hdiv-conforming functions (6), thus expanding our study to IE-FEM (with quadrilateral and hexahedral elements) do not present any difficulty. However, when the coating is thin compared to the wavelength, the use of high-order hexahedrons leads to an over-discretized thickness. To overcome this problem, we’ve implemented high-order anisotropic hexahedral elements: $p$-order in tangential directions and $r$-order in normal direction, with $r \leq p$ (Fig. 2).

![Figure 2: coated scatterer.](image)

As an example, we’ve run simulations on a sphere (1GHz), coated with a uniform dielectric coating: $R_{\Gamma_0} = 0.564$, $R_{\Gamma} = 0.594$, $\varepsilon_r = 4.0$. The coating acts like a waveguide, thus yielding accurate results with low-order methods is difficult, in comparison with the anisotropic spectral method (Fig. 3).

![Figure 3: Comparison between classical low-order functions and high-order spectral methods on the coated sphere.](image)

Conclusion
The spectral element method has been expanded to the first Nédélec family of Hdiv conforming functions. We’ve discussed the drawbacks of the method, and slight modifications have been introduced to ensure precision and fast pre-computation time. Finally, our study has been extended to IE-FEM, with spectral anisotropic finite elements, for simulations performed on coated scatterers. Significant decrease of the number of unknowns has been observed through numerical examples.
References


A BE-Topology Optimization Method Enhanced by Topological Derivatives

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Abstract. The BE-based topology optimization using the level set method is enhanced by introducing the topological derivatives. To avoid the dependency on initial topology, and to reduce the computational cost, holes are nucleated inside the structural domain based on the derivatives. In this study the topological derivative is formulated based on the boundary element equations. A hole creating method in which a large hole is given by a contour line of the topological derivative is proposed. Application to a numerical example is demonstrated and the performance of the approach is discussed.

Introduction

The topology optimization can be a useful tool to find an optimal layout in the conceptual design process. During the past two decades, various finite element (FE) methods[1,2,3] have been developed for the topology optimization. These FE-based topology optimization methods are realized by introducing the Eulerian mesh. Because of this, even by the level set method, the accuracy of boundary values such as displacement and stresses may be deteriorated.

Since, in the boundary element (BE) analysis, the boundary of a structure is defined explicitly, the method provides higher accuracy than the FE-based optimization methods. Therefore, the BE-based topology optimization method can be a beneficial means.

The authors[4] have developed a topology optimization method using the BEM. In the method the level set function is used to capture the topological change. Since the all coordinates of element junctions are used as design variables, the material layout is to be released from any geometrical restrictions. Despite of this advantage, no hole can be nucleated in a simply connected region as in the FE-based level set approach. Hence, the topological evolution should proceed from a multi-connected region. While the increase of the number of initial holes accelerates the topological convergence, since the number of boundary elements is also increased, it spends more computational cost. Besides of this, the final layout strongly depends on the initial topology. Therefore, it is desirable to create holes without any intuitive approaches.

To solve these issues, in this study the enhancement of the BE-based topology optimization method is attempted by the aid of the topological derivatives[5]. The rate of an objective function due to the growth of an infinitesimal hole is evaluated inside the structural domain. To achieve this, the topological derivative is formulated within the framework of the BE analysis. In this study the shape of hole is given by a contour line of the topological derivative. Once the domain is opened by the holes, the topological evolution can be driven by the advective motion of the level set function. Through a numerical example, the validity of the developed method is discussed.

Topology Optimization Based on the Level Set Method

In this section the outline of the topology optimization using the level set method developed in [4] is described.

Optimization Problem. Let us consider an optimization problem for two-dimensional elastostatics
defined by

$$
\min_{X_0} J(W; X_0) := F(W; X_0) + [\lambda]^T \{ HU - GP \} + \lambda_i (V - V_{max}),
$$

subject to $[\lambda]^T \{ HU - GP \} = 0, \{ U \} = \{ U \} \text{ on } \Gamma_u, \{ P \} = \{ P \} \text{ on } \Gamma_p \forall (\lambda),

$$
\lambda_i (V - V_{max}) = 0, \quad \lambda_i \geq 0.
$$

where $J$ is the objective function, $F$ is a cost function which will be minimized in the original optimization problem, $\{ \lambda \}$ is a vector of Lagrange multiplier, $[H]$ and $[G]$ are matrices of boundary element equations, $\{ U \}$ and $\{ P \}$ are nodal displacement and traction vectors, these are prescribed on sub-boundaries $\Gamma_u$ and $\Gamma_p$, respectively. $\lambda_i$ is a Lagrange multiplier, $V$ is the volume of the structure and $V_{max}$ is an allowable volume limit. $\{ X_0 \}$ is the design variable vector consisting of the coordinates of boundary element junctions. $\{ W \}$ is a vector given by unknown parts of $\{ U \}$ and $\{ P \}$.

**Design Sensitivity Analysis.** The variation of $J$ due to the change of configuration $\{ \Delta X_0 \}$ is expressed as

$$
\Delta J = [\frac{\partial F}{\partial W}] \{ \Delta W \} + [\frac{\partial F}{\partial X_0}] \{ \Delta X_0 \}
$$

$$
+ [\lambda]^T \{ [\frac{\partial H}{\partial X_0} \cdot \Delta X_0][U] - [\frac{\partial G}{\partial X_0} \cdot \Delta X_0][P] + [\lambda][\Delta W] \} + \lambda_i [\frac{\partial V}{\partial X_0}] \{ \Delta X_0 \},
$$

where $[\lambda]$ is the solving matrix for the boundary element equation. $\partial / \partial W$ stands for the Fréchet derivative. $\{ \Delta W \}$ is the change of boundary element solution resulted from the geometrical change $\{ \Delta X_0 \}$.

In order to eliminate the terms associated with $\{ \Delta W \}$, the following adjoint equation is introduced:

$$
[\lambda]^T \{ \lambda \} = -[\frac{\partial F}{\partial W}].
$$

Once the Lagrange vector $\{ \lambda \}$ is obtained from eq(3), eq(2) can be recast by

$$
\Delta J = [\beta + \lambda_i \frac{\partial V}{\partial X_0}] \{ \Delta X_0 \},
$$

where

$$
[\beta]^T \{ \Delta X_0 \} = [\frac{\partial F}{\partial X_0}] \{ \Delta X_0 \} + [\lambda]^T \{ [\frac{\partial H}{\partial X_0} \cdot \Delta X_0][U] - [\frac{\partial G}{\partial X_0} \cdot \Delta X_0][P] \}.
$$

**Topology Optimization by Level Set Method.** The shape optimization can be conducted by the velocity at each boundary element junction obtained from eq(4):

$$
\{ V_0 \} = -[\beta + \lambda_i \frac{\partial V}{\partial X_0}].
$$

However, to achieve the topology optimization using the level set function, the velocity field should be extended to the design domain. The level set function $\psi$ is defined as a signed distance function from the boundary in which $\psi > 0$ in the material region. The structural configuration is given by the zero contour lines of $\psi$. In order to define the level set function, a grid is introduced into a design domain. The level set function is assigned at each fixed grid point(Fig.1). Once the contour lines are drawn on the background grid, the element junctions are equidistantly distributed along the contour. The boundary element discretization is then achieved by connecting the element junctions with each other. The change of the shape and topology arises from the advection of the level set function governed by the Hamilton-Jacobi equation

$$
\frac{\partial \psi}{\partial t} = -\nabla \cdot \nabla \psi,
$$
where \( v \) denotes a convective velocity.

In order to update \( \psi \), the velocity field should be assigned to the grid points locating near the boundary. However, as mentioned above, the velocity obtained from eq(6) is distributed only on the boundary. The velocity at each grid point is produced by the projection of normal velocity on its nearest contour segment.

Since the level set function \( \psi \) is defined by a distance function, \(|\nabla \psi| = 1\) holds for eq(7). Consequently, the level set function at the \( n + 1 \)th step can be updated by

\[
\psi^{n+1}_i = d_i + v_{G_i} \Delta t,
\]

where \( \psi^{n+1}_i \) and \( v_{G_i} \) are the level set function and the velocity at the \( i \)th grid point, respectively. \( d_i \) is the distance from the grid point to the nearest segment. \( \Delta t \) is time increment, and is determined adaptively at each time step so that the numerical stability can be assured.

Once the volume \( V \) reaches the allowable limit, it should be kept at \( V_{\text{max}} \). The Lagrange multiplier \( \lambda_i \) which meets this condition is given by

\[
\lambda_i = -\left[ \frac{\partial V}{\partial X_k} \right]^{-1} \left[ \frac{\partial V}{\partial X_k} \right] - \lambda_i.
\]

**Topological Derivatives for Boundary Element Equation**

**Topological Derivatives.** The topological derivative is originally defined by the sensitivity of the cost function when a small hole is introduced in the domain[6]. However, due to the mathematical difficulties of the comparison between before and after the hole nucleation, the new definition has been proposed by Novotny et al.[5]. In their paper the topological derivative is given by the rate of cost function resulted from the growth of an infinitesimal hole. Since, in this case, the topological change is not considered, the derivative can be discussed within the framework of the shape sensitivity analysis.

**Formulation of Topological Derivatives.** Consider an elastic body with a small hole of radius \( \rho \) centered at \( \bar{x} \). We assume that the topological derivative \( D_T \) can be regularized for a cost function \( F \) by[7]

\[
D_T(x) = \lim_{\rho \to 0} \frac{F(W + \Delta W; \rho + \Delta \rho) - F(W; \rho)}{\Delta V_P},
\]

where \( \Delta V_P \) is the change in volume due to the introduction of a small hole of radius \( \rho \).
where $V_0$ is the volume of the hole, therefore, $\Delta V_0 = 2\pi \rho \Delta \rho$. Since the numerical solution is obtained by BE equations, the sensitivity should be derived in the context of BE analysis. The boundary integral equation at a smooth boundary on $\Gamma = \Gamma_u + \Gamma_p$, is given by

$$\frac{1}{2} \mathbf{u} = - \int_{\Gamma} \mathbf{p}^* \cdot \mathbf{u} \, d\Gamma + \int_{\Gamma} \mathbf{u}^* \cdot \mathbf{p} \, d\Gamma - \int_{\Gamma_p} \mathbf{p}^* \cdot \mathbf{u} \, d\Gamma,$$

where $\mathbf{u}^*$ and $\mathbf{p}^*$ are the fundamental solutions, and $\mathbf{p} = 0$ is assumed on the boundary of hole $\Gamma_p$.

The discretization of eq(11) leads to the boundary element equation,

$$[\mathbf{H}][\mathbf{U}] - [\mathbf{G}][\mathbf{P}] + \{\mathbf{P}\} = 0,$$

where $\{\mathbf{P}\}$ is a vector corresponding to the last term of the right-hand side of eq(11).

The objective function $J$ is defined as

$$J(W; \rho) = F(W; \rho) + [\lambda]^T [\mathbf{H} \mathbf{U} - \mathbf{G} \mathbf{P} + \mathbf{P}].$$

Notice that, since the contribution of hole to the volume is independent of its position, in the evaluation of the topological derivative the consideration of the volume is not essential.

The variation $\Delta J$ due to the increment $\Delta \rho$ is given by

$$\Delta J = \frac{\partial F}{\partial W} \{\Delta W\} + [\lambda]^T \{\mathbf{A}\} \{\Delta W\} + \{\Delta \mathbf{P}\}.$$  

In eq(14), $\partial F/\partial \rho=0$ is assumed. The terms associated with $\{\Delta W\}$ can be eliminated by the aid of eq(3). The topological derivative is then obtained by

$$D_T(\mathbf{x}) = [\lambda]^T \{\mathbf{P}\} = \lim_{\Delta \rho \to 0} \frac{\{\Delta \mathbf{P}\}}{\Delta \rho}.$$  

**Evaluation of $\{\mathbf{P}\}$**. The fundamental solution $\mathbf{p}^*$ is given by $\mathbf{\sigma}^* \cdot \mathbf{n}$, where $\mathbf{\sigma}^*$ and $\mathbf{n}$ are the stress tensor of the fundamental solution and the unit outward normal vector, respectively. $\mathbf{\sigma}^*$ and $\mathbf{u}$ can be expressed in terms of $\rho$ as

$$\mathbf{\sigma}^* = \sigma_0^* + \rho \Sigma^* + \cdots, \quad \mathbf{u} = \mathbf{u}_0 + \rho \mathbf{U},$$

where $\sigma_0^*$ and $\mathbf{u}_0$ are quantities evaluated at $\mathbf{x}$. Using eq(16) and taking the limit of eq(15), we obtain the following equation:

$$\{\mathbf{P}\} = \frac{1}{\pi} \sigma_0^* \int_0^{2\pi} \mathbf{n} \otimes \mathbf{U} \, d\theta.$$  

$U[8]$ in eq(16) is given by the displacement on a unit circular hole embedded in an infinite body subjected to the uniform stress corresponding to the one at $\mathbf{x}$. The integration in eq(17) can be derived as

$$\int_0^{2\pi} \mathbf{n} \otimes \mathbf{U} \, d\theta = \frac{\pi (1 + \chi)}{\delta \mu} [[\mathbf{tr}\mathbf{\sigma}]I - 4\mathbf{\sigma}],$$

where $I$ and $\mu$ are the identity tensor and the shear modulus, respectively. $\chi$ is defined by

$$\chi = \begin{cases} 3 - 4\nu, & \text{(plane strain)} \\ 3 - \nu, & \text{(plane stress)} \end{cases}.$$
Numerical Example

Minimum Compliance Problem. The developed method is applied to the minimum compliance problem. The cost function for this problem is given by

$$ F = \int_{\Gamma_p} \mathbf{p} \cdot \mathbf{u} \, d\Gamma, $$

(20)

where $\Gamma_p$ consists of sub-boundaries of $\Gamma_u$, on which non-zero traction is prescribed. In the following, it is assumed that the geometry of $\Gamma_p$ is fixed during the optimization process. Therefore, the variation $\Delta F$ is expressed by

$$ \Delta F = \int_{\Gamma_p} \mathbf{p} \cdot \Delta \mathbf{u} \, d\Gamma. $$

(21)

Creation of Holes. In many topology optimization methods using the topological derivatives, the topological evolution is achieved by opening small holes gradually. Since these holes are located in some regions having smaller topological derivatives for minimization problems, the accumulation of them will trace a contour line of the distribution of the derivative. Hence, it can be expected that the creation of a large hole in the shape of a contour line may lead to a similar result. Our method is capable to implement this strategy without any difficulty. The large hole can be created by utilizing the topological derivative as a level set function. Even if an oversize hole is made, the detail of shape and the topology will be restored during the topology optimization process conducted by the level set method.

Analytical Conditions. A rectangular cantilever beam of $70h \times 36h$ is considered. Here $h$ is the grid size. The boundary is discretized by piecewise constant elements with the size of about $h$. As shown in Fig.2, the left side is fixed and a vertical load $P$ is subjected at the middle of the right-hand side. In the BE analysis the load $P$ is distributed on two elements. The initial region is also used as the design domain for the layout optimization. The plane strain condition with the Poisson’s ratio of 0.3 is assumed. The volume restriction is set by $V_{\text{max}} = 0.4V_0$, where $V_0$ is the initial volume.

Topology Optimization of a Cantilever Beam. Fig.3 is showing the optimization process in the earlier stage and the topology at the 1500th step. The optimal shape given in [4] can be reproduced by the present method. The creation of holes is achieved every 20 step unless the volume $V$ reaches the limit $V_{\text{max}}$. The shape of hole is given by a contour line which corresponds to 60% of the mean value of $D_F$. Figures on the left side are showing the distribution of the topological derivative. The right figures are shapes just after the creation of hole. From these results it can be found that, though the shape convergence needs a number of iterations, the final topology has already been formed at the 40th step. We confirmed that the sensitivity of the final shape to the parameters such as the frequency of creation of holes and the level of contour is weak. Through these experiments and application to other examples, the robustness of the developed method was evidenced.
Conclusions

The enhancement of the BE-based topology optimization using the level set method was attempted by means of the topological derivatives. These quantities are formulated within the framework of the boundary element analysis. Holes are adaptively nucleated along a contour line of the topological derivative. Once the holes are opened, the optimization is led by the successive level set method. Therefore, it can be recognized that, in the proposed method, the role of topological derivative is to produce early topology. Application to a minimum compliance problem proved the capability of the method.

References
Simulation of transient flows in highly heterogeneous media with the flux Green element method

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Abstract
In many geological formations, rapidly varying medium properties are encountered. In certain instances the variability of the medium properties is of significant orders of magnitude to cause numerical difficulties to conventional numerical schemes because of the sharp discontinuity in the gradient of the potential or pressure. The flux-based Green element method (GEM), that implements the singular boundary integral equations in an element-by-element manner, is used to simulate transient flow in heterogeneous media where changes of several orders of magnitude of the hydraulic conductivity are encountered. The formulation is flexible to accommodate either type 1 heterogeneity in which there is spatial distribution of the medium properties or type 2 in which abrupt changes of medium properties exist at certain boundaries. Only examples of type 2 heterogeneity are addressed in this paper; those of type 1 have earlier been addressed. The current formulation is tested on two numerical examples, and it is observed that high accuracy is achieved with coarse discretization of the domain.

1. Introduction
Previous Green element formulations have been applied to transient potential flow problems in which the medium parameters depend on the primary or dependent variable (nonlinear potential problem) and/or have a functional distribution with respect to space (type 1 heterogeneous problem) [1-4]. This paper addresses type 2 heterogeneous problems in which the medium parameters exhibit sharp discontinuities at certain parts of the medium and also referred to as zoned problems. Boundary element simulations of steady flows in zoned media have previously been solved [5]. Lorincki et al. [6] had solved the steady state form of this problem with a flux-based GEM which retains only three unknowns (the primary variable, and fluxes in \( x \) and \( y \) directions) at the internal nodes. Although problems solved were handled with rectangular elements, the authors demonstrated how triangular elements can be incorporated into the formulation. Here another flux based GEM formulation is used in solving the heterogeneous problem with sharp discontinuities in the medium parameter. It is a more direct approach that retains the primary variable and all normal directional fluxes at the internal nodes. Although problems solved were handled with rectangular elements, the authors demonstrated how triangular elements can be incorporated into the formulation. Here another flux based GEM formulation is used in solving the heterogeneous problem with sharp discontinuities in the medium parameter. It is a more direct approach that retains the primary variable and all normal directional fluxes at the internal nodes. Although problems solved were handled with rectangular elements, the authors demonstrated how triangular elements can be incorporated into the formulation. Here another flux based GEM formulation is used in solving the heterogeneous problem with sharp discontinuities in the medium parameter. It is a more direct approach that retains the primary variable and all normal directional fluxes at the internal nodes. Whereas it generates a lot more unknowns at the internal nodes, its high accuracy using coarse grids makes it quite attractive. The compatibility equation at an internal node shared by elements with different conductivity values is derived and used in resolving the closure problem at such nodes. Its derivation enables the evaluation of the integral of the normal fluxes over a closed circular contour centred at the internal node, taking into consideration that the fluxes are continuous across boundaries with sharp discontinuity in medium properties. Two examples of transient potential flows in zoned media are solved with the current formulation.

2. Flow equation and flux GEM
The equation that governs transient potential flow in a heterogeneous media is given by

\[
\nabla \cdot (K \nabla h) = \frac{\partial h}{\partial t} + f
\]

(1)
where $V$ is the 2-D gradient operator, $h$ is the potential, $K$ is the hydraulic conductivity which is homogeneous in each zone but exhibits sharp discontinuities at the inter-zonal boundaries of the flow domain, $t$ is time and $f$ is a forcing term. The usual boundary conditions of the Dirichlet, Neumann and Cauchy types are accommodated at different sections of the external boundaries of the medium, while $h$ is prescribed everywhere in the domain at the initial time. Because $K$ is homogeneous in each zone, Eq. (1) can be treated as a diffusion equation of the form

$$K\nabla^2 h = \frac{\partial h}{\partial t} + f$$  (2)

Compatibility conditions for $h$ and its normal flux are admitted at the inter-zonal boundaries. Applying Green’s theorem to Eq. (2) gives the integral equation commonly encountered in boundary element circles.

$$-2Kh = \left[ (Kh \nabla \cdot n + Gq)_ds + \frac{\partial h}{\partial t} \right] dA = 0$$  (3)

where $G = \ln(r - r_i)$ is the fundamental solution to the Laplacian operator, $q = -K\nabla h \cdot n$, the subscript $i$ denotes the source point $r_i = (x_i, y_i)$ and $\delta$ is the nodal angle at $r_i$ that is obtained from a Cauchy integration of the Dirac delta function about the source point. The integral equation is implemented in a subdomain or element that is used to discretize $\Lambda$, while the quantities $h$ and $q$ are interpolated by Lagrange-type interpolation functions. The discrete form of Eq. (3) in a subdomain or element is

$$R_i h_i + L_i q_i + W_i \left( \frac{\partial h_i}{\partial t} + f_i \right) = 0$$  (4)

where the indices take values equivalent to the number of nodes in the element, and

$$R_i = K \left[ N_i \nabla G(r, r_i) \cdot n \right] ds + \delta_j \delta_i$$

$$L_i = \int N_i G(r, r_i) ds,$$

$$W_i = \int G(r, r_i) N_i dA.$$  (5)

$N_j$ is the interpolation function with respect to node $j$ of the element. All the element integrations in Eq. (5) are evaluated analytically for the 4-node linear and 8-node quadratic rectangular elements, and as well as the 3-node linear and 6-node quadratic elements. Achieving the exact integrations by GEM is largely due to the fact that the source node is always located in the element over which integration proceeds. Approximating the time derivative with a difference expression $\frac{dh}{dt} = (h^{(n+1)} - h^{(n)}) / \Delta t$ (where $\Delta t = t_2 - t_1$), using the generalized finite difference scheme with weighting factor $\theta \in [0, 1]$ and aggregating the discrete element equations for all the elements used in discretizing the domain produces a matrix equation of the form

$$V \phi^{(2)} = S_i$$  (6)

where $\phi^{(2)} = [h^T, q^T]^T = [h(t_2), q(t_2)]^T$ is a mixed vector of unknowns. The matrix Eq. (6) is readily solved to produce the solution for $h$ and $q$ at all nodal points at the current time $t_2$. Thus
far, the formulation has proceeded in the usual way with previous flux GEM applications [1-3]. For nodes located at the interface of two zones with abrupt changes in hydraulic conductivity $K$, the compatibility relations for the potential and the flux are applied [5]. What is new in this work is the treatment given to internal nodes located where more than two zones of different $K$ values meet. The equation that is applicable for such nodes is presented in the subsequent section.

4. Compatibility equation at internal nodes where more than 2 zones meet

This situation is illustrated for a general case where 5 zones meet. The number of generated integral equations falls short by one the number of unknowns, resulting in a closure anomaly. This anomaly is resolved by introducing an additional equation for the normal directional fluxes meeting at the node. At the same time it should be recognized that the $K$ values for the 5 zones are different. We introduce a circle $C$ centred at the internal node $P$ of infinitesimally small radius $R$ that defines a surface $S$. It is observed that the normal directional fluxes at the segments emanating from the internal nodes are tangential to the circle (Figure 1a). The integral of the normal directional fluxes around this circle is given by

$$\oint_{C} q ds = M \oint_{C} \frac{n h K h K q}{n} ds$$

Where $M$ for the illustrated case is 5. Using the expression for the normal directional flux, $q = -Kvh \cdot n = -Kvh / \partial n$, Eq. (7) becomes

$$\oint_{C} q ds = -\sum_{j=1}^{M} K_j \oint_{C_j} \frac{\partial h}{\partial n} ds$$

Noting that the normal $n$ on the segments of the elements is also the tangential direction $s$ on the circle $C$, Eq. (8) becomes

$$\oint_{C} q ds = -\sum_{j=1}^{M} K_j \oint_{C_j} dh$$

Since $h$ is continuous at the node, then

$$\oint_{C} q ds = -\sum_{j=1}^{M} K_j \oint_{C_j} dh = 0$$

The numerical implementation of Eq. (10) follows:

$$\oint_{C} q ds = R \sum_{j=1}^{M} \oint_{\beta_j} q d\beta = 0$$

where $\beta$ is the included angle at the internal node and $\beta_{i+1} = \beta_i$. Since the radius $R$ is a constant, it is factored out, so that Eq. (11) becomes

$$\sum_{j=1}^{M} \oint_{\beta_j} q d\beta \Delta \beta_j = 0$$
Where \( \overline{q}_j = (q_j + q_{j+1})/2 \) is taken as the average flux in the \( j^{th} \) zone within the included angle \( \Delta \beta_j = \beta_{j+1} - \beta_j \). It is this additional Eq. (12) that is implemented in resolving the closure problem at an internal node where more than two zones meet.

As in previous flux GEM formulations, the complete solution information is made available for each element after solving the global matrix equation (6). That allows for the calculation of the solution at any point to involve only integrations on the element in which the point is located. The preclusion of other elements from the integration process greatly enhances the efficiency of the formulation. The direct calculation of the normal fluxes with an interpolation distribution that is of the same order as that of the primary variable enhances the formulation’s accuracy but gives rise to an escalation in the number of unknowns. The enhanced accuracy allows for the use of coarse grids (demonstrated in the next section) that compensates for the increased number of unknowns.

Figure 1: Establishing the compatibility equation at an internal node where more than 2 zones meet.

3. Simulated Examples

The current flux GEM is applied to two examples. These two examples had been solved with a steady flux GEM formulation proposed by Lorinci et al. [6]. Here the first example is solved in a transient manner with the current formulation, while in both cases much fewer elements are used because of the high accuracy associated with the current formulation.

3.1 Example 1

In this example, discontinuities of hydraulic conductivity are of the order of magnitude of \( 10^4 \). The domain is rectangular \([0,1] \times [0,2]\) with four zones with hydraulic conductivity values: \( K_1 = 1 \) for the zone \([0,0.5] \times [0,1]\), \( K_2 = 10^{-2} \) for the zone \([0.5,1] \times [0,1]\), \( K_3 = 10^{-4} \) for the zone \([0,0.5] \times [1,2]\) and \( K_4 = 10^{-1} \) for the zone \([0.5,1] \times [1,2]\). There is no forcing term, so \( f = 0 \). The example is simulated as a transient problem by considering the potential \( h \) to be everywhere zero initially, and thereafter suddenly raised to unit value at \( y = 0 \). The boundary conditions are:

\[
q(0, y, t) = q(1, y, t) = 0, \quad 0 \leq y \leq 2
\]
\[
h(x,0,t) = 1, \quad h(x,2,t) = 0, \quad 0 \leq x \leq 1
\]

A uniform time step \( \Delta t = 10 \) and the fully implicit scheme are adopted in the simulations. The domain is discretized into \( 8 \times 16 \) linear rectangular elements. The simulation is carried out till steady state is achieved. This was achieved when the time is about 2000. The results at time of 400
is presented in Fig. 2a while the steady state solution is presented in Fig. 2b. The steady state result agrees with that of Lorinci et al. [6] which used 64 × 128 linear rectangular elements or 64 times more number of elements than employed in the current formulation. As expected in the results presented in Figs. 2a and 2b, the higher hydraulic conductivity at the lower half of the medium allows flow to quickly proceed into the medium so that steady state is more quickly attained than in the upper half with lower hydraulic conductivity.

3.2 Example 2
In this example, two faults are in a rectangular domain [0,1] × [0,2]. Faults are common features in geological formations and they are shear planes when tectonic movements occur. Each fault is 0.75 long and 10⁻³ wide. The first fault is located at [0.25,1] × [0.4995,0.5005] and the second at [0,0.75] × [1.4995,1.5005]. The domain has a uniform hydraulic conductivity value of unity, while two cases are simulated with the faults having hydraulic conductivity values: \( K_f = 10^{-3} \) and \( K_f = 10^{-4} \). The boundary conditions are:

\[
q(0, y) = q(1, y) = 0, \quad 0 \leq y \leq 2
\]
\[
h(x,0) = 1, \quad h(x,2) = 0, \quad 0 \leq x \leq 1
\]

The steady state simulation of the problem is carried with 8 × 14 linear rectangular elements, and the results for the two cases presented in Figs. 3a and 3b. The steady state result for the first case agrees with that of Lorinci et al. [6] which used 16 × 34 linear rectangular elements. The constriction to the flow by the faults is observed by the steep gradient in the potential across the faults, and this more pronounced for the case with \( K_f = 10^{-4} \).

4. Conclusion
A flux GEM formulation has been applied to transient flow problem in heterogeneous media in which abrupt changes of several orders of magnitude of the hydraulic conductivity are encountered.
It calculates the potential and normal fluxes at every grid point using the same order of interpolating functions for both quantities. The closure anomaly at internal nodes where more than 2 zones of different hydraulic conductivity meet is resolved by an additional equation that is derived in this paper. It is derived in a general way such that it is independent of whether or not one uses regular or irregular grids. The accuracy of the formulation is evident by the coarse grid that is used to achieve accuracy comparable to those from other methods which use much finer grid. The coarseness of the discretization makes up for the large number of unknowns generated by the formulation due to the direct calculation of the normal fluxes at every grid point.

Figure 3: Contour of potential for medium with 2 faults, (a) $K_f = 10^{-3}$, (b) $K_f = 10^{-4}$.

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6. References
A procedure for 2-D elastostatic analysis of functionally graded materials by the Boundary Element-Free Method

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Abstract. A technique based on the coupling of standard boundary integral equations (BIE) and meshless interpolation schemes is proposed for 2-D elastostatic analysis applied to functionally graded materials (FGM). FGM are non-homogenous materials in which some properties (e.g. density, Young modulus, Poisson rate, etc.) vary as a function of spatial coordinates. Here, the numerical method adopted is the boundary element-free method (BEFM), which is a meshless technique based on BIE, such as the Local Boundary Integral Equation (LBIE) method and the Boundary Node Method (BNM), differing from them with respect to the integration domain and the approximation scheme.

Introduction

Conventional computational mesh-based methods, such as Finite Element Method (FEM) and Boundary Element Method (BEM), are not well suited for a certain number of problems involving remeshing processes, like large deformation analysis and crack propagation.

To deal with this drawback, mesh-free counterparts of well-established numerical formulations have been developed. A series of so-called Meshless Local Petrov-Galerkin (MLPG) methods [1] has been developed in recent years for a great number of engineering applications. From the classical Boundary Integral Equation (BIE) formulation derives the Local Boundary Integral Equation (LBIE) method [2], which has been recently applied to potential problems with discontinuities [3], fracture mechanics problems [4] and non-homogeneous material analysis [5].

Other BIE-based meshless methods are the Boundary Node Method (BNM) [6] and the Boundary Element-Free Method (BEFM) [7]. Basically, they differ from LBIE in the chosen boundaries of integration, which in BNM and BEFM are the real boundary of the problem and in LBIE are circular (2-D) or spherical (3-D) sub-boundaries inside the global boundary, which characterizes LBIE as a domain method. In all three methods, the interpolation scheme used is based on least-squares approximations.

In BNM and LBIE, the scheme used is the so-called Moving Least-Squares (MLS) [8] and in BEFM is used the Orthogonal Moving Least-Squares (OMLS), also called Improved Moving Least-Squares (IMLS) [7]. Both schemes are mesh-independent and the main difference between MLS and OMLS is the basis functions used in each one. In MLS the basis is composed by a set of complete linear independent monomial functions whereas in OMLS such functions are orthogonal polynomials.

One of the recent sources of research for meshless applications is the analysis of Functionally Graded Materials (FGM’s) [9]. FGM’s are non-homogenous materials in which some properties (density, Young modulus, Poisson rate, etc.) vary as a function of spatial coordinates. This characteristic is very interesting for industry in general. Several FGM analyses have been proposed such as LBIE for heat conduction [10], elasticity [5] and viscoelasticity [11], among others.

In this work a complete meshless technique based on BEFM for 2-D elastostatic analysis applied to FGM’s is proposed. The mathematical basis, computational implementation and some illustrative examples comparing results with other well-established techniques are presented.

BEM formulation for non-homogeneous analysis

For the solution of general elastostatic functionally graded material problems by the BE technique, an alternative boundary integral equation, not directly derived from the partial differential equation, can be obtained through weighted residual procedures or in the light of simple reciprocal statements as seen in [12]. Herein, only its final form is shown (body forces are omitted for simplicity),
where Ω represents the domain of the body, Γ its boundary and \( c_{ij} \) is the usual free coefficient found in elastic analysis. The following notation is used:

- \( u_j, p_j \) and \( \sigma_{ij} \): displacement, traction and strain components at point \( x \) due to a unit concentrated load applied in “i” direction at point \( \xi \) (homogeneous fundamental solution).

- \( u_j, p_j \) and \( \sigma_{ij} \): displacements, tractions and fictitious “correcting” stresses that compensate for the variation of functional properties of the problem to be solved.

It is worth mentioning that eq.(1) is meant to be valid for general 3D and 2D problems, provided subscripts are assumed to vary between 1-3 and 1-2 respectively.

Since in the case of FG’s the constitutive equations are functions of the position within the body, the computation of actual internal stresses is of great importance in the solution procedure. Therefore, the derivatives of eq.(1), written for \( \xi \in \Omega \), can be combined to represent the internal stresses in the form

\[
\sigma_{ij}(\xi) = \int_\Omega [u_j(\xi,x)p_j(x) + p_j(\xi,x)u_j(x) + \int_\Omega \epsilon_{ij}^p(\xi,x)\sigma_{ij}^p(x) dx] \, dx
\]

where the last two terms introduce the fictitious correcting stress influence. It should be mentioned that the derivatives of the domain integral of eq.(1) need careful evaluation and generate a Cauchy principal value integral (third integral on the right) together with a free term represented by the coefficient \( g_{ij} \).

Two-dimensional OMLS approximation scheme

Just like in non-linear inelastic analysis, all boundary methods based on BIE (e.g., BEM [12], BNM [6], BEFM [7]), require that certain variable values must be determined within the domain of the problem for the complete solution (in certain applications, this computation can be carried out with the aid of reciprocity procedures). To deal with this, two kinds of interpolation are commonly used: domain interpolation, which is made based on the same dimension of the problem, and boundary interpolation, which is parameterized in one dimension less than the problem.

The scheme adopted in this work to approximate boundary and domain variables is the so-called orthogonal moving least-squares (OMLS), also found in literature as improved moving least-squares (IMLS) [7]. This scheme is based on a standard moving least-squares method (MLS), differing only on the basis used. In MLS, the basis is composed by a complete set of linear independent monomial functions

\[
p_j(\xi,x) = \left[ 1, \gamma(\xi), \gamma(\xi)^2, \ldots, \gamma(\xi)^{m-1} \right], \quad j = 1, 2, \ldots, m-1
\]

for boundary interpolation (\( m \) defines the order of the interpolation – linear, quadratic, etc.; \( \gamma(\xi) \) is the 1-D parameterized co-ordinate of node \( i \)) and

\[
p_j^\tau(\xi_i,x) = \left[ 1, \gamma_i, y_i \right] \quad \text{for linear basis (} m = 3 \]

\[
p_j^\tau(\xi_i,x) = \left[ 1, \gamma_i, y_i, x_i, x_i y_i, x_i^2, y_i^2 \right] \quad \text{for quadratic basis (} m = 6 \]

for domain interpolation, where \( x_i \) is the real 2-D coordinates of node \( i \), \( \gamma_i = x_i - x_i^\tau \).

The basis used in OMLS is a complete set of weighted orthogonal polynomial functions that can be generated from the basis in eq.(3) by the Schmidt method as follows

\[
p_i^\tau = \frac{1}{\sum_{j=1}^{m-1} (p_i, p_j)} p_j, \quad i = 2, 3, \ldots, m
\]

where \( (p_i, p_j) \) denotes the weighted inner product as follows [7]
If the set of polynomial functions is orthogonal then
\[
(p_i, p_j) = \begin{cases} 
0 & \text{if } i \neq j \\
\lambda_{ij} & \text{if } i = j 
\end{cases} 
\]

The above interpolation leads to an approximation scheme that neither deals with fictitious quantities nor requires further post-processing of solutions, which are inconveniences of the MLS. This procedure is the same for boundary and domain interpolation.

The OMLS approximation has the following form for displacements
\[
\mathbf{u}_i(\mathbf{x}) = \mathbf{\Phi}(\mathbf{x}) \cdot \mathbf{u}_i \\
\mathbf{u}_j(\mathbf{x}) = \mathbf{\Phi}(\mathbf{x}) \mathbf{u}_j, \quad j = 1, 2, ..., n
\]

where \(\mathbf{u}_i\) is the vector containing the nodal values of displacement in \(i\) direction, \(n\) is the number of nodes that contribute to the interpolation at node \(i\) (i.e., the number of nodes at the domain of definition of node \(i\)) and \(\mathbf{\Phi}\) is the shape function of OMLS. The exact same procedure is followed for traction interpolation.

\[
\mathbf{\Phi}(\mathbf{x}) = \mathbf{p}^T(\mathbf{x}) \mathbf{X}(\mathbf{x}) \mathbf{B}(\mathbf{x})
\]

where
\[
\mathbf{X}(\mathbf{x}) = \mathbf{P}^T \mathbf{W}(\mathbf{x}) \mathbf{P}
\]

\[
\mathbf{B}(\mathbf{x}) = \mathbf{P}' \mathbf{W}(\mathbf{x})
\]

\[
\mathbf{p} = \begin{bmatrix} 
\mathbf{p}'(\mathbf{x}_1) \\
\mathbf{p}'(\mathbf{x}_2) \\
\vdots \\
\mathbf{p}'(\mathbf{x}_n) 
\end{bmatrix}
\]

\[
\mathbf{W}(\mathbf{x}) = \begin{bmatrix} 
w_1(\mathbf{x}) & 0 & \ldots & 0 \\
0 & w_2(\mathbf{x}) & \ldots & 0 \\
\vdots & \ldots & \ldots & \vdots \\
0 & \ldots & 0 & w_n(\mathbf{x}) 
\end{bmatrix}
\]

\(w_i(\mathbf{x})\) is the weight function, here chosen to be a Gaussian distribution function, as follows
\[
w_i(\mathbf{x}) = \begin{cases} 
e^{-\left(\frac{d_i}{c_i}\right)^2} & \text{for } 0 \leq d_i \leq r_i \\
e^{-\left(\frac{d_i}{c_i}\right)^2} & \text{for } d_i \geq r_i
\end{cases}
\]

\(d_i = \|\mathbf{x} - \mathbf{x}_i\|, c_i\) is a constant that controls the shape of \(w_i\), \(r_i\) is the size of the support of \(w_i\) associated with \(\mathbf{x}_i\) (see Fig.1 for boundary interpolation and Fig.2 for domain interpolation) and \(k\) is a parameter here chosen as 1. There are many other functions suitable for use in MLS approximation, like cubic and quadric spline functions [1,13], but for many applications found in the literature, the best results were obtained with the Gaussian function. The domain of definition of a node \(i\) is composed by all nodes that contain it inside their supports.

Figure 1: Support of node \(i\) for boundary interpolation
Because of the basis orthogonality, matrix $\mathbf{A}(\mathbf{x})$ is diagonal, being naturally well conditioned and very easy to invert. In MLS, matrix $\mathbf{A}(\mathbf{x})$ does not have this property and sometimes its conditioning is found poor [7]. The normal strain along the boundary, $\varepsilon_{\gamma\gamma}(\mathbf{x})$, can be calculated as follows

$$ \varepsilon_{\gamma\gamma}(\mathbf{x}) = \frac{\partial \mathbf{B}(\mathbf{x})}{\partial \gamma}(\mathbf{x}) \mathbf{u}_i $$

where $\mathbf{u}_i$ contains the value of the displacements in the direction of the boundary and the derivative term is

$$ \frac{\partial \mathbf{B}(\mathbf{x})}{\partial \gamma}(\mathbf{x}) = \frac{\partial \mathbf{B}(\mathbf{x})}{\partial \gamma}(\mathbf{x}) \mathbf{X}^{-1}(\mathbf{x}) \mathbf{B}(\mathbf{x}) + \mathbf{p}(\mathbf{x}) \frac{\partial \mathbf{X}^{-1}(\mathbf{x})}{\partial \gamma}(\mathbf{x}) \frac{\partial \mathbf{B}(\mathbf{x})}{\partial \gamma}(\mathbf{x}) $$

in which

$$ \frac{\partial \mathbf{X}^{-1}(\mathbf{x})}{\partial \gamma}(\mathbf{x}) = -\mathbf{X}^{-1}(\mathbf{x}) \frac{\partial \mathbf{X}(\mathbf{x})}{\partial \gamma}(\mathbf{x}) \mathbf{X}^{-1}(\mathbf{x}) $$

Spatial discretization

A node cloud is spread over the boundary $\Gamma$ and domain $\Omega$ of the problem. The boundary is divided in sub-boundaries $\Gamma_n, n = 1, 2, \ldots, N$ ($N$ is the total number of sub-boundaries (see Fig.3)), each connecting two neighbour boundary nodes. It is important to mention that the OMLS shape functions do not depend on them; their only purpose is to define the boundary geometry for numerical integration.
only in the form of determination of the normal strains along the boundary, which is here carried out according to eq. (13).

Substituting the displacements, tractions and fictitious stresses by their respective interpolated expressions in Eqs. (1) and (2) lead to

\[ e_{ij}(\xi) = \sum_{j=1}^{n_1} \sum_{k=1}^{n_2} \int_{\Gamma} \left( e_{ijkl}(\xi, x) \right) \phi_j(x) \left( \sigma_{ij}(x) \right) \right) \, d\Gamma(x) \]

(16)

\[ \sigma_{ij}(\xi) = \sum_{j=1}^{n_1} \sum_{k=1}^{n_2} \int_{\Gamma} \left( e_{ijkl}(\xi, x) \right) \phi_j(x) \left( \sigma_{ij}(x) \right) \right) \, d\Gamma(x) \]

(17)

where \( n_1 \) and \( n_2 \) indicate the number of nodes in the domain of definition of \( \xi \) respectively, for boundary and domain interpolation and \( (\cdot)_{\xi} \) and \( (\cdot)_{\alpha} \) are the boundary and domain OMLS shape functions. The matrix forms of eqs. (16) and (17) are, respectively

\[ H_u = Gp + Q\sigma^n \]

(18)

and

\[ \sigma = G'p - H'u + Q'^*\sigma^n \]

(19)

where matrices \( H, H', Q, \) etc. are classical boundary element matrices. Note that matrix \( Q' \) also includes the contributions of the free coefficient \( g_{ij} \).

After the application of the displacement and traction boundary conditions, eqs. (18) and (19) can be written as

\[ Ay = f + Q\sigma^n \]

(20)

and

\[ \sigma = -A'\sigma + f' + Q'^*\sigma^n \]

(21)

Equation (20) can then be solved for the boundary unknowns included in vector \( y \)

\[ y = K\sigma^n + m \]

(22)

where \( m \) represents the homogeneous material solution to the boundary problem. Substituting (22) in (21) and rearranging,

\[ \sigma = Sm^n + n \]

(23)

in which vector \( n \) represents the solution in terms of stresses for the homogeneous medium and

\[ K = A'^*Q \]

\[ S = Q' - A'K \]

\[ m = A'^*f \]

\[ n = f' - A'm \]

Non-homogeneous material modelling

Consider that the Young modulus of a certain functionally graded material \([9]\) varies as a function of spatial coordinates as

\[ E(x) = E^* - E\hat{E}(x) \]

(24)

where \( E \) is a constant and \( E\hat{E}(x) \) is the part that varies with the spatial coordinates. Hence, the real stresses can be written as

\[ \sigma(E(x)) = \sigma^*(E) - \sigma^*(E\hat{E}(x)) \]

(25)
where $\sigma'$ is the standard or “reference” stress obtained considering only $\sigma_v$ and $\sigma'$ is the correcting part calculated with $E(x)$. If $\sigma$ contains the real strains, then

$$\sigma = D(\varepsilon') \sigma'(E)$$

where $\sigma$ is the elastic tensor that relates the real strains with the reference stresses.

The relation between the real stresses and strains is as follows

$$\sigma(E(x)) = C(E(x))\varepsilon$$

where $C(E(x))$ is the space varying operator calculated with $E(x)$.

Substituting eq.(26) in (27) leads to

$$\sigma(E(x)) = C(E(x))D(\varepsilon')\sigma'(E)$$

and considering Eq.(25), the following relation arises

$$\sigma'(E(x)) = \left[1 - C(E(x))D(\varepsilon')\right] \sigma'(E)$$

which is the relation between the correcting stresses and the reference or standard stresses involved in the solution process.

The reference stresses can be computed if eq.(23) is modified as follows

$$\sigma'(S + I) = n$$

and substituting eq.(29) in (30) leads to

$$\sigma'(S + I)\left[1 - C(E(x))D(\varepsilon')\right] \sigma'(E) + n$$

Hence, rearranging

$$B\sigma'(E) = n$$

where matrix $B$ is inherently well-conditioned, due to its very nature, as seen below

$$B = I - (S + I)\left[1 - C(E(x))D(\varepsilon')\right]$$

After solving the system of equations defined in eq.(32), the real stresses can be calculated using eq.(28) and the strains using eq.(26).

**Examples**

**Elastic strip.** The first example is defined as the elastic strip depicted in Fig.4. It was previously analyzed in [11] and is subjected to a uniform unitary load on one edge and presents $x$-displacements restricted on another.

![Figure 4: Example 1: Elastic catulled strip in tension.](image)

In this example $\nu = 0.0$ and the Young modulus varies as follows

$$E(x) = E_0 e^{\gamma x}$$

Here, $E_0 = 1.0$ and $\gamma = 0.2$ in the first analysis and $\gamma = 0.4$ in the second. The node cloud used has 52 boundary nodes and 57 domain nodes. The comparison of the displacements $u_x$ of the nodes lying originally in $x = 3.0$ (the loading edge) obtained with this technique and the one presented graphically in [11] is shown in Table 1, in which the difference in percentage is computed with reference to the LBIE results.

<table>
<thead>
<tr>
<th>$\gamma$ values</th>
<th>LBIE</th>
<th>BEFM</th>
<th>variation</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>2.251</td>
<td>2.255</td>
<td>0.17%</td>
</tr>
<tr>
<td>0.4</td>
<td>1.738</td>
<td>1.746</td>
<td>0.40%</td>
</tr>
</tbody>
</table>
Square plate. Consider the square plate under uniform unitary loading applied over the horizontal edges, as seen in Fig.5, with $v = 0.35$ and Young modulus defined as follows ($E = 1.23 \times 10^5$):

$$E(x) = 1.23 \times 10^5 \cdot \left(1 + 0.5x_1^2\right)$$

The node cloud used in this problem was composed by 44 boundary nodes and 81 domain nodes.

![Figure 5: Example 2: Square plate in tension.](image)

The results obtained by BEFM technique are compared with the presented in [5]. Three results are analyzed: $u_1$ and $u_2$ displacements along the right vertical edge (Figs.6a and 6b) and $\sigma_{22}$ stress along the superior horizontal edge (Fig.6c).

![Figure 6: (a) $u_1$ displacements; (b) $u_2$ displacements; (c) $\sigma_{22}$ stress](image)

Conclusions

In the present work, a complete meshless technique based on BEFM for 2-D elastostatic analysis applied to FGM’s is proposed. Because of their thermal and mechanical characteristics, FGM’s are very interesting for industry and the research on their applications is far from ending since new applications are continuously being developed for a number of engineering applications.

As indicated by the results, the technique described is seen to be suitable for non-homogeneous media analyses. In addition, the development of new FGM applications with BEFM, as well as using other approaches like LBIE, for inelastic material behaviour, like plasticity, is currently under way and will be the object of future publications.

References


A comparative study of two time-domain BEM for 2D dynamic crack problems in anisotropic solids

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Keywords: Hypersingular time-domain BEM, anisotropic elastic solids, dynamic stress intensity factors, Galerkin-method, collocation method, convolution quadrature

Abstract. This paper presents two time-domain boundary element methods (TDBEM) for dynamic crack analysis in two-dimensional (2D), homogeneous, anisotropic and linear elastic solids subjected to an impact loading. A combination of the classical displacement boundary integral equations (BIEs) and the hypersingular traction BIEs is applied. The spatial discretization is performed by a Galerkin-method in both cases. A collocation method is adopted for the temporal discretization in the first TDBEM, while the convolution quadrature of Lubich is implemented for the temporal discretization in the second TDBEM. An explicit time-stepping scheme is developed to compute the unknown boundary data and the crack-opening-displacements numerically.

Introduction

This paper presents a transient dynamic crack analysis for 2D, homogeneous, generally anisotropic and linear elastic solids. Stationary cracks in both infinite and finite solids under impact loading are considered. Two different time-domain BEM are developed and compared. A combination of the strongly singular displacement BIEs and the hypersingular traction BIEs is applied in both TDBEM. On the external boundary of the cracked solid the strongly singular displacement BIEs are used, while on the crack-faces the hypersingular traction BIEs are implemented. In the first TDBEM, time-domain elastodynamic fundamental solutions for anisotropic solids derived by Wang and Achenbach [4] via Radon transform are implemented, while the Laplace-transformed elastodynamic fundamental solutions for anisotropic solids given by Wang and Zhang [5] are used in the second TDBEM. To solve the time-domain BIEs numerically, an explicit time-stepping scheme is developed. The first TDBEM uses a collocation method for the temporal discretization. By using linear temporal shape-functions, time integrations in the system matrices can be carried out analytically. The second TDBEM uses the convolution quadrature of Lubich [2,3] for the temporal discretization. The spatial discretization is performed by the Galerkin-method in both TDBEM. Strongly singular and hypersingular boundary integrals are computed by special analytical and numerical techniques. To describe the local behavior of the crack-opening-displacements (CODs) at the crack-tips properly, square-root crack-tip shape-functions are applied for elements on the crack-faces near the crack-tips, while linear shape-functions are chosen for all other boundary elements. The arising line-integrals in the elastodynamic fundamental solutions over a unit circle are computed numerically by standard Gaussian quadrature. Several numerical examples for computing transient elastodynamic stress intensity factors (SIFs) are presented and discussed to verify and compare the accuracy, the stability and the efficiency of the two different time-domain BEM.

Problem formulation and time-domain BIEs

Let us consider a homogeneous, anisotropic and linear elastic solid with a crack of arbitrary shape. In the absence of body forces, the cracked solid satisfies the equations of motion
Hooke’s law

\[ \sigma_{\alpha\beta}(x,t) = \rho \ddot{u}_{\alpha}(x,t), \]

the initial conditions

\[ u_{\alpha}(x,t) = 0 \quad \text{for} \quad t \leq 0, \]
\[ t_{\alpha}(x,t) = 0 \quad \text{for} \quad t \leq 0, \]

and the boundary conditions

\[ t_{\alpha}(x,t) = \Gamma_{\alpha}(x,t), \quad x \in \Gamma_{b}, \]
\[ u_{\alpha}(x,t) = \Gamma_{u}(x,t), \quad x \in \Gamma_{u}. \]

Here, \( u, \sigma, t = \sigma_{\alpha\beta}\epsilon_{\alpha\beta} \) represent the displacement, the stress and the traction components, \( \epsilon_{\alpha\beta} \) is the outward normal vector, \( \rho \) is the mass density, \( C_{\alpha\beta\gamma\delta} \) is the elasticity tensor, \( \Gamma_{\alpha} \) denotes the upper and the lower crack-faces, \( \Gamma_{b} \) and \( \Gamma_{u} \) stand for the external boundaries where the tractions \( t \) and the displacements \( u \) are prescribed. A comma after a quantity represents spatial derivatives while a dot over a quantity denotes time differentiation. Greek indices take the values 1 and 2, while Latin indices take the values 1, 2 and 3. Unless otherwise stated, the conventional summation rule over repeated indices is implied.

In the sense of the weighted residual, the time-domain displacement Galerkin-BIEs for a cracked solid can be written as

\[ \int_{\Gamma_{b}} \psi(x) u_{\alpha}(x,t) \, d\Gamma = \int_{\Gamma_{b}} \psi(x) \left[ \int_{\Gamma_{c}} u_{\alpha}^{G}(x,y,t) * t_{\alpha}(y,t) \, d\Gamma_{c} \right] \, d\Gamma \]

\[ + \int_{\Gamma_{b}} \psi(x) \left[ \int_{\Gamma_{c}} u_{\alpha}^{G}(x,y,t) * \Delta u_{\alpha}(y,t) \, d\Gamma_{c} \right] \, d\Gamma, \]

where \( u_{\alpha}^{G}(x,y,t) \) and \( t_{\alpha}^{G}(x,y,t) \) are the elastodynamic displacement and traction fundamental solutions, \( \psi(x) \) is the test function, \( \Delta u_{\alpha} \) are the crack-opening-displacements, \( \Gamma_{c} = \Gamma_{b} + \Gamma_{u} \), and an asterisk denotes the Riemann convolution which is defined by

\[ g(x,t) * h(x,t) = \int_{0}^{t} g(x,t - \tau) h(x,\tau) \, d\tau. \]

The time-domain traction Galerkin-BIEs can be obtained by substituting eq (7) into Hooke’s law (2), taking the limit process \( x \rightarrow \Gamma_{c}^{+} \) and considering the boundary conditions (5)-(6)

\[ \int_{\Gamma_{b}} \psi(x) t_{\alpha}(x,t) \, d\Gamma = \int_{\Gamma_{b}} \psi(x) \left[ \int_{\Gamma_{c}} v_{\alpha}^{G}(x,y,t) * t_{\alpha}(y,t) \, d\Gamma_{c} \right] \, d\Gamma \]

\[ + \int_{\Gamma_{b}} \psi(x) \left[ \int_{\Gamma_{c}} w_{\alpha}^{G}(x,y,t) * \Delta u_{\alpha}(y,t) \, d\Gamma_{c} \right] \, d\Gamma, \]

where \( v_{\alpha}^{G}(x,y,t) \) and \( w_{\alpha}^{G}(x,y,t) \) are the traction and the higher-order traction fundamental solutions, which are defined by

\[ v_{\alpha}^{G}(x,y,t) = -C_{\alpha\beta\gamma\delta}(x) u_{\beta}^{G}(y), \]
\[ w_{\alpha}^{G}(x,y,t) = C_{\alpha\beta\gamma\delta}(x) C_{\gamma\rho\delta\epsilon}(y) u_{\beta\rho}(x,y,t). \]

The displacement BIEs (7) are strongly singular, while the traction BIEs (9) are hypersingular. The elastodynamic fundamental solutions for homogenous, anisotropic and linear elastic solids derived by Wang and Achenbach [4] are implemented in the first TDBEM, while the Laplace-transformed elastodynamic fundamental solutions for anisotropic solids derived by Wang and Zhang [5] are used in the second TDBEM. It should be noted here that both elastodynamic fundamental solutions cannot be given in closed-forms in contrast to that for homogeneous, isotropic and linear elastic solids. In 2D case, they can be represented by line-integrals over a unit circle. Both the time-domain and the Laplace-transformed fundamental solutions can be divided into singular static and regular dynamic parts. The singular static parts of the elastodynamic fundamental solutions can be reduced to an explicit expression and correspond to the elastostatic fundamental solutions. The regular dynamic parts can be given only as line-integrals over a unit circle.
Numerical solution of the time-domain BIEs

To solve the strongly singular displacement BIEs (7) and the hypersingular traction BIEs (9), two different numerical solution methods are implemented. In both TDBEM, the Galerkin-method is applied for the spatial discretization. At the crack-tips, special crack-tip elements are used to describe the local behavior of the CODs properly. This ensures an accurate and direct calculation of the dynamic stress intensity factors from the numerically computed CODs. Strongly singular and hypersingular boundary integrals are computed analytically and numerically by special techniques. The temporal discretization is performed by a collocation method in the first TDBEM. By using linear temporal shape-functions, time integrations can be performed analytically. In the second TDBEM, the convolution quadrature of Lubich [2,3] is implemented for the temporal discretization, which requires the Laplace-transformed instead of the time-domain fundamental solutions. This special feature distinguishes the second TDBEM from the first TDBEM and the classical time-domain BEM. The line-integrals over the unit circle arising in the elastodynamic fundamental solutions are computed numerically by using standard Gaussian quadrature formula. The essential features of the two TDBEM are summarized in Table 1.

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Table 1: Two different TDBEM

After temporal and spatial discretizations and invoking the initial conditions (3) and (4), an explicit time-stepping scheme can be obtained as

\[ \mathbf{A}^k \mathbf{u}^k = \mathbf{B}^k \mathbf{t}^k + \sum_{k=1}^{K} (\mathbf{B}^{k-k+1} \mathbf{t}^k - \mathbf{A}^{k-k+1} \mathbf{u}^k), \]  

(12)

where \( \mathbf{A}^k \) and \( \mathbf{B}^k \) are the system matrices, \( \mathbf{u}^k \) is the vector containing the boundary displacements and the CODs, and \( \mathbf{t}^k \) is the traction vector for the external boundary and the crack-faces. By considering the boundary conditions (5) and (6), eq (12) can be rearranged as

\[ \mathbf{x}^k = (\mathbf{C})^{-1} \left[ \mathbf{D} \mathbf{y}^k + \sum_{k=1}^{K} (\mathbf{B}^{k-k+1} \mathbf{t}^k - \mathbf{A}^{k-k+1} \mathbf{u}^k) \right], \]  

(13)

in which \( \mathbf{x}^k \) represents the vector with the unknown boundary data, while \( \mathbf{y}^k \) denotes the vector with the prescribed boundary data. Eq. (13) is an explicit time-stepping scheme for computing the unknown boundary data including the CODs time-step by time-step.

Numerical results

In the explicit time-stepping scheme, the accuracy and the stability of the TDBEM are dependent on the used time-step. The following relation between the temporal and the spatial discretization is introduced to assess the stability of the present TDBEM

\[ \lambda = \frac{c_{\text{max}}^2 N}{l}, \quad c_L = \sqrt{C_{11}/\rho}, \quad c_T = \sqrt{C_{66}/\rho}. \]  

(14)

In eq (14), \( l \) represents the element-length, \( c_{\text{max}} \) is the larger one of the wave velocities \( c_L \) and \( c_T \), and \( \rho \) is the mass density. For isotropic solids it is often reported in the literature that good results can be obtained by using \( \lambda = 1 \).

To compare the two time-domain BEM, several numerical examples are investigated. For convenience, the following normalized dynamic SIFs are introduced

\[ K_{1i}(t) = K_{1i}(t)/\sigma_0 \sqrt{na}, \quad K_{2i}(t) = K_{2i}(t)/\sigma_0 \sqrt{na}, \]  

(15)

where \( \sigma_0 \) is the loading amplitude.

As first example, we consider a finite crack of length 2a in an infinite, homogeneous, anisotropic and linear elastic solid as depicted in Fig. 1. The crack is subjected to an impact tensile crack-face loading of the form...
\( \sigma(t) = \sigma_{22} H(t) \), where \( \sigma_{22} \) is the loading amplitude and \( H(t) \) is the Heaviside step function. A Graphite-epoxy composite with 65\% graphite and 35\% epoxy is considered, which has the following elastic constants

\[
C_{ij} = \begin{bmatrix}
155.43 & 3.72 & 3.72 & 0 & 0 & 0 \\
3.72 & 4.96 & 0 & 0 & 0 & 0 \\
16.34 & 0 & 0 & 0 & 0 & 0 \\
3.37 & 0 & 0 & 0 & 0 & 0 \\
7.48 & 0 & 0 & 0 & 0 & 0 \\
7.48 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}_{\text{sym}} \quad \text{GPa}
\]

(16)

and the mass density \( \rho = 1600 \text{kg/m}^3 \). The elasticity matrix (16) is rotated with the angle \( \theta = 30^\circ \) around \( x_3 \)-axis to obtain a general material anisotropy. The crack is divided into 20 elements. The effects of the time-step on the stability and the accuracy of the two different TDBEM for a fixed spatial discretization are investigated. Numerical calculations are carried out for the following normalized time-steps: \( c \Delta t/a = 0.04, 0.06, 0.08, 0.1, 0.16, 0.2, 0.3 \) and 0.4. According to eq (14) these values lead to \( \lambda = 0.4, 0.6, 0.8, 1, 1.6, 2, 3 \) and 4 respectively. Plane strain condition is assumed.

The normalized dynamic SIFs obtained by the first time-domain BEM (TDBEM 1) and the second one (TDBEM 2) are presented in Fig. 2 and Fig. 3 for different values of \( \lambda \).

Stable numerical results for the normalized mode-I and mode-II dynamic SIFs are obtained and they show a good agreement for all \( \lambda \)-values. The maximum value of \( K(t) \) provided by TDBEM 2 is slightly higher. The effects of the numerical damping for large time-steps can be clearly recognized by \( K_{II}(t) \) in Fig. 2. A choice of the normalized time-steps \( c \Delta t/a = 0.06 \) corresponding to \( \lambda = 0.6 \) leads to unstable numerical results in TDBEM 1. In contrast, TDBEM 2 is stable for all considered values of \( \lambda \). Therefore TDBEM 2 is more stable than TDBEM 1.
In the second example, let us consider a finite crack of length 2a in a finite rectangular plate as shown in Fig. 4. The plate is subjected to an impact tensile loading of the form \( \sigma(t) = \sigma_0 H(t) \).

The geometry of the cracked plate is determined by \( h=20\,\text{mm}, \quad 2w=h \) and \( 2a=4.8\,\text{mm} \). The same elastic constants as given in eq (16) are assumed. To show the influence of the material anisotropy on the dynamic SIFs, numerical calculations are carried out for the rotation angles \( \theta = 0^\circ, 30^\circ, 60^\circ \) and \( 90^\circ \). The spatial discretization of the external boundary is done by an element-length of 1mm for \( \theta = 0^\circ, 30^\circ, 60^\circ \) and 1.2mm for \( \theta = 90^\circ \), the crack is divided into 12 elements, and a time-step of \( \Delta t=0.22\,\mu\text{s} \) is used. According to eq (14) different \( \lambda \)-values of 0.7, 1, 1.6 and 2.2 are obtained for \( \theta = 0^\circ, 30^\circ \), 60\(^\circ\) and 90\(^\circ\). The normalized dynamic SIFs obtained by both TDBEM and the FEM using ANSYS under plane strain condition are presented in Fig. 5 for the angles \( \theta = 0^\circ, 30^\circ, 60^\circ \) and 90\(^\circ\).
A comparison of the normalized mode-I and mode-II dynamic SIFs obtained by both TDBEM and the FEM shows a very good agreement for all considered angles. For $\theta=0^\circ$ and $\theta=90^\circ$ the mode-II dynamic SIF vanishes, since no shear stress components are induced in the case of an orthotropic material behaviour.

In the last example we consider a rectangular orthotropic plate with an edge crack under an impact tensile loading $\dot{\sigma}(t) = \sigma_0 H(t)$ as shown in Fig. 6. Numerical calculations are carried out for the following geometrical parameters: $2w=20\text{mm}$, $2h=40\text{mm}$ and $2a=4.8\text{mm}$. Orthotropic material properties and plane stress condition are assumed. In particular, the following material constants are taken:

$$
E_1 = 118.3\text{GPa}, \quad E_2 = 54.3\text{GPa}, \quad G_{12} = 8.79\text{GPa}, \quad \nu_{12} = 0.083, \quad \rho = 1900\text{kg/m}^3.
$$

The external boundary is divided into elements with a length $1.5\text{mm}$, the crack is discretized by 20 elements, and a time-step $\Delta t=0.3\mu\text{s}$ is chosen in the numerical calculations. Numerical results for the normalized dynamic SIFs obtained by the two different TDBEM are presented in Fig. 6.

Here again, a comparison of the numerical results provided by both TDBEM with that of García-Sánchez et al. [1] and the FEM results using ANSYS shows a very good agreement.

**Summary**

This paper presents a comparative study of two different time-domain BEM for transient elastodynamic crack analysis in homogeneous, generally anisotropic and linear elastic solids. A combination of the classical displacement BIEs and the hypersingular traction BIEs is applied in both time-domain BEM. A Galerkin-method is used for the spatial discretization, while two different variants for the temporal
discretization are implemented. The first variant uses a collocation method while the second variant applies the convolution quadrature of Lubich for the temporal discretization of the time-domain BIEs. Numerical results for the dynamic stress intensity factors are presented and compared with reference solutions. To compare the stability behavior and the sensitivity with respect to the used time-steps of the two time-domain BEM, comprehensive numerical tests have been performed. Both TDBEM are quite stable and less sensitive to the used time-steps, but TDBEM 2 is more stable than TDBEM 1. On the other hand, TDBEM 2 needs more computational time than TDBEM 1.

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References
Transient Thermal Bending Problem by Local Integral Equation Method

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Abstract. A meshless method based on the local Petrov-Galerkin weak-form is developed to solve plate bending problems described by the Reissner-Mindlin theory under a thermal shock. Uncoupled thermoelasticity is considered here. The Reissner-Mindlin theory reduces the original 3-d thick plate problem to a 2-d problem. Nodal points are randomly distributed over the mean surface of the considered plate. Each node is the center of a circle surrounding this node. In the first step thermal distribution is computed from transient heat conduction equations. In the second step the governing equations for Reissner-Mindlin plate bending theory are solved. A unit test function is used in the local weak-form. Applying Gauss divergence theorem to the weak-form, the local boundary-domain integral equations are derived. The local integral equations (LIEs) have a very simple nonsingular form. The numerical integration of the domain integrals does not give rise to difficulties if the meshless approximation based on the Moving Least-Squares (MLS) method is applied.

Introduction

Plate structural elements are widely used in many engineering structures developed in aircraft, civil and ship engineering. They are often subjected to combinations of lateral pressure and thermal loading. The first attempt to analyze thin plates under a thermal load was given by Marguerre [1]. Investigations dealing with static and dynamic behaviour of isotropic and anisotropic thermoelastic plates have been discussed by Tauchert [2]. Kamiya et al. [3] introduced the boundary integral equation method to analyze thermal bending problems. De Leon and Paris [4] developed boundary element method based on the decomposition of Kirchhoff’s governing equation for plate deflection into a pair of harmonic equations. Tauchert [5] gave a nice overview of thermally induced flexure, buckling and vibration of plates described by the Kirchhoff theory.

Among many thick plate theories available, those by Reissner [6], Mindlin [7], and higher order shear theories [8] are widely accepted and have found applications in many engineering problems. Thermoelastic analyses including transverse shear effects were performed by Das and Rath [9], and Bapu Rao [10]. Reddy and Hsu [11] presented analytical solution for simply supported rectangular cross-ply laminated plates under sinusoidal mechanical load and temperature is assumed to be varying linearly through the thickness. Rolfes et al [12] studied both the transverse shear and normal stresses in laminated plates subjected to mechanical and thermal loads. The analysis is based on the first-order shear deformation theory and the plate is discretized by using a single field displacement finite element model.

Recently, functionally graded materials (FGMs) have been extensively used for engineering structures under a severe thermal load. FGMs are multi-phase materials with the phase volume fractions varying gradually in space, in a pre-determined profile. FGMs possess some advantages over conventional composites because of their continuously graded composition and properties [13]. Due to the high mathematical complexity of the initial-boundary value problems, analytical approaches for the FGM bodies are restricted to simple geometry and boundary conditions. Thus, analyses in FGM demand accurate and efficient numerical methods. Praveen and Reddy [14] analyzed the thermomechanical response of thick plates with continuous variation of properties through the plate thickness. The FEM has been applied to isotropic plates with a simple power law distribution of ceramic and metallic constituents. Vel and Batra
[15] obtained an exact solution for three-dimensional deformations of a simply supported functionally graded rectangular plates subjected to mechanical and thermal loads on its top and bottom surfaces.

In the present paper, the authors have developed a meshless method based on the local Petrov-Galerkin weak-form to solve thermal problems of orthotropic thick plates with material properties continuously varying through the plate thickness. The Reissner-Mindlin theory reduces the original 3-d thick plate problem to a 2-d problem. The Meshless Local Petrov-Galerkin (MLPG) method has attracted much attention during the past decade [16] for many problems of continuum mechanics. Nodal points are randomly distributed over the mean surface of the considered plate. Each node is the center of a circle surrounding this node. Long and Atluri [17] applied the meshless local Petrov Galerkin method to solve the bending problem of a thin plate. The MLPG method has been also applied to Reissner-Mindlin plates under dynamic load by Sladek et al. [18]. Soric et al. [19] have performed a three-dimensional analysis of thick plates, where a plate is divided by small cylindrical subdomains for which the MLPG is applied. Homogeneous material properties of plates are considered in previous papers. Recently, Qian et al. [20] extended the MLPG to 3-d deformations in thermoelastic bending of functionally graded isotropic plates.

In this paper, the weak form of governing differential equations for Reissner-Mindlin plate bending theory with Duhamel-Neumann constitutive equations is written on each subdomain. A unit test function is used in the local weak-form. Applying Gauss divergence theorem to the domain integral, the local boundary-domain integral equations are derived. The meshless approximation based on the Moving Least-Squares (MLS) method is used for all quantities.

**Local boundary integral equations**

Consider an elastic orthotropic plate of constant thickness $h$, with the mean surface occupying the domain $\Omega$ in the plane $(x_1, x_2)$. The plate is subjected to thermal loading with the temperature field $T(x_1, x_2, t)$. The Reissner-Mindlin plate bending theory [6,7] is used to describe the plate deformation. The spatial displacement field, due to transverse loading and expressed in terms of displacement components $u_1$, $u_2$, and $u_3$, has the following form [8]

$$u_1(x_1, x_2, t) = x_1 w_1(x_1, t), \quad u_2(x_1, x_2, t) = x_2 w_2(x_1, t), \quad u_3(x_1, x_2, t) = w_3(x_1, t), \tag{1}$$

where $w_1(x_1, x_2, t)$ and $w_3(x_1, x_2, t)$ represent the rotations around the in-plane axes and the out-of-plane deflection, respectively.

The linear strains are given by

$$\begin{align*}
\varepsilon_{11}(x_1, t) &= x_1 w_{11}(x_1, t), \\
\varepsilon_{22}(x_1, t) &= x_2 w_{22}(x_1, t), \\
\varepsilon_{12}(x_1, t) &= (w_{12}(x_1, t) + w_{21}(x_1, t))/2, \\
\varepsilon_{31}(x_1, t) &= (w_{13}(x_1, t) + w_{31}(x_1, t))/2, \\
\varepsilon_{32}(x_1, t) &= (w_{23}(x_1, t) + w_{32}(x_1, t))/2. \tag{2}
\end{align*}$$

In the case of orthotropic materials, the relation between the stress $\sigma_{ij}$ and the strain $\varepsilon_{ij}$ when temperature changes are considered, is governed by the well known Duhamel-Neumann constitutive equations for the stress tensor

$$\sigma_{ij}(x_1, t) = c_{ijkl} \varepsilon_{ij}(x_1, t) - \gamma_0 \theta(x_1, x_2, t), \tag{3}$$

where $c_{ijkl}$ are the material stiffness coefficients. The stress-temperature modulus can be expressed through the stiffness coefficients and the coefficients of linear thermal expansion $\alpha_T$

$$\alpha_T = c_{ijkl} \alpha_T \delta_{ij}. \tag{4}$$

Next, we assume that the material properties are graded along the plate thickness, and we represent the profile for volume fraction variation by

$$P(x_3) = P_0 + (P_b - P_0) V \quad \text{with} \quad V = \left( \frac{x_3}{h} + \frac{1}{2} \right)^n, \tag{5}$$

where $P$ denotes a generic property like modulus, $P_0$ and $P_b$ denote the property of the top and bottom faces of the plate, respectively, and $n$ is a parameter that dictates the material variation profile. Poisson’s ratios are assumed to be uniform.

The bending moments $M_{i\theta}$ and the shear forces $Q_{i\theta}$ are defined as
\[
\begin{bmatrix}
M_{11} \\
M_{22} \\
M_{12}
\end{bmatrix}
= \int_{\Omega} \begin{bmatrix}
\sigma_{n} \\
\sigma_{m} \\
\sigma_{p}
\end{bmatrix} \cdot \left[ \begin{array}{c}
\rho \ddot{w}_n \\
\rho \ddot{w}_m \\
\rho \ddot{w}_p
\end{array} \right] d\Omega,
\]
and
\[
\begin{bmatrix}
Q_{x} \\
Q_{y}
\end{bmatrix}
= \kappa \int_{\Omega} \begin{bmatrix}
\sigma_{n} \\
\sigma_{m}
\end{bmatrix} \cdot \left[ \begin{array}{c}
\rho \ddot{w}_n \\
\rho \ddot{w}_m
\end{array} \right] d\Omega,
\]
(6)
where \( \kappa = 5/6 \) in the Reissner plate theory.

Substituting equations (3) and (2) into moment and force resultants (6) allows the expression of the bending moments \( M_{12} \) and shear forces \( Q_{x} \) and \( Q_{y} \) for \( \alpha, \beta = 1, 2 \), in terms of rotations, lateral displacements of the orthotropic plate and temperature. In the case of considered continuous gradation of material properties through the plate thickness, one obtains

\[
M_{12} = D_{12} \left( w_{x,\alpha} + w_{y,\beta} \right) + C_{12} w_{z,\gamma} - H_{12},
\]
(7)

where

\[
D_{12} = \int_{\Omega} \frac{x_{2} E_{2}(x_{1})}{e} dx_{1}, \quad D_{22} = \int_{\Omega} \frac{x_{1} E_{1}(x_{2})}{e} dx_{2}, \quad D_{11} = \int_{\Omega} \frac{x_{1} x_{2} G_{2}(x_{1},x_{2})}{e} dx_{1} dx_{2}, \quad C_{a} = k \int_{\Omega} G_{a}(x_{1},x_{2}) dx_{1} dx_{2}.
\]

with \( E_{a} \) being the Young’s moduli referring to the axes \( x_{a}, a = 1, 2, \) and \( G_{12}, G_{13}, G_{23} \) are shear moduli, \( \nu_{12} \) Poisson’s ratios and \( e = 1 - \nu_{12} \nu_{21} \).

Thermal processes are relatively slow with respect to elastic waves and stationary Reissner’s equations of motion may be considered [6]

\[
\begin{aligned}
Q_{x,t} &= 0; \\
M_{12,t} - Q_{x} &= 0 \quad \text{or in view of (8a):} \\
\left( M_{12}(x,t) - x_{1} Q_{x}(x,t) \right)_{\beta} &= 0 \quad \text{in} \; \Omega,
\end{aligned}
\]
(8a)

A pure thermal load is considered here. All quantities are time dependent due to variation of temperature in time.

In the MLPG methods the weak-form over local subdomains such as \( \Omega \), is constructed [16]. The local subdomains could be of any geometrical shape and size. In the current paper, the local subdomains are taken to be of circular shape. The local weak-form of the governing equations (8) for \( \chi \in \Omega_{i} \) can be written as

\[
\int_{\Omega_{i}} \left( M_{12}(x,t) - x_{1} Q_{x}(x,t) \right) w_{x}(x) d\Omega = 0,
\]
(9)

\[
\int_{\partial \Omega_{i}} Q_{x,t} w_{x}(x) d\Omega = 0,
\]
(10)

where \( w_{x}(x) \) and \( w_{z}(x) \) are weight or test functions.

Applying the Gauss divergence theorem to Eqs. (9) and (10) and considering unit step functions for the test functions, one obtains the local integral equations

\[
\int_{\partial \Omega_{i}} \left( M_{12}(x,t) - x_{1} Q_{x}(x,t) \right) n_{1} \tau_{1} d\Omega = 0,
\]
(11)

\[
\int_{\partial \Omega_{i}} Q_{x,t} n_{1} \tau_{1} d\Omega = 0,
\]
(12)

where \( \partial \Omega_{i} \) is the boundary of the local subdomain and \( M_{12}(x,t) = M_{12}(x,t) \tau_{1} \).

The trial functions are approximated by the Moving Least-Squares (MLS) method over a number of nodes spread within the domain of influence. The approximated functions for generalized displacements can be written as [16]

\[
\tilde{\mathbf{w}}^{h}(x,t) = \mathbf{w}^{h}(x,t) - \sum_{k=1}^{n} \mathbf{b}^{\nu}(x) \tilde{\mathbf{w}}^{\nu}(t),
\]
(13)
where the nodal values $\hat{w}(t)$ are fictitious parameters, and $\phi'(x)$ is the shape function associated with the node $a$. The number of nodes $n$ used for the approximation is determined by the weight function $v^r(x)$. A 4th order spline-type weight function is applied in the present work.

Substituting the approximation (13) into the definition of the normal bending (7), one obtains

$$
\mathbf{S}_i(x,s) = N_i \sum_{a=1}^n \mathbf{B}'_a(x)w^r(t) + N_i \sum_{a=1}^n \mathbf{B}'_a(x)w^r(t) - \mathbf{P}(x,t) = N_i \sum_{a=1}^n \mathbf{B}'_a(x)w^r(t) - \mathbf{P}(x,t),
$$

(14)

where $\mathbf{P}(x,t) = [P_{11} n_1, P_{22} n_2]^T$, the matrices $N_i(x)$ are related to the normal vector $n(x)$ on $\partial \Omega_s$ by

$$
N_i(x) = \begin{bmatrix} n_1 & n_2 & 0 \\ 0 & n_2 & n_1 \end{bmatrix} \quad \text{and} \quad N_2(x) = \begin{bmatrix} C_{11} & 0 \\ 0 & C_{22} \end{bmatrix} \begin{bmatrix} n_1 & n_2 & 0 \\ 0 & n_2 & n_1 \end{bmatrix},
$$

and the matrices $\mathbf{B}'_a$ are represented by the gradients of the shape functions as

$$
\mathbf{B}'_a(x) = \begin{bmatrix} 2D_{1i} \phi'_i & 0 \\ 0 & 2D_{2i} \phi'_i \\ D_{1i} \phi'_i & D_{2i} \phi'_i \end{bmatrix}, \quad \mathbf{B}'_2(x) = \begin{bmatrix} \phi'_1 & 0 \\ 0 & \phi'_2 \end{bmatrix}.
$$

The influence on the material gradation is incorporated in $C_{11}$ and $D_{11}$. Similarly, one can obtain the approximation for the shear forces term

$$
\hat{w}(t) = G(x) \sum_{i=1}^n \phi'(x)w^r(t) + F'(x)\tilde{\omega}_1'(t),
$$

(15)

where

$$
G(x) = (n_1 C_{11}(x), n_2 C_{22}(x)), \quad F'(x) = \left[ \phi'_1, \phi'_2 \right].
$$

Furthermore, in view of the MLS-approximations for the unknown quantities in the local boundary-domain integral equations (11) and (12), we obtain their discretized forms as

$$
\sum_{a=1}^n \left[ \int_{\Gamma_a} \mathbf{N}_i(x) \mathbf{B}'_a(x) d\Gamma_a - \int_{\partial \Omega_s} XG(x)\phi'(x)d\Gamma \right] w^r(t) = \int_{\partial \Omega_s} XG(x)F'(x)d\Gamma,
$$

(16)

$$
\sum_{a=1}^n \left[ \int_{\Gamma_a} G(x)\phi'(x)d\Gamma \right] w^r(t) + \sum_{a=1}^n \tilde{\omega}_1'(t) \int_{\partial \Omega_s} G(x)F'(x)d\Gamma = 0,
$$

(17)

where $X = [x_1, x_2]^T$. Eqs. (16) and (17) are considered on the sub-domains adjacent to interior nodes as well as to the boundary nodes on $\Gamma_{MD}$. The essential boundary conditions on $\Gamma_{MD}$ (part of the global boundary with prescribed rotations or displacements) can be imposed directly, using the interpolation approximation (13).

**Numerical examples**

A simply supported square plate with a side-length $a = 0.254 m$ and the plate thicknesses $h/a = 0.05$ is considered. On the top surface of the plate a thermal shock $\theta = H(t - 0)$ with Heaviside time variation is applied. If the ends of the plate are thermally insulated, a uniform temperature distribution is imposed on plate surfaces are given. The bottom surface is thermally insulated too. In such a case the temperature distribution is given by Carslaw and Jaeger [21]

$$
\theta(x,y,t) = 1 - \frac{4}{\pi} \sum_{n=1}^{\infty} (-1)^n \frac{\exp \left[ -\frac{(2n+1)^2 \pi^2 k t}{4h^2} \right]}{2n+1} \frac{(2n+1) \pi x}{2h},
$$

(18)
where diffusivity coefficient $\kappa = k / \rho c$, with thermal conductivity $k = 1000 W / m^2 K$, mass density $\rho = 7500 kg / m^3$ and specific heat $c = 400 W s / kg deg$.

Isotropic material parameters are considered: Young’s moduli $E_1 = E_2 = 0.6895 \times 10^8 N/m^2$, Poisson’s ratios $\nu_{12} = \nu_{21} = 0.3$, the thermal expansion coefficients $\alpha_{11} = \alpha_{22} = 1 \times 10^{-5} deg^{-1}$. The used shear moduli correspond to Young’s modulus $E_s$, namely, $G_{12} = G_{13} = G_{23} = E_s / 2(1 + \nu_{12})$.

![Fig. 1 Time variation of the central deflection in the plate with vanishing heat flux](image1)

![Fig. 2 Time variation of the bending moment in the plate with vanishing heat flux](image2)

Homogeneous and FGM properties with $n=2$ are considered here. Numerical results for the central plate deflection are presented in Fig. 1. Deflections are normalized by the central deflection corresponding to stationary thermal distribution with $\theta = 1 deg$ on the top plate surface and vanishing temperature on the bottom surface. For homogeneous material properties the corresponding stationary deflection is $w_{stat} = 0.4829 \times 10^{-2} m$. One can observe that in the whole time interval deflection for a homogeneous plate is lower than in a stationary case. The stiffness of the FGM plate is lower than for a homogeneous one.

For homogeneous plate the deflection is approaching to zero for late time instants. It is opposite to FGM plate. The bending moment at the center of the plate $M_{11}^{stat} = 0.4699 Nm$ is used as a normalized parameter in Fig. 2. The peak values of the bending moments for both homogeneous and FGM plates are almost the same. Since the deflection is not vanishing for late time instants at FGM plate, the bending moment is finite too.

In the next example, a thermal shock with Heaviside time variation is considered again on the top surface of the plate. The bottom surface is kept at vanishing temperature. Again in this case a uniform temperature distribution on plate surfaces are obtained. The analytical expression for temperature is given by [21]

$$\theta(x, t) = \frac{x}{h} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\cos nx}{n} \exp \left( -\frac{n^2 \pi^4 t}{h^2} \right) \sin \frac{n \pi x}{h}$$

![Fig. 3 Time variation of the deflection in the plate with vanishing temperature on the bottom surface](image3)
The time variation of the normalized plate deflection for vanishing temperature on the bottom side of the plate is given in Fig. 3. One can observe that the deflection is approaching the stationary value at late time instants. No peak is observed in this case. Since the plate stiffness for the FGM plate is lower than in homogeneous case, the plate deflections are larger for the FGM case.

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BEM for thin vortex layers

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Abstract: A simple model for a thin shear layer in two-dimensional flow is to consider a constant vorticity distribution between two interfaces. Boundary integral methods have desirable qualities when applied to the evolution of the interfaces. Unfortunately, when the layer is very thin, high resolution demanded for accuracy of the numerical quadrature mitigates the value of this approach. A new numerical quadrature will be introduced that allows high accuracy with modest numbers of markers on the boundary. The method is based on the removal of the nearby singularity in the integrands.

Key–Words: Thin vortex layers, numerical quadrature, analytic continuation.

1 Introduction

The simplest model for a thin shear layer is a vortex sheet [1]. In the limit of small thickness, a $\delta$-distribution of vorticity lies along the centerline of the layer. Unfortunately, evidence is now very strong that vortex sheets develop curvature singularities in finite time [2]. On the other hand, the motion of thin layers of vorticity exists globally in time [3]. The question then arises “what does a thin layer look like after the time a singularity would form on a vortex sheet?”

The first study [4] that attempts to address this question found that the nature of the thin layer changes dramatically. Near the place where the vortex sheet singularity would form, the thin layer develops an elliptically shaped bulge in which shearing motion is replaced by rotating motion. But details of the motion, such as the scaling of the blob size with the mean thickness, could not be calculated when the mean thickness is very small. The difficulty is associated with the accuracy of the numerical method.

The motion of the layer is determined completely by the motion of its boundaries. Boundary integrals can be used to calculate the velocity along the boundaries, but when the boundaries are close together for thin layers, rapid variation in the integrands demand excessive high resolution for accuracy, a common difficulty for boundary integral methods for thin regions [5].

A new numerical method for the treatment of the integrals is proposed here as an adaption of a method developed for thin annular regions [6]. First, we provide a new description of the formulation of the vortex layer in terms of boundary integrals that has the advantage of clarifying how the method is a superposition of a particular solution with a homogeneous solution. This approach allows greater generality than the standard derivation of the boundary integrals. After the description of the numerical approximations to the boundary integrals is given, an analysis is performed to determine exactly the errors the arise for a perfectly flat layer. Large errors arise when the layer is thin. The source of the error is clearly identified and parallels previous results for thin annular regions [7]. Finally, a new method is proposed that removes the source of the large errors.

2 Mathematical formulation

A horizontal vortex layer of constant vorticity $\omega = 2U/H$ lies between two interfaces separated by a mean thickness $H$ as illustrated in Fig. 1. The bounding interfaces are prescribed by the parametric form $(x_j(t), y_j(t))$ where the subscripts $j = 1, 2$ will
refer to the bottom, top interfacial location respectively.

The distribution of vorticity will satisfy the Euler equations of motion for an incompressible fluid if the bounding interfaces move with the velocity of the fluid. The velocity with components \((u, v)\) may be determined through the streamfunction \(\psi\) as

\[
\begin{align*}
\psi &= \psi_y, \quad v = -\psi_x, \\
\end{align*}
\]

where the streamfunction satisfies

\[
\nabla^2 \psi = -\omega. \tag{2}
\]

Continuity of the velocity at the bounding surfaces then impose the requirements that the streamfunction and its normal derivative are continuous at the interfaces. The typical way to solve (2) is to express the solution is of course a solution to Laplace’s equation and its normal derivative are continuous at the interfaces. The homogeneous solution is of course a solution to Laplace’s equation and can be readily expressed in terms of boundary integrals.

For the vortex layer, a choice for the particular solution is

\[
\psi^{(p)} = \begin{cases} 
-Uy + \frac{UH}{4} & \text{above top interface,} \\
\frac{UH}{4} & \text{between interfaces,} \\
Uy + \frac{UH}{4} & \text{below bottom interface.}
\end{cases} \tag{3}
\]

The choice is motivated by the exact solution when the bounding interfaces are flat \(y_1 = -H/2\) and \(y_2 = H/2\). When the interfaces are no longer flat, the particular solution jumps by

\[
\psi_{2}^{(p)} - \psi_{1}^{(p)} = \frac{U}{H}y_1^2 - Uy_1 - \frac{UH}{4} \tag{4}
\]

at the bottom interface, and by

\[
\psi_{3}^{(p)} - \psi_{2}^{(p)} = \frac{U}{H}y_2^2 - Uy_2 + \frac{UH}{4} \tag{5}
\]

at the top interface. The normal velocity component also jumps. The downward normal derivative at each interface has the form

\[
\frac{\partial \psi}{\partial n} = \frac{y_1}{s_1} \frac{\partial \psi}{\partial x} - \frac{x_1}{s_1} \frac{\partial \psi}{\partial y}, \tag{6}
\]

where the subscripts \(p\) refer to differentiation and

\[
s_p = x_p^2 + y_p^2. \tag{7}
\]

So the jump in the normal derivative of \(\psi\) is

\[
\frac{\partial \psi}{\partial n} - \frac{\partial \psi}{\partial n} = \frac{x_1}{s_1} \left( U + \frac{2U}{H} y_1 \right) \tag{8}
\]

at the bottom interface, and by

\[
\frac{\partial \psi}{\partial n} - \frac{\partial \psi}{\partial n} = \frac{x_2}{s_2} \left( U - \frac{2U}{H} y_2 \right) \tag{9}
\]

The solution to the homogeneous problem may be expressed in terms of dipole \(\mu_2\) and source \(\sigma_2\) distributions along each interface [8].

\[
\psi^{(h)} = \sum_{k=1}^{2} \left\{ \frac{1}{2\pi} \int \mu_2(q) \frac{N(X_k, x)}{D(X_k, x)} dq \right. + \left. \frac{1}{4\pi} \int \sigma_2(q) \ln[D(X_k, x)] s_{k,p}(q) dq \right\}, \tag{10}
\]

where

\[
N(X, x) = Y_2(q) \left[ X(q) - x \right] - X_2(q) \left[ Y(q) - y \right], \tag{11}
\]

and

\[
D(X, x) = (X - x)^2 + (Y - y)^2. \tag{12}
\]

The dipole strength is given by the jump in \(\psi^{(h)}\) across the interface from above to below and is chosen to cancel the jump in \(\psi^{(p)}, (4)\) and (5).

\[
\mu_1 = \frac{U}{H}y_1^2 + Uy_1 + \frac{UH}{4}, \tag{13}
\]

\[
\mu_2 = -\frac{U}{H}y_2^2 + Uy_2 - \frac{UH}{4}. \tag{14}
\]

Similarly, the source strength is given by the jump in the normal derivative of \(\psi^{(h)}\) across the interface from below to above and is chosen to cancel the jump in normal derivatives in \(\psi^{(p)}, (8)\) and (9).

\[
\sigma_1 = -\frac{x_1}{s_1} \left( U + \frac{2U}{H} y_1 \right), \tag{15}
\]

\[
\sigma_2 = -\frac{x_2}{s_2} \left( U - \frac{2U}{H} y_2 \right). \tag{16}
\]

The integrals in (10) can be written in a simple, more compact form by introducing the complex location \(z = x + iy\). Then,

\[
\psi^{(h)} = \Re \left\{ \frac{1}{2\pi} \int \mu_2(q) \frac{z_{k,p}(q)}{(z_k(q) - z)} dq \right. + \left. \frac{1}{2\pi} \int \sigma_2(q) s_{k,p}(q) \ln(z_k(q) - z) dq \right\}. \tag{17}
\]
where the complex streamfunction is \( \psi = \psi - i\phi \) and \( \phi \) is the velocity potential. The superscript * indicates the operation of complex conjugation. The homogeneous complex velocity is needed at the interfaces (19) and, after an integration by parts, becomes

\[
\psi^{(h)} = \sum_{k=1}^{2} \frac{1}{2\pi i} \int M_k(q) \ln(z_k(q) - z) \, dq,
\]

where

\[
M_k(q) = i\sigma_k(q) z_k(q) - \mu_k(q).
\]

In particular,

\[
M_1(q) = i m_1(q) z_1(q),
\]

\[
m_1(q) = U \left( 1 + \frac{2}{\pi} \psi_1(q) \right),
\]

\[
M_2(q) = i m_2(q) z_2(q),
\]

\[
m_2(q) = U \left( 1 - \frac{2}{\pi} \psi_2(q) \right).
\]

By differentiating (19), the homogeneous complex velocity is

\[
w^{(h)} = \sum_{k=1}^{2} \frac{1}{2\pi i} \int \frac{m_k(q) z_k(q)}{z_k(q) - z} \, dq.
\]

The complex velocity is needed at the interfaces in order to advance them. The homogeneous complex velocity is given by a Cauchy integral whose evaluation at the interfaces is discontinuous. Its principal-valued part gives the average complex velocity at the interface and its jump has been designed to cancel the jump in the particular solution at the interface. Thus by simply adding the averaged velocity from the particular solution, the interfacial velocity is determined. Specifically,

\[
w_1(p) = \frac{U}{2\pi i} \sum_{k=1}^{2} \left\{ \frac{1}{2\pi i} \int \frac{m_k(q) z_k(q)}{z_k(q) - z_1(p)} \, dq \right\},
\]

\[
w_2(p) = \frac{2U}{\pi i} \sum_{k=1}^{2} \left\{ \frac{1}{2\pi i} \int \frac{m_k(q) z_k(q)}{z_k(q) - z_2(p)} \, dq \right\}.
\]

There is one further simplification that arises from the result

\[
\frac{1}{2\pi i} \int \frac{z(q)}{z(q) - z} \, dq = \begin{cases} 1/2 & \text{for } y > 0, \\ 0 & \text{for } y = 0, \\ -1/2 & \text{for } y < 0. \end{cases}
\]

By applying this result to (24) and (25),

\[
w_1(p) = \frac{U}{\pi H} \int \frac{y_1(q) - y_1(p)}{z_1(q) - z_1(p)} \, dq - \frac{U}{\pi H} \int \frac{y_2(q) - y_2(p)}{z_2(q) - z_2(p)} \, dq,
\]

\[
w_2(p) = \frac{U}{\pi H} \int \frac{y_1(q) - y_1(p)}{z_1(q) - z_1(p)} \, dq - \frac{U}{\pi H} \int \frac{y_2(q) - y_2(p)}{z_2(q) - z_2(p)} \, dq.
\]

This is exactly the same result found in [4].

The approach followed here is less straightforward than the standard approach but has the advantage of combining a particular solution with a homogeneous solution, a process that can be applied whenever a particular solution is known. The procedure may also be generalized to certain three-dimensional flows, such as axisymmetric vortex rings [10].

As it stands, the integrals in (27) and (28) range over \((-\infty, \infty)\). Fortunately, there is much interest in the evolution of periodic disturbances to the flat layer, and a simple periodic Greens function is available in two dimensions. With the assumption that the layer is \(2\pi\)-periodic \(x(p+2\pi) = x(p), y(p+2\pi) = y(p)\), all that is required is to make the replacement

\[
\frac{1}{2\pi i} \int \frac{1}{z(q) - z} \, dq = \frac{1}{4\pi} \cot \left( \frac{z(q) - z(p)}{2} \right),
\]

and to use the range \((0, 2\pi)\) for the integrals.

## 3 Numerical Approximation of the Boundary Integrals

The method of lines is quite natural as a numerical approximation to (27) and (28). The interfacial locations are represented by \(N\) equally-spaced points;

\[
x_{j,n} = x_j(nh), y_{j,n} = y_j(nh), \quad n = n/N.
\]

An approximation to the derivatives that is spectrally accurate is easily obtained through use of the Fast Fourier Transform (FFT). Let

\[
z(p) = p + \sum_{k=-\infty}^{\infty} A_k e^{ikp}.
\]
The Fast Fourier Transform can be used to determine \( A_k \), subject only to aliasing errors. The derivative is then calculated from
\[
    z_{p}(p) = 1 + \sum_{k=-\infty}^{\infty} \frac{ikA_ke^{i\alpha k}}{N1}.
\]
(31)
The inverse transform determines \( z_{p,n} \) as required. Spectral accuracy is preserved.

The final step is the approximation for the boundary integrals. Here the trapezoidal rule is an obvious choice since it is spectrally accurate for periodic integrands [11]. Care must be taken though since some of the integrands have an indefinite form at \( q = p \). The limiting values are easily determined, though, and so the boundary integrals in (27) are approximated by
\[
    \frac{U}{\pi H} \sum_{m=-\infty}^{\infty} \frac{y_{1,m} - y_{1,n}}{z_{1,m} - z_{1,n}} z_{1,q,m} + \frac{y_{1,p,n}}{N1},
\]
(32)
and
\[
    \frac{U}{\pi H} \sum_{m=-\infty}^{\infty} \frac{y_{2,m} - y_{1,n}}{z_{2,m} - z_{1,n}} z_{2,q,m}.
\]
(33)
Similar approximations apply to the integrals in (28).

The accuracy in the approximations to the integrals is formally \( O(\exp(-\alpha N)) \) but this does not mean the errors can’t be large for a fixed \( N \). The errors and the spectral accuracy can be illustrated directly for the special case of a flat layer: \( \chi_1 = p + Ut \), \( y_1 = -H/2, \chi_2 = p - Ut, \) \( z_2 = H/2 \). Only the second integral in (27) contributes a non-trivial result,
\[
    \frac{U}{\pi H} \int \frac{H}{q - p - 2Ut + iH} dq = -U.
\]
(34)
The corresponding sum (33) can also be evaluated exactly,
\[
    \frac{U}{\pi H} \sum_{m=-\infty}^{\infty} \frac{H}{mb - nh - 2Ut + iH} = -U - U \frac{e^{-\alpha H + 2Ut}}{1 - e^{-\alpha H + 2Ut}}.
\]
(35)
The error is clearly spectral, but \( \alpha = H \) and it may take large \( N \) to achieve good accuracy when \( H \) is very small. For example, when \( t = 0 \), the error behaves as \( U/NH \) for small \( NH \). If \( H = 0.01 \), \( N \approx 1000 \) is needed for good accuracy. The same loss of accuracy will occur for the first integral in (28).

The way passed this loss of accuracy has been presented before [7],[6] for thin annular regions. The adaption of the idea to a thin vortex layer will be described for the second integral in (27). The application to the first integral in (28) is similar. The first step is to analytically continue \( z_{2}(q) \) and find \( Q \) such that \( z_{2}(Q) = z_{1}(p) \). In other words, we have found the location of the pole in the integrand in the complex \( q \)-plane. As demonstrated in (34), this pole should be about a distance \( H \) from the real axis of \( q \). The second step is to modify the second integral in (27) by using (26):
\[
    \frac{U}{\pi H} \int \frac{y_{2}(q) - y_{1}(p)}{z_{2}(q) - z_{1}(p)} z_{2,q}(q) dq = \frac{U}{\pi H} \int \frac{y_{2}(q) - y_{2}(Q)}{z_{2}(q) - z_{1}(p)} z_{2,q}(q) dq - \frac{U}{\pi H} (y_{2}(Q) - y_{1}(p)).
\]
(36)
The pole in the integrand had been removed!
The analytic continuation of \( z(q) \) is easily achieved by use of the Fourier series (30).
\[
    z(Q) = Q + \sum_{k=-\infty}^{\infty} A_{k} e^{i\alpha k}.
\]
(37)
This sum must be performed numerically since \( Q \) will be complex. Also, it is important to suppress any \( A_{k} \) for large \( k \) that are at the round-off level of the computer.

Preliminary results indicate a vast improvement in accuracy in the evolution of the interfaces; the results will be reported elsewhere.

4 Conclusion

Analytic continuation can be used to improve the accuracy of calculating boundary integrals in the motion of a thin vortex layer. In essence, a nearby pole in the complex integration variable is removed so that the resulting integrand is much smoother and easier to integrate numerically.

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References:


A boundary element method for simultaneous Poisson’s equations

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Keywords: Boundary Element Method, Simultaneous Poisson’s Equations, Iterative Solver

Abstract A boundary element method based on a set of simultaneous Poisson’s equations is presented. The source term of the original Poisson’s equation is approximated with a simultaneous Poisson’s equations. The fundamental solutions for the set of Poisson’s equations are presented to formulate a boundary integral equation for solving the original Poisson’s equation. A representative length parameter is introduced in the derived fundamental solution to improve the convergence property of the iterative solution. The effectiveness of the present approach is demonstrated through several numerical examples for Poisson’s equation with source distributions.

Introduction

The boundary integral representation for Poisson’s equation has a domain integral term originated from the source term. This domain integral can be converted to boundary integrals if the source satisfies Laplace’s equation. For more general source distribution, the domain integral can be converted by mean of the dual reciprocity method (DRM)\cite{1, 2}, the multiple reciprocity method (MRM) \cite{3} and its variant\cite{4, 5, 6}.

In the DRM, the value of the source term at an arbitrary point in the domain is approximated with a linear combination of radial basis functions (RBF) whose collocation points are placed in the domain and on the boundary. In order to convert the domain integral term originated from the source term of Poisson’s equation, particular solutions corresponding to the radial basis function are required. Also, the coefficients of the source term approximation have to be determined in advance by collocation method which requires fully populated matrix to solve and is unstable for large scale problems. On the other hand, MRM requires particular solutions for sources corresponding to a series of fundamental solutions. By using these particular solutions, the original domain integral term can be converted to a series of boundary integrals and a domain integral. The resulting domain integral could be neglected after a sufficient number of integrations by parts. The boundary integrals for the source involve the boundary values of the source and its derivatives whose correct values have to be given for accurate analysis. Ochiai proposed a variant of MRM, called triple reciprocity BEM\cite{4, 5, 6}, which applies the reciprocity formulation only three times. Instead of using the correct values of the derivatives of the source, they are roughly estimated to be zero. The error of the derivative of the source on the boundary is compensated by giving the values of the source at collocation points in the domain instead.

In this study, we consider Poisson’s equation and approximate the source term in terms of simultaneous coupled Poisson’s equations. Using the fundamental solutions of the simultaneous Poisson’s equations, a boundary integral equation is derived. A standard BEM procedure can be applied to discretize the obtained boundary integral equation and an iterative solver is used to solve the resulting set of linear algebraic equations. In the
present formulation, the higher-order fundamental solutions, polyharmonic functions, are modified so that the coefficient matrix of the algebraic equations can be preconditioned easily. The effectiveness of the present formulation is demonstrated through a numerical test example. Although the present approach is equivalent to the original MRM and triple reciprocity BEM, it gives an alternative interpretation of MRM.

Boundary Integral Formulation Based on Simultaneous Poisson’s Equations

Consider a potential problem governed by Poisson’s equation

\[ \nabla^2 u_1(x) + u_2(x) = 0, \quad x \in \Omega \]  

with the boundary condition

\[ u_1(x) = \tilde{u}_1(x), \quad x \in \Gamma_u \]  
\[ q_1(x) = \frac{\partial u_1(x)}{\partial n} = \tilde{q}_1(x), \quad x \in \Gamma_q \]

where \( \Omega \) is the domain under consideration and \( \Gamma = \Gamma_u \cup \Gamma_q \) is its boundary, as shown in Fig.1; \( u_1(x) \) denotes the potential and \( u_2(x) \) the source term. Also, \( q_1(x) = \partial u_1(x)/\partial n \) is the derivative of \( u_1(x) \) in the outward normal direction to the boundary, and \( \tilde{u} \) and \( \tilde{q} \) are given functions prescribed on the specified boundaries, respectively.

![Fig. 1 Domain and boundary.](image)

The source term \( u_2(x) \) is assumed also to be a known function whose values are given both in \( \Omega \) and on \( \Gamma \) as

\[ u_2(x) = \tilde{u}_2(x), \quad x \in \Omega, \Gamma. \]  

We approximate the source term \( u_2(x) \) into the following steps of simultaneous Poisson’s equations:

\[ \begin{align*}
\nabla^2 u_1(x) + u_2(x) &= 0, \\
\nabla^2 u_2(x) + u_3(x) &= 0, \\
\nabla^2 u_3(x) + u_4(x) &= 0, \\
&\vdots \\
\nabla^2 u_N(x) + u_{N+1}(x) &= 0.
\end{align*} \]  

\( \nabla^2 u_{l-1}(x), \ l = 3, 4, 5, \cdots, N \) correspond to curvatures of \( u_{l-1}(x) \) on the boundary, and can be assumed as known. If we use the values of \( u_2 \) at sufficient number of collocation points in the domain as additional information, we can roughly estimate the amount of the curvatures of \( u_{l-1}(x) \), \( l = 3, 4, 5, \cdots, N \) negligible\[4, 5, 6\], as follows:

\[ u_l(x) = -\nabla^2 u_{l-1}(x) = 0, \quad x \in \Gamma. \]
The integral representations for the simultaneous Poisson’s equations (5) can be derived by starting from the following weighted residual form:

\[
\int_{\Omega} \sum_{i=1}^{N} u_{m}^{\alpha}(x, y) \left[ \nabla^2 u_{\alpha}(x) + u_{\alpha+1}(x) \right] d\Omega(x) = 0, \tag{7}
\]

where \( u_{m}^{\alpha}(x, y) \) \((m = 1, 2, \ldots, N)\) are weighting functions.

By integrating by parts Eq.(7) repeatedly, we obtain

\[
\sum_{i=1}^{N} \left\{ \int_{\Omega} u_{m}^{\alpha}(x, y) q_{i}(x) d\Gamma(x) - \int_{\Gamma} q_{i}^{\alpha}(x, y) u_{\alpha}(x) d\Gamma(x) \right\} + \sum_{i=1}^{N} \int_{\Omega} \nabla^2 u_{m}^{\alpha}(x, y) u_{\alpha}(x) d\Omega(x)
\]

\[
+ \sum_{i=1}^{N} \int_{\Gamma} \left\{ \nabla^2 u_{m}^{\alpha}(x, y) + u_{\alpha-1}^{\alpha}(x, y) \right\} u_{\alpha}(x) d\Gamma(x) + \int_{\Omega} u_{N}^{\alpha}(x, y) u_{N+1}(x) d\Omega(x) = 0. \tag{8}
\]

We now consider particular solutions for the weighting functions \( u_{m}^{\alpha}(x, y) \) of the following adjoint form of the original simultaneous Poisson’s equations: Eq.(5).

\[
L u^* + \delta(x - y) = \begin{bmatrix}
\nabla^2 & 0 & 0 & \cdots & 0 \\
0 & \nabla^2 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & \nabla^2 & 0
\end{bmatrix}
\begin{bmatrix}
u_{11}^* \\
u_{12}^* \\
u_{13}^* \\
\vdots \\
u_{N}^*
\end{bmatrix} + I \delta(x - y) = 0, \tag{9}
\]

where \( u^* = u_{m}^{\alpha}(x, y) \mu x \) denotes the unity matrix and \( \delta(x - y) \) is Dirac’s delta function.

By applying the property of Dirac’s delta function, the second and third terms of Eq.(8) becomes

\[
\sum_{i=1}^{N} \int_{\Omega} \nabla^2 u_{m}^{\alpha}(x, y) u_{\alpha}(x) d\Omega(x) + \sum_{i=1}^{N} \int_{\Gamma} \left\{ \nabla^2 u_{m}^{\alpha}(x, y) + u_{\alpha-1}^{\alpha}(x, y) \right\} u_{\alpha}(x) d\Gamma(x) = -u_{m}(y). \tag{10}
\]

Now, we consider \( K \) collocation points \( x^k \), \((k = 1, \ldots, K)\) in the domain \( \Omega \) and approximate \( u_{N+1}(x) \) with a summation of concentrated sources. It is written by a linear combination of Dirac’s delta functions one of which arguments is place at the collocation points, as follows:

\[
u_{N+1}(x) = \sum_{k=1}^{K} a^k \delta(x - x^k), \tag{11}
\]

where \( a^k \), \((k = 1, \ldots, K)\) are unknown coefficients which must be determined in the same way like \( u \) and \( q \).

Substituting Eq.(11) into the fourth term of the left-hand side of Eq.(8) gives

\[
\int_{\Omega} u_{N}^{\alpha}(x) u_{N+1}(x) d\Omega(x) = \sum_{k=1}^{K} a^k u_{N}^{\alpha}(x^k, y). \tag{12}
\]

Substituting Eqs.(10) and (12) int Eq.(8), we finally obtain an integral representation for Eq.(5) as follows:

\[
c(y) u_{\alpha}(y) = \sum_{i=1}^{N} \left\{ \int_{\Omega} u_{m}^{\alpha}(x, y) q_{i}(x) d\Gamma(x) - \int_{\Gamma} q_{i}^{\alpha}(x, y) u_{\alpha}(x) d\Gamma(x) \right\} + \sum_{k=1}^{K} a^k u_{N}^{\alpha}(x^k, y), \tag{13}
\]

where \( c(y) = 1 \) for \( y \in \Omega \) and \( c(y) = \frac{1}{r^2} \) for \( y \) located on a smooth part of \( \Gamma \).

The fundamental solutions \( u_{m}^{\alpha} \) are given by

\[
u^* = L^* \phi(x, y), \tag{14}
\]
where \( L^r \) is the cofactor of \( L \), and \( \phi(x, y) \) is the fundamental solution for the determinant operator of \( L \), i.e.,

\[
\det L \phi(x, y) + \delta(x - y) = 0. \tag{15}
\]

For \( N = 4 \), for example, \( u^* \) becomes as follows:

\[
u^* = \begin{bmatrix}
\frac{1}{4\pi r} & 0 & 0 \\
-\frac{r}{8\pi} & \frac{1}{4\pi r} & 0 \\
\frac{r^3}{96\pi} & -\frac{r}{8\pi} & \frac{1}{4\pi r} \\
-\frac{r^5}{288\pi} & \frac{r^3}{96\pi} & \frac{1}{4\pi r}
\end{bmatrix}.
\tag{16}
\]

We observe from Eq.(16) that the fundamental solutions \( u^*_m \) for \( i > m \) are polyharmonic functions and become zero for \( r = 0 \).

**Discretization of the boundary integral equation and representative length parameter**

Discretizing Eq.(13) by using constants elements gives

\[
\frac{1}{2} u_{m}(y^') = \sum_{i=1}^{N} \sum_{j=1}^{N_B} \left( \int_{\Gamma_j} u_{m}(x^i, y^j) d\Gamma_j(x^i) \right) q_i(x^j) - \left( \int_{\Gamma_j} q^*_m(x^i, y^j) d\Gamma_j(x^i) \right) u_i(x^j) + \sum_{k=1}^{K} \alpha^k u^*_m(z^k, y^j), \tag{17}
\]

where \( N_B \) denotes the number of boundary elements, \( x^i/\text{and } y^j \) denote the centers of geometry of elements \( j \) and \( l \), respectively.

In Eq.(17), the unknowns are \( u_i(x^i), q_j(x^j), \) and \( \alpha^k \), and their total number becomes \( N \times N_B + K \).

Although from Eq.(17), we obtain \( N \times N_B \) equations, we still need \( K \) equations in order to solve for \( N \times N_B + K \) unknowns. Therefore, we use additional \( K \) collocation points in the domain and specify the values for the source term \( u_2(z^k), (k = 1, \ldots, K) \). It is achieved by using the component for \( m = 2 \) of Eq.(17) and substituting \( z^k, (k = 1, \ldots, K) \) into \( y^j \)

\[
u_2(z^k) = \sum_{i=1}^{N} \sum_{j=1}^{N_B} \left( \int_{\Gamma_j} u^*_2(x^i, z^k) d\Gamma_j(x^i) \right) q_i(x^j) - \left( \int_{\Gamma_j} q^*_2(x^i, z^k) d\Gamma_j(x^i) \right) u_i(x^j) + \sum_{k=1}^{K} \alpha^k u^*_2(z^k, z^k). \tag{18}
\]

The final linear algebraic equations can be written, by collecting all the unknown quantities on the left-hand side, as

\[
[A] [x] = [b], \tag{19}
\]

where \([A]\) is the coefficient matrix, \([x]\) the unknown vector, and \([b]\) is the right-hand side vector calculated from known quantities.

Iterative solvers are often used for solving equations like Eq.(19). Some of the rows of Eq.(19) are generated by applying Eq.(18) at \( K \) internal collocation points. Since \( u_2 \) is prescribed at these points in Eq.(18), the unknowns are the coefficients \( \alpha^k, (k = 1, 2, \ldots, N) \). We observe that \( u^*_2(z^k, z^k) \) becomes zero for \( z^k = z^l \) and the corresponding diagonal components of the matrix \([A]\) corresponding to \( \alpha^k, (k = 1, 2, \ldots, N) \) also become zero. This is inappropriate for preconditioning of \([A]\) to obtain fast convergence of iterative solution. Therefore, we modify the fundamental solution so that it becomes maximum for \( r = 0 \) and decreases
monotonously in accordance with the increase of $r$ by adding some of homogeneous solutions of Eq.(9). The modified fundamental solutions contain a parameter $L$, a representative length of the domain, and can be given as follows:

$$u^* = \begin{bmatrix} \frac{1}{4\pi r} & 0 & 0 & 0 \\ \frac{r}{8\pi} + \frac{L}{8\pi} & \frac{1}{4\pi r} & 0 & 0 \\ \frac{L^3}{192\pi} - \frac{r^2 L}{48\pi} + \frac{L^5}{96\pi} & -\frac{r}{8\pi} + \frac{L}{8\pi} & \frac{1}{4\pi r} & 0 \\ -\frac{r^3}{2880\pi} + \frac{L r^4}{960\pi} - \frac{L^2 r^2}{576\pi} + \frac{L^3 r}{96\pi} & -\frac{L r^2}{48\pi} + \frac{L^3}{96\pi} & -\frac{r}{8\pi} + \frac{L}{8\pi} & \frac{1}{4\pi r} \end{bmatrix}$$

(20)

Numerical Examples
Consider a cubic region having a edge of 10 as shown in Fig.2. All the sides are divided uniformly into 1200 triangular constant elements with 602 nodes. Also, 27 internal collocation points are placed uniformly.

Following Poisson's equation is calculated for the cubic region using the above mentioned method

$$\nabla^2 u_1(x) + 10 \cos \frac{\pi z}{10} = 0$$

(21)

The boundary conditions are given as follows:

$$u_1(x) = \begin{cases} 0 & \text{at } z = -5 \\ 100 & \text{at } z = 5 \end{cases}$$

(22)

$$q_1(x) = 0 \text{ at } x, y = \pm 5$$

(23)

The number of simultaneous Poisson's equations used in the present example is $N = 3$. The representative length $L$ is taken as the length of the diagonal line of the cubic region. GMRES is used as the iterative solver with the convergence criterion $\varepsilon = 10^{-5}$ and Point-Jacobi preconditioner.

Fig. 3 shows the results for $u_1(z)$ at the internal collocation points. The results obtained by the present approach show good agreements with the exact solutions.

Fig. 4 shows the change in the residual against the number of iterations. We observe that the results obtained by the method which uses the modified fundamental solutions with a representative length parameter can reduce the number of iterations by applying the preconditioner.
Concluding Remarks
A boundary element method for Poisson’s equation has been presented. The source term is approximated with simultaneous coupled Poisson’s equations. The fundamental solutions of the coupled Poisson’s equations have been derived and the corresponding boundary integral equation has been presented. Some homogeneous solutions of the simultaneous Poisson’s equations are added to the fundamental solutions so that the convergence property of the iterative solutions is improved. The numerical test example has demonstrated the effectiveness of the present approach.

References

Fig. 3 Results for $u_1(z)$ at internal collocation points.

Fig. 4 Residuals versus number of iterations.
Scaled Boundary Finite-Element Analysis of Dynamic Stress Intensity Factors and T-stress

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Abstract: The dynamic stress intensity factors and T-stress at crack tips are determined by using the scaled boundary finite-element method. Only the boundary of a bounded domain is discretized with elements. The inertial effect at high frequencies is represented by expressing the dynamic stiffness as continued fractions. By introducing auxiliary variables, an equation of motion is formulated with a high-order static stiffness matrix and a high-order mass matrix. Standard methods in structural dynamics can be directly applied to perform modal, frequency-domain and time-domain analyses. The dynamic stress intensity factors and T-stress are determined directly from their definitions. This technique does not require a fundamental solution or an asymptotic expansion of the singular stress field. Numerical examples demonstrate the simplicity and high accuracy of the present technique.

Introduction

For the dynamic analysis of fracture problems, the inertial effect has to be considered in computing the stress fields and extracting the stress intensity factors and the T-stress. The mesh and time step should be sufficiently fine to represent the highest frequency of interest. Many studies on the dynamic stress intensity factors for cracks in homogeneous plates have been reported (see, e.g., [1–5]). Literature reviews can be found in [5, 6]. The T-stress at interface cracks is rarely studied [7, 8].

The scaled boundary finite-element method has emerged as an attractive alternative to model problems with singularities. In a single analysis, this method obtains [9]: (a) orders of singularity; (b) power-logarithmic singularities, when exist; (c) stress intensity factors; (d) T-stress and higher order terms in the asymptotic expansion; (e) angular distributions of stresses corresponding to individual terms in the asymptotic expansion. In addition, the stress intensity factors and T-stress are extracted directly based on their definitions without evaluating singular functions close to a singular point. Crack propagation is modeled in [10]. In this paper, a numerical procedure to determine the dynamic stress intensity factors and the dynamic T-stress is presented.

Summary of the Scaled Boundary Finite-Element Method

A so-called scaling centre O is chosen in a zone from which the total boundary must be visible (Fig. 1). Without losing generality, the origin of the Cartesian coordinates \( \hat{x}, \hat{y} \) is chosen at the scaling centre. The boundary \( S \) is discretized with line elements. The geometry of an element is interpolated using the mapping functions \( [N(\eta)] \) in the local coordinate \( \eta \) and the nodal coordinates \{x\}, \{y\}. The domain \( V \) is described by scaling the boundary with a dimensionless radial coordinate \( \xi \) pointing away from the scaling centre \( O \). \( \xi = 0 \) at \( O \) and \( \xi = 1 \) on the boundary is chosen. A point \( (\hat{x}, \hat{y}) \) inside the domain is expressed as

\[
\begin{align*}
\hat{x}(\xi, \eta) &= \xi [N(\eta)] \{x\}; \\
\hat{y}(\xi, \eta) &= \xi [N(\eta)] \{y\}
\end{align*}
\] (1)
The scaled boundary coordinates \( (ξ, η) \) are called the scaled boundary coordinates in two dimensions. They resemble the polar coordinates \( (r, θ) \). The polar coordinates \( (r, θ) \) are expressed in the scaled boundary coordinates as (Fig. 1b)

\[
\begin{align*}
\hat{r}(ξ, η) &= ξr(η) = ξ \sqrt{x^2(η) + y^2(η)}; \\
θ(η) &= \arctan(y(η)/x(η))
\end{align*}
\] (2)

A bi-material wedge shown in Fig. 2 is conveniently defined by constant values of the scaled boundary coordinates when the scaling centre is chosen at the vertex. The two straight edges and material interface passing through the scaling center are defined by constant \( η \) and are not discretized.

Along the radial lines passing through the scaling centre \( O \) and a node on the boundary (Fig. 1) the nodal displacement functions \( \{u(ξ)\} \) are introduced. The displacements at a point \( (ξ, η) \) are interpolated from the functions \( \{u(ξ)\} \)

\[
\{u(ξ, η)\} = [N^w(η)]\{u(ξ)\} = [N_1(η)]\{u\}_1 + [N_2(η)]\{u\}_2 + \ldots + [N\}_l\{u\}_l
\]

where \([l]\) is a \( 2 \times 2 \) identity matrix. The strains are expressed in the scaled boundary coordinates as

\[
\varepsilon(ξ, η) = \{B^1(η)\} \{u(ξ)\}_ξ + \{B^2(η)\} \{u(ξ)\}_η
\] (4)

The expressions for \([B^1(η)]\) and \([B^2(η)]\) can be found in Ref. [11]. The scaled boundary finite-element equation in displacement is expressed as [12]

\[
[E^0](ξ^2) \{u(ξ)\}_ξ + ([E^0] - [E^1] + [E^1]^T) \{u(ξ)\}_ξ - [E^2] \{u(ξ)\} + (ωξ)^2[M^0] \{u(ξ)\} = 0
\] (5)

where \( ω \) is the excitation frequency. The coefficient matrices \([E^0]\), \([E^1]\), \([E^2]\) and \([M^0]\) are given in Ref. [11]. Determining the nodal forces on radial lines and defining the dynamic stiffness matrix \([S(ω, ξ)]\) yield [12]

\[
\{Q(ξ)\} = [E^1]^T \{u(ξ)\}_ξ + [E^1]^T \{u(ξ)\} = [S(ω, ξ)]\{u(ξ)\}
\] (6)

The scaled boundary finite-element equation in dynamic stiffness is a function of \((ωξ)\) [12]

\[
([S(ωξ)] - [E^1][E^0]^{-1}([S(ωξ)] - [E^1]^T) - [E^2] + (ωξ)^2[S(ωξ)], (ωξ) + (ωξ)^2[M^0] = 0
\] (7)

**Continued Fraction Solution for Dynamic Stiffness Matrix**

A continued fraction solution of the dynamic stiffness matrix is obtained in Ref. [13] by addressing the scaled boundary finite-element equation in dynamic stiffness (Eq. 7). The continued fraction solution of order \( M_d \) is expressed as

\[
[S(ωξ)] = [K] - (ωξ)^2[M] - (ωξ)^4([S^{(1)}] - (ωξ)^2[S^1]) - (ωξ)^4([S^{(2)}] - (ωξ)^2[S^2]) - \ldots - (ωξ)^4([S^{(M_d)}] - (ωξ)^2[S^M])^{-1} \ldots )^{-1}^{-1}
\] (8)
where \([K]\) and \([M]\) are the static stiffness and mass matrices, respectively. They represent the low frequency expansion of the dynamic stiffness matrix and are routinely used in structural dynamics. \([S_0^{(i)}]\) and \([S_1^{(i)}]\) \((i = 1, 2, \ldots, M_d)\) are coefficient matrices of the high-order terms, which represent the high-frequency response. Substituting Eq. (8) into Eq. (7), these coefficient matrices are determined as follows.

The following real Schur decomposition of a Hamiltonian matrix is performed

\[
\begin{bmatrix}
E^0 - 1 & E^1 T \\
-E^0 + 1 & -E^0 - 1
\end{bmatrix} \begin{bmatrix}
V_{11} & V_{12} \\
V_{21} & V_{22}
\end{bmatrix} = \begin{bmatrix}
V_{11} & V_{12} \\
V_{21} & V_{22}
\end{bmatrix} \begin{bmatrix}
|S_{11}| & |S_{12}| \\
0 & |S_{22}|
\end{bmatrix}
\]

(9)

The real Schur form matrix \([S]\) is a quasi-upper triangular matrix. \([V]\) is a transformation matrix. \([S]\) and \([V]\) are arranged in such a way that the real parts of the eigenvalues of \([S_{11}]\) are non-positive \((\lambda, |S_{11}| \leq 0)\) and \([V_{11}]\) contains the displacement modes of the bounded domain. To extract the stress intensity factors and \(T\)-stress, \([S_{11}]\) is block-diagonalized leading to

\[
[S_{11}] = \text{diag}([S_{a1}], -[I], [S^{(i)}], 0); \quad [V_{11}] = ([\Psi_{a1}], [\Psi^{(T)}], [\Psi^{(i)}], [\Psi^{(f)}])
\]

(10)

The zero eigenvalues and \([\Psi^{(f)}]\) in Eq. (10) describe the translational rigid body motions. The real parts of all the eigenvalues \(\lambda\) of \([S^{(i)}]\) are between \(-1\) and \(0\) (i.e., \(-1 < \text{Re}(\lambda([S^{(i)}])) < 0\) \([S^{(i)}]\) and \([\Psi^{(i)}]\) represent the singular stress terms. The \(T\)-stress and the rotational rigid body motion are described by the diagonal block \(-[I]([I]\) is an \(2 \times 2\) identity matrix) and \([\Psi^{(T)}]\). The remaining diagonal blocks are grouped together and denoted as \([S_{a1}]\) \((\text{Re}(\lambda([S_{a1}])) < -1)\) and the corresponding submatrix of \([V_{11}]\) is denoted as \([\Psi_{a1}]\).

The semi-positive definite static stiffness matrix \([K]\) is expressed as

\[
[K] = [V_{21}] [V_{11}]^{-1}
\]

(11)

The solution for mass matrix \([M]\) is expressed as

\[
[M] = [V_{11}]^{-T} [m] [V_{11}]^{-1}; \quad \text{with} \quad ([I] - [S_{11}]^T [m] + [m] ([I] - [S_{11}]) = [V_{11}]^T [M^0] [V_{11}]
\]

(12)

The coefficient matrices of high-order terms \([S_0^{(i)}]\) and \([S_1^{(i)}]\) \((i = 1, 2, \ldots, M_d)\) are obtained recursively. Defining the constant matrices for the case \(i = 1\)

\[
\begin{align*}
[a^{(1)}] &= [E^0 - 1]; & [V^{(1)}] &= [V_{11}]; & [U^{(1)}] &= 2[I] - [S_{11}]
\end{align*}
\]

(13a)

\[
\begin{align*}
[b_0^{(1)}] &= [E^0 - 1] [M]; & [c^{(1)}] &= [M] [E^0 - 1] [M]
\end{align*}
\]

(13b)

The solution for \([S_0^{(i)}]\) is equal to

\[
[S_0^{(i)}] = [V^{(i)}] [Y_0^{(i)}]^{-1} [V^{(i)}]^T; \quad \text{with} \quad [V_0^{(i)}] [U^{(i)}] + [U_0^{(i)}] [Y_0^{(i)}]^T = [V^{(i)}]^T [c^{(i)}] [V^{(i)}]
\]

(14)

with a quasi-upper triangular coefficient matrix \([U^{(i)}]\) (Eq. (13)). The solution for \([S_1^{(i)}]\) is

\[
[S_1^{(i)}] = [V^{(i+1)}] [Y_0^{(i+1)}]^{-1} [V^{(i+1)}]^T
\]

(15a)

where \([V^{(i+1)}]\) is defined in Eq. (16) below and \([Y_0^{(i)}]\) is obtained from

\[
([I] + [U^{(i)}]^T) [Y_0^{(i)}] + [Y_0^{(i)}] ([I] + [U^{(i)}]) = [V^{(i+1)}]^T ([b_0^{(i)}] [S_0^{(i)}] + [S_0^{(i)}] [b_0^{(i)}]^T) [V^{(i+1)}]
\]

(15b)

The constant matrices for the next recursion are expressed as

\[
\begin{align*}
[a^{(i+1)}] &= [c^{(i)}]; & [V^{(i+1)}] &= [S_0^{(i)}]^{-1} [V^{(i)}]; & [U^{(i+1)}] &= 2[I] + [U^{(i)}]
\end{align*}
\]

(16a)

\[
\begin{align*}
[b_1^{(i+1)}] &= -[b_0^{(i)}]^T + [c^{(i)}]; & [S_1^{(i)}] &= [a^{(i)}] - [b_1^{(i)}] [S_1^{(i)}] - [S_1^{(i)}] [b_1^{(i)}]^T + [S_1^{(i)}] [c^{(i)}] [S_1^{(i)}]
\end{align*}
\]

(16b)
Applying Eqs. (14) and (15) and updating the constant matrices using Eq. (16) repeatedly, \( [S_i^{(j)}] \) can be determined up to a specified order \( M_d \).

Substituting the continued fraction solution for the dynamic stiffness matrix in Eq. (8) into the force-displacement relationship in Eq. (6) leads to the equation of motion

\[
([K_0] - (\omega \bar{\xi})^2 [M_0]) \{\gamma(\bar{\xi})\} = \{F(\bar{\xi})\}
\]

with (The column concatenation is denoted by semicolons)

\[
\{\gamma(\bar{\xi})\} = \{[u(\xi_1)]; [u(\xi_2)]; \ldots; [u(M_d)](\bar{\xi})\}
\]

\[
\{F(\bar{\xi})\} = \{[q(\xi_1)]; [q(\xi_2)]; \ldots; [q(M_d)]\}
\]

\[
[K_0] = \text{diag}([K], [S_0^{(1)}], [S_0^{(2)}], \ldots, [S_0^{(M_d)}])
\]

\[
[M_0] = \begin{bmatrix}
[M] & -[I] & 0 & \ldots & 0 \\
-[I] & [S_0^{(1)}] & -[I] & \ldots & 0 \\
0 & -[I] & [S_0^{(2)}] & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & [S_0^{(M_d)}]
\end{bmatrix}
\]

where \( \{u^{(i)}(\bar{\xi})\} (i = 1, 2, \ldots, M_d) \) are auxiliary variables. It can be verified by eliminating the auxiliary variables \( \{u^{(i)}(\bar{\xi})\} \) that Eq. (17) is equivalent to Eqs. (6) and (8). The high-order static stiffness matrix \([K_0]\) and mass matrix \([M_0]\) are symmetric [13]. In the time domain, Eq. (17) is written as

\[
[K_0]\{\gamma(\bar{\xi})\} + \bar{\xi}^2 [M_0]\{\gamma(\bar{\xi})\} = \{F(\bar{\xi})\}
\]

When formulated on the boundary of a bounded domain, i.e., \( \bar{\xi} = 1 \), Eqs. (17) and (19) represent the equation of motion of the bounded domain. It can be assembled with the equations of other bounded domains or of standard finite elements. Standard procedures in structural dynamics can be applied to determine the natural frequencies and the nodal responses in the time and frequency domains. It is shown in Ref. [13] that \( 3 \rightarrow 4 \) terms are sufficient to model one wavelength.

**Evaluation of Dynamic Stress Intensity Factors and T-stress**

After solving the equation of motion of the global system, the nodal displacements \( \{u(\bar{\xi} = 1)\} \) and auxiliary variables at the boundary \( \{u^{(i)}(\bar{\xi} = 1)\} (i = 1, 2, \ldots, M_d) \) of a bounded domain are determined based on element connectivity as in the finite element method. To extract the stress intensity factors and T-stress directly from their definitions, the stress field at the singular point is determined.

The singularity in \( \{u(\bar{\xi})\}_{\bar{\xi}} \) is removed analytically by the variable transformation [14]

\[
\{v(\bar{\xi})\} = \bar{\xi}^{[S]} [V_1]^{-1} [u(\bar{\xi})]
\]

The matrix \([S]\) partitioned conformably with \([S_1]\) in Eq. (10) is introduced for conciseness

\[
\bar{\xi}^{[S]} = \text{diag}( [\xi], [\xi^{-1}], [\xi^{[S]}], [\xi] ) ; \quad \text{with} \quad [S_i] = \text{diag}(0, -[I], [S^{(i)}], 0)
\]

where the diagonal block \([S^{(i)}]\) corresponds to the singular term. Equation (17) is rewritten by introducing the variable transformation in Eq. (20) as

\[
([A] - \omega^2 [B(\bar{\xi})]) \{D(\bar{\xi})\} = [C]\bar{\xi} [D(\bar{\xi})]_{\bar{\xi}}
\]

where \([D(\bar{\xi})]\) is a column vector formed by concatenating \( \{v(\bar{\xi})\} \), and the auxiliary variables \( \{u^{(i)}(\bar{\xi})\} \)

\[
\{D(\bar{\xi})\} = \{ \{v(\bar{\xi})\}; \{u^{(1)}(\bar{\xi})\}; \{u^{(2)}(\bar{\xi})\}; \ldots \}
\]
The coefficient matrices are written as

\[
[A] = \text{diag}(-[S_{11}]) + [S_1], [S_0^{(1)}], [S_0^{(2)}], \ldots
\]

\[
[B(\xi)] = \xi^2 \begin{bmatrix}
-\xi[S_{11}]^{-1}[E_0]^{-1}[-M] & \xi[S_{11}]^{-1} & 0 & \ldots \\
0 & -[I] & 0 & \ldots \\
\vdots & \vdots & \ddots & \ddots
\end{bmatrix}
\]

\[
[C] = \begin{bmatrix} [I] & 0 & 0 \end{bmatrix}
\]

The initial condition of \(D(\xi)\) is determined at \(\xi = 1\) from Eq. (20)

\[
\{v(\xi = 1)\} = [V_{11}]^{-1}\{u(\xi = 1)\}
\]

and the solution for auxiliary variables on the boundary \(\{u(i)(\xi = 1)\}\) \((i = 1, 2, \ldots, M)\).

The functions \(D(\xi)\) are solved numerically at discrete values of radial coordinate \(\xi_j\) \((j = 1, 2, \ldots, n_2)\) ranging from \(\xi_1 = 1\) on the boundary to \(\xi_{n_2}\) at the scaling center in the frequency or time domain. The function \(\{v(\xi)\}\) is extracted (Eq. (23)). The asymptotic solution for \(\{u(\xi)\}\) at \(\xi \rightarrow 0\) is expressed as [9]

\[
\{u(\xi)\} = [V_{11}]\xi^{-[S_{11}]}\{c\} + O(\xi^2)
\]

where \(\{c\}\) are the integration constants whose values depend on the displacement amplitudes on the boundary and the excitation frequency \(\omega\). Substituting the block-diagonal form of \([S_{11}]\) and the corresponding partition of \([V_{11}]\) in Eq. (10) leads to

\[
\{u(\xi)\} = [\Psi^{(0)}](\xi)\{s(0)\} + [\Psi^{(i)}]\xi^{-[S_{11}]}\{c(i)\} + [\Psi^{(T)}]\xi^{-[S_{11}]}\{e(i)\} + O(\xi^2)
\]

where the integration constants \(\{c\}\) is partitioned as

\[
\{c\} = \{\{e_{n_1}\}; \{e^{(0)}\}; \{e^{(i)}\}\}
\]

On the right-hand side, the first term represents the translational rigid body motion, the second term the singular stresses \((0 < \text{Re}(\lambda(-[S_{11}]))) < 1\), and the third term the rigid body rotation and the \(T\)-stress. The calculation of stresses from the displacements is similar to that in the finite element method. At a specified local coordinate \(\eta\) of an element, the stresses are obtained by substituting Eq. (27) into Eq. (4) and using the Hooke’s law \(\{\sigma(\xi, \eta)\} = [D]\{\varepsilon(\xi, \eta)\}\) \(([D]\) is the elasticity matrix)

\[
\{\sigma(\xi, \eta)\} = [\Psi^{(0)}](\eta)\xi^{-[S_{11}]}\{c(0)\} + [\Psi^{(i)}](\eta)\xi^{-[S_{11}]}\{c(i)\} + [\Psi^{(T)}](\eta)\{e(0)\} + O(1)
\]

where \([\Psi^{(0)}](\eta)\) and \([\Psi^{(i)}](\eta)\) are the singular stress and \(T\)-stress modes [9], respectively

\[
[\Psi^{(0)}](\eta) = [D][-B^1(\eta)][\Psi^{(0)}] + [B^1(\eta)][\Psi^{(0)}]
\]

\[
[\Psi^{(i)}](\eta) = [D][B^1(\eta)][\Psi^{(i)}] + [B^1(\eta)][\Psi^{(i)}]
\]

The solution at the scaling center \(\xi = 0\), \(\{v(\xi = 0)\}\), is extracted from \(D(\xi = 0)\) (Eq. (23)). Using Eqs. (26) and (28), Eq. (20) is written at \(\xi = 0\) as

\[
\{v(\xi = 0)\} = \{0; \{e(0)\}; \{e^{(0)}\}; \{e^{(i)}\}\}
\]

The integration constants \(\{e^{(0)}\}\) and \(\{e^{(i)}\}\) are obtained directly from Eq. (31). The stress field around the singular point can then be calculated by using Eq. (29).
A procedure to determine the stress intensity factors and $T$-stress directly from Eq. (29) and known integration constants \{$c^{(i)}\}$ and \{$e^{(i)}\}$ is developed in [9] for power (real and complex) and power-logarithmic singularities. It is summarized in the following for the cases of square-root and complex power singularities. Without losing generality, a crack lying on the $x$-coordinate is addressed (Fig. 2). For the case of square-root singularity, the diagonal block $[S^{(1)}]$ is equal to diag(−0.5, −0.5).

At the point in front of the crack on the boundary (point $P$, where $r = l_f$, in Fig. 2), the singular stress term in Eq. (29) is written as

$$\{\sigma^{(i)}(\xi = 1, \theta = 0)\} = [\Psi_{\sigma}^{(i)}(\eta(\theta = 0))]\{e^{(i)}\}$$

(32)

where \([\Psi_{\sigma}^{(i)}(\eta(\theta = 0))]\) are obtained by interpolating the stress modes \([\Psi_{\sigma}^{(i)}(\eta)]\) evaluated at the discrete Gauss points, i.e., at discrete values of $\theta$. The stress intensity factors are equal to

$$K_I = \sqrt{\pi l_f} \sigma_{\sigma}^{(0)}(\xi = 1, \theta = 0); \quad K_{II} = \sqrt{\pi l_f} \sigma_{\sigma}^{(s)}(\xi = 1, \theta = 0)$$

(33)

The $T$-stress term in Eq. (29) is independent of the radial coordinate $\xi$ (or $r$). At a point in front of the crack tip it is written as

$$\{\sigma^{(T)}(\xi, \theta = 0)\} = [\Psi_{\sigma}^{(T)}(\eta(\theta = 0))]\{e^{(T)}\}$$

(34)

The $T$-stress is simply the $xx$ component of \(\{\sigma^{(T)}(\xi, \theta = 0)\}\)

$$T = \sigma_{xx}^{(T)}(\xi, \theta = 0)$$

(35)

Note that the stress intensity factors and $T$-stress are evaluated directly from their definitions based on stresses. The type and order of singularity and the angular variation are obtained as a part of the scaled boundary finite-element solution without using an asymptotic solution of the stress field. No singular functions are evaluated numerically.

**Numerical examples**

**Rectangular plate with a central crack**

A rectangular plate of height $2h = 40$mm and width $2w = 20$mm is shown in Fig. 3(a). A horizontal crack of length $2a = 4.8$mm exists at its center. The material of the plate is characterized by shear modulus $G = 76.923$GPa, Poisson’s ratio $\nu = 0.3$ and mass density $\rho = 5 \times 10^3$kg/mm$^3$. Plane strain condition is considered. The plate is loaded at the top and bottom edges at time $t = 0$ by uniform tractions with Heaviside function time dependence $p(t) = PH(t)$.

Due to symmetry, only half of the plate is discretized. The mesh, as illustrated in Fig. 3(b), consists of three subdomains. The scaling centers of the subdomains are indicated by the markers “□”. The boundary of subdomains are discretized with 7-node elements with Gauss-Lobatto-Legendre shape functions [13]. The two nodes at the ends of an element are shown by the circles. The length of all the line elements is equal to 5mm. The crack tip is at the the scaling center of subdomain 1. A time step $\Delta t = 0.05$us is selected. The order of continued fraction is chosen as $M_c = 4$ for all subdomains. The distance from the crack tip to the point on the boundary of subdomain 1 in front of the crack tip is $l_f = w - a = 7.6$mm.
The dynamic stress intensity factor is plotted in Fig. 4(a) as the solid line. The finite element result in Ref. [5, Figure 12] is digitized and shown in Fig. 4(a) as the dash-dotted line. Very good agreement between the two results is observed. The small amplitude oscillations at very high frequency (the period is about 0.3s, which is shorter than the shortest wave the mesh can model) in the reference solution do not appear in the present results. The dynamic $T$-stress obtained with the present scaled boundary finite-element method is presented in Fig. 4(b) by the solid line. The time-domain boundary element result in [8, Figure 5] is digitized and plotted in Fig. 4(b) as the dotted line. For comparison of results, the dynamic stress intensity factor presented in Ref. [8, Figure 5] is also shown in Fig. 4(a). As the boundary element results are obtained with a larger time step $\Delta t = 0.3\mu s$, the response history is smoother. The amount of difference between the boundary element result and the present solution in the dynamic $T$-stress is similar to that in the dynamic stress intensity factor.

Rectangular plate with an inclined central crack

A rectangular plate of width $2w = 30\text{mm}$ and height $2h = 60\text{mm}$ is shown in Fig. 5(a). A crack of length $2a = 10\sqrt{2}\text{mm}$ exists. The crack passes through the center of the plate with an inclination angle of $45^\circ$. The material properties are shear modulus $G = 76.923\text{GPa}$, Poisson’s ratio $\nu = 0.3$ and mass density $\rho = 5 \times 10^{-6}\text{kg/mm}^3$. Plane strain condition is considered. The plate is loaded at time $t = 0$ by a uniform traction with Heaviside function time dependence $p(t) = PH(t)$. A mesh of 8 subdomains is shown in Fig. 5(b). Each edge of the subdomains is modeled with one 10-node elements. The scaling centers of subdomains 1 and 2 are chosen at the two crack tips. A time step $\Delta t = 0.1\mu s$ is selected. The order of the continued fraction is chosen as $M_{cf} = 4$ for all of the subdomains.

The mode I and II stress intensity factors normalized with $P\sqrt{\pi a}$ are plotted in Fig. 6a and Fig. 6b as solid lines. The finite element result [5, Figure 21] (for $K_I$ only) and the boundary element result [3, Figure 9] are digitized and shown as reference solutions. The present result for $K_I$ is in very good agreement with the finite element result. The difference with the boundary element result [3, Figure 9] is larger but still insignificant. The present result for $K_{II}$ are in good agreement with the boundary element result [3, Figure 9] other than at the very late time. The dynamic $T$-stress is shown in Fig. 6c as the solid line. The path independent integral result reported in Ref. [8, Figure 7] is digitized and plotted as the dotted line. The response history is evaluated by using the boundary element method with a time step $\Delta t = 0.4\mu s$. The dynamic stress intensity factors are not reported in this reference. Considerable difference between the present result and the reference solution exists. Additional analyses with increasing mesh density are performed by using the scaled boundary finite-element method. No appreciable difference with the present result in Fig. 6 is observed.
Conclusions

The dynamic stress intensity factors and $T$-stress for two-dimensional in-plane problems are evaluated by the scaled boundary finite-element method. The continued fraction solution for the dynamic stiffness matrix of a bounded domain containing a stress singularity point is obtained to represent the inertial effect at high frequencies. No internal mesh is required, which reduces the effort on mesh generation. The equation of motion of the bounded domain is expressed using a high-order static-stiffness and mass matrices. Efficient analysis methods in structural dynamics can be directly applied to perform response history analyses. The dynamic stress intensity factors and $T$-stress are extracted directly from their definitions. No singular function is computed close to the singular point. Numerical examples demonstrate the simplicity and accuracy of the scaled boundary finite element method.

References

The BEM applied to an inverse procedure for the determination of the acoustic pressure distribution of a radiating body

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Abstract. This article describes the application of the Boundary Element Method to solve an inverse problem of stationary acoustics. The acoustic potential around a vibrating radiator is determined by using a boundary integral relation between the actual radiator and an auxiliary, closed surface, surrounding the vibrating body. The radiator is assumed to be a two-dimensional body and it is embedded in an unbounded acoustic domain. The geometric data of the radiator external surface is known. A closed auxiliary measuring surface is set around the body at a specified distance. At this measuring surface the velocity potential of the radiated field is determined. The known Dirichlet data (velocity potential) on the auxiliary surface, in association with the given geometric surface of the radiator, allow to establish a boundary integral relation between the velocity potential on the auxiliary surface and the actual radiator surface. The resulting algebraic system is usually rectangular and very ill-conditioned. The solution of the ill-conditioned system is performed by two distinct regularization strategies. The first approach applies the concept of the Singular Value Decomposition (SVD) and the second approach uses the Tikhonov regularization technique. Both approaches are implemented and compared. It will be shown that these techniques are feasible for determining the acoustic pressure distribution on a given radiating source.

Introduction

Consider a radiating body $\Omega_x$ enclosed by a bounded surface $\Gamma_x$ and immersed in an unbounded acoustic domain $\Omega_{\infty}$, as shown in Fig. 1. The 2D stationary acoustic phenomena is governed by the Helmholtz operator [1]. The starting point of the present formulation is a boundary integral representation relating an acoustic potential $u(x)$ at a point $x$ with the acoustic potential $u(x)$ and its derivative normal to the surface of the radiating body $\partial u(x)/\partial n (x \in \Gamma_x)$:

$$c(x) u(x) = \int_{\Gamma_x} u'(x,x') \frac{\partial u(x)}{\partial n} d\Gamma - \int_{\Gamma_x} \frac{\partial u'}{\partial n}(x,x') u(x') d\Gamma(x)$$

(1)

In equation (1) the functions $u'(x,x')$ and $\partial u'(x,x')/\partial n$ are the kernels of the classical 2D fundamental solution for the Helmholtz equation and $c(x)$ is the free term [1]. If the boundary $\Gamma_x$ is divided in $n$ boundary elements, a standard BE discretization leads to the algebraic system:

$$[H]_{max} \{u(x)\}_{max} = [G]_{max} \left\{ \frac{\partial u(x)}{\partial n} \right\}_{max}$$

(2)

Assuming that the matrix $[G]$ can be inverted, equation (2) may be recast as:
Starting from the Boundary Integral Equation (1) it is possible to determine a vector containing the acoustic potential \( u_i(x) \) in \( m \) points \((i=1,m)\) within the unbounded acoustic domain \((\Omega_{\text{inf}})\):

\[
\begin{bmatrix} \partial u(x) \overline{c_n} \\ \end{bmatrix} = \left[ G \right]^{-1} \left[ H \right] \{ u(x) \}
\]

(3)

The vector containing the Neumann data on the boundary \( \Gamma_x \), \( \{ \partial u(x) / \overline{c_n} \} \), given in (3), may be inserted into expression (4) leading to:

\[
\{ u(x) \}_{\text{ext}} = \left[ \hat{G} \right]_{\text{aux}} \left[ G \right]^{-1} \left[ H \right]_{\text{aux}} \{ u(x) \}_{\text{ext}} - \left[ \hat{H} \right]_{\text{aux}} \{ u(x) \}_{\text{ext}} = \left[ R \right]_{\text{aux}} \{ u(x) \}_{\text{ext}}
\]

(4)

(5)

with

\[
\{ R \}_{\text{aux}} = \left[ \hat{G} \right]_{\text{aux}} \left[ G \right]^{-1} \left[ H \right]_{\text{aux}} - \left[ \hat{H} \right]_{\text{aux}}
\]

(6)

In equation (5) the radiating operator \([ R ]\), defined in (6), relates the acoustic potential in \( n \) points at the boundary \( \Gamma_x \) to the acoustic potential of \( m \) points within the domain \( \Omega_{\text{inf}} \).

The inverse problem consists of determining the acoustic potential \( \{ u(x) \} \) over the \( n \) points of the boundary \( \Gamma_x \) from the known \( m \) points \( \{ u(x) \} \) on the domain \( \Omega_{\text{aux}} \). Formally this can be accomplished by inverting the radiating operator \([ R ]\) in (5):

\[
\{ u(x) \}_{\text{ext}} = \left[ R \right]^{-1} \left[ u(x) \right]_{\text{aux}}
\]

(7)

The Inverse Problem.
In practice this task requires the inversion of a rectangular and ill-conditioned matrix \([R]\). Two distinct schemes have been devised to improve or to regularize the inversion of the operator \([R]\). The idea is to decrease the condition number \(\kappa\) of the matrix \([R]\), which can defined as the ratio between the largest and the smallest singular values of \([R]\), \(\kappa(R) = \frac{\sigma_{\max}}{\sigma_{\min}}\) [2, 3].

**The Singular Value Decomposition – SVD.** The generalized inverse, or pseudo-inverse, \([R]^+\) of a rectangular matrix \([R]\) can be expressed as a function of its singular values as [2,3]:

\[
[R]^+ = [V']\Sigma \left[U^T\right]
\]

The matrix \([\Sigma']\) is a diagonal matrix, the elements of which are the inverse of the singular values of \([R]\). It can be shown that the regularization consists in deleting from \([\Sigma']\) the singular values which are smaller than a given value. The success of the regularization procedure lays exactly on the proper choice of the smallest, non-negligible, singular value \(\sigma_i\) [3]. A strategy to determine this value will be given in the next session.

**The Tikhonov regularization procedure.** The scheme proposed by Tikhonov [2, 4] requires the determination of the smallest singular value \(\sigma_i\) which allows the inversion of the matrix \([R^H R]\). The matrix \([R^H]\) is the Hermitian of \([R]\). Introducing a unit diagonal matrix \([I]\), the complete inversion equation for the Tikhonov scheme may be written as [4]:

\[
\{u(\Sigma)\} = \left[[R^H]^+ [R] + \sigma_i[I]\right]^{-1} [R^H] \{u(\Sigma, i)\}
\]

**Regularization Strategy.**

The solution strategy proposed in this article is illustrated in figure 2. The idea is to recover the acoustic velocity potential \(\{u(\Sigma)\}\) at the boundary of the radiating body \(\Sigma \in \Gamma_s\) from acoustic potential values measured, or numerically determined, over a closed surface that circumscribes the body \(\{u(\Sigma, i)\}\), \(\Sigma, i \in \Gamma_{aux1}\). To establish the smallest singular value \(\sigma_i\) of \([R]\) to be considered in the SVD or in the Tikhonov regularization strategies, leading to an accurate inversion of \([R]\), the acoustic potential is measured, or numerically determined, at a second closed surface \(\Gamma_{aux2}\), which lays closer to the radiating object. This second auxiliary surface \(\Gamma_{aux2}\) is used to calibrate the inversion parameters, that is, the smallest singular value \(\sigma_i\) to be considered. Once this calibration is performed, the pseudo-inverse of \([R]\) is used to determine the acoustic potential on the actual boundary \(\Gamma_s\).

**Numerical examples.**

**A vibrating cylinder.** Consider a vibrating cylinder with unit diameter \(a=1m\), as shown in figure 3. The boundary \(\Gamma_s\) is discretized in \(n=80\) constant elements, upon which Neuman boundary conditions are applied, \(\partial u / \partial n = \cos \theta\). The unbounded domain \(\Omega_{ext}\) is considered to be air with sound velocity \(c_s=345m/s\). The acoustic potential \(\{u(\Sigma, i)\}\) is numerically determined through equation (4) for \(m=120\) equally spaced points at the auxiliary surface \(\Gamma_{aux1}\) with radius \(\xi_1=2.0m\) and with \(m=80\) points at the closer auxiliary surface \(\Gamma_{aux2}\) with radius \(\xi_2=1.05m\). In the second step, to calibrate the regularization processes, the values of the \(m=80\) points on auxiliary surface...
\( \Gamma_{aux} \) are determined as an inverse problem by means of equations (7) and (9) from the known data \( m=120 \) already known on \( \Gamma_{aux} \).

Table 1 shows the (largest) errors between the values of the acoustic potential \( \{u(x,y)\} \), directly determined by the discretized integral equation (4) and the values recovered on \( \Gamma_{aux} \) by inverting the radiating operator \([R]\) using the SVD strategy. The comparison is performed for distinct values of the classical dimensionless wave number \( ka \) [1]. The accuracy of the recovered solution is determined for distinct limiting singular values \( \sigma_x \). For each limiting singular value \( \sigma_x \) and wave number \( ka \), the system condition number \( \kappa(R) = \sigma_{min} / \sigma_{max} \) is also furnished.

An analysis reveals that for low wave numbers even relative small values of \( \sigma_x \) furnishes a good solution of the inverse problem on the auxiliary surface. As expected lower values of the condition number \( \kappa(R) \) results in more accurate solutions. Table 2 furnishes the results for the same type of analysis but considering the Tikhonov regularization procedure. The Tikhonov strategy does not present the same behavior of the SVD solution. It can be clearly seen that there is an optimum limiting value for the singular value \( \sigma_x \). For this example the best solution is obtained for \( \sigma_x=10^{-3} \).

After this calibration procedure the acoustic potential may be recovered at the actual boundary \( \Gamma_x \) of the vibrating cylinder using both inversion strategies. The acoustic potential recovered using the Tikhonov strategy with limiting value \( \sigma_x=10^{-3} \) and for the wave number \( ka=1.0 \), may be seen in figure 4. The potential on all the \( n=80 \) points of the cylinder boundary are shown. The largest error of the acoustic potential determined by this procedure is smaller than 0.2%. The SVD solution is even more accurate for this case.

A loudspeaker profile. The recovery of the acoustic potential on a 2D surface that resembles a loudspeaker profile, shown in figure 6, is discussed next. The loudspeaker profile (surface \( \Gamma_x \)) with dimensions \( H=2.4m, B=2.0m \) and \( L_1=1.5m \) and immersed in air with \( c_s=345m/s \) is divided into \( n=34 \) constant boundary elements. On this surface unit Neumann boundary conditions are imposed \( \partial u / \partial n = 1 \). An auxiliary surface \( \Gamma_{aux} \) is established with a radius \( \xi=2.0m \). Upon this surface the acoustic potential is determined in \( n=102 \) points. Table 3 shows the largest errors for the potential recovered at the original \( n=34 \) boundary points on \( \Gamma_x \) for 3 distinct limiting singular values. The dimensionless wave numbers are \( ka=10 \) and \( ka=20 \).
Table 1: Estimation of parameter $\sigma_S$ in SVD regularization

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Table 2: Estimation of parameter $\sigma_S$ for Tikhonov regularization

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Figure 5: Comparison between recovered potential (Tikhonov $\sigma_s=10^{-3}$) and direct numerical calculation on the surface of the vibrating cylinder.

Solutions for SVD and Tikhonov strategies are reported. Both strategies furnish accurate results. Nevertheless, for this example the SVD strategy is more precise. A directional pattern of the recovered potential on the loud speaker surface can be seen in Fig 7.

Figure 6: loud speaker profile

Figure 7: Directional representation of the acoustic potential recovered boundary ($ka=20$)

Table 3: Accuracy of recovered potential on $n=34$ boundary points

<table>
<thead>
<tr>
<th>$ka$</th>
<th>SVD - error %</th>
<th>TIKHONOV - error %</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>$\sigma_s = 10^{-3}$</td>
<td>$\sigma_s = 10^{-4}$</td>
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<tr>
<td>10.0</td>
<td>0.002</td>
<td>0.002</td>
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<tr>
<td>20.0</td>
<td>0.0005</td>
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Concluding remarks.

The BEM has been applied to recover acoustic potentials on the surface of a radiating body by inverting data, measured or numerically determined, at a given auxiliary closed surface. Two regularization strategies have been applied to solve the inversion problem. A second auxiliary surface has been proposed to calibrate the regularization strategies. Presented data show that the procedure leads to accurate results. Further investigations are being conducted to assess the role of discretization densities on the actual an auxiliary surfaces.

Acknowledgements.

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Error controlled analysis of non-linear problems using the Boundary Element Method

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Keywords: non-linear analysis, initial stress approach, internal cells, error controlled adaptivity, elasto-plasticity.

Abstract. This paper presents a new method to control the error of a non-linear boundary element analysis. For the evaluation of the domain integral, which arises from the initial stress formulation of non-linear problems, a mesh of either area or volume cells is used. The area and volume cells used in the formulation are of the type $C^0$, and therefore, they cannot reproduce the expected initial stress field in the non-linear zone. This discontinuity between the cells is used to estimate the error and to refine the cell mesh. It is a general method which can be used to solve 2-dimensional as well as 3-dimensional problems. Moreover, it is not restricted to certain non-linear models. The applicability and the accuracy of this new method will be demonstrated on some elasto-plastic problems.

Introduction

The Boundary Element Method (BEM) is a useful tool for solving linear as well as non-linear numerical problems. With this method, usually only the boundary of the problem has to be discretized. Thus, the dimension of the problem is reduced by one. This is a significant advantage regarding mesh generation, data storage, post-processing and the dimension of the system of equations.

When dealing with non-linear problems, not only boundary integrals but also domain integrals arise. In the standard 2d or 3d approach, a mesh of area or volume cells is used respectively for the evaluation of the domain integrals (see Beer et al. [2], Venturini [12]). By using this method the main advantage of the BEM with respect to mesh generation is partially lost because cells have to be generated by the user, either in the whole domain or in parts of the domain that are expected to behave non-linearly. However, the size of the system of equations does not depend on the domain discretization, which means that no additional degrees of freedom will be introduced using internal cells. In addition, internal cells can be generated automatically (see Ribeiro et al. [10]).

Basic equations

For the derivation of the boundary integral equation for small strain non-linear problems, the following basic equations are relevant:

- the Cauchy’s infinitesimal strain tensor, also called Green’s strain tensor

\begin{equation}
\varepsilon_{ij} = \frac{1}{2}(u_{ij} + u_{ji})
\end{equation}
- the total strain, which is expressed as a sum of the elastic and non-linear components
  \[ \varepsilon_{ij} = \varepsilon_{ij}^e + \varepsilon_{ij}^0, \]  
  \hspace{1cm} (2) \hspace{1cm}

- the Navier-Cauchy equation of equilibrium for the stress tensor
  \[ \sigma_{ij,i} + b_j = 0 \]  
  \hspace{1cm} (3) \hspace{1cm}

- the equilibrium condition on the boundary
  \[ t_j = \sigma_{ij}n_i \]  
  \hspace{1cm} (4) \hspace{1cm}

- and as far as the elastic strains are concerned, generalized Hooke's law
  \[ \sigma_{ij} = D_{ijkl} \varepsilon_{kl}. \]  
  \hspace{1cm} (5) \hspace{1cm}

By using eq. 2 and eq. 5 the non-linear part can be separated as follows

\[ \sigma_{ij} = D_{ijkl}(\varepsilon_{kl}^e - \varepsilon_{kl}^0) = \sigma_{ij}^e - \sigma_{ij}^0 \]  
\hspace{1cm} (6) \hspace{1cm}

where \( \sigma_{ij}^e \) is a notational elastic stress (corresponding to the strain \( \varepsilon_{kl}^e \)), and \( \sigma_{ij}^0 \) is termed the “initial” stress. However, the negative sign in this equation emphasizes the difference between this rather artificial decomposition of the stress (see Fig. 1).

The non-linear part \( \sigma_{ij}^0 \) depends in general on the stress-strain relationship of the chosen non-linear model. Thus, the initial stress \( \sigma_{ij}^0 \) is related via the elastic constitutive relationship to the irrecoverable component of the strain \( \varepsilon_{kl}^0 \) (e. g. Telles [11], Herding and Kuhn [7]).

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Fig. 1: Schematic stress decomposition where \( \sigma^p, \sigma^d \) or \( \Delta\sigma \) correspond to the initial stress \( \sigma^0 \)

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**Non-Linear Boundary Integral Formulation**

For a domain \( \Omega \) with boundary \( \Gamma \), the standard integral representations are derived by applying Betti’s reciprocal principle or Green’s second identity. In particular, displacement, strain and stress integral representations are easily derived and may be found in Bonnet [3].
In the non-linear case, compared to the linear case, we have to deal with incremental quantities denoted by the superposed period (for example $\dot{\sigma}$). The initial stress integral equation, which does not consider body forces, can be expressed in the incremental form as

$$\dot{u}_j(p) = \int_{\Gamma} U_{ij}(p, Q)\dot{u}_j(Q)\,d\Gamma - \int_{\Gamma} T_{ij}(p, Q)\dot{u}_j(Q)\,d\Gamma + \int_{\Omega} E_{ijk}(p, q)\dot{\sigma}_{0jk}(q)\,d\Omega \quad (7)$$

where $Q$ is a point on the boundary $\Gamma$ and $p$ and $q$ are points in the domain $\Omega$. $\dot{u}_j(Q)$ is the displacement field, $T_{ij}(Q)$ the traction field and $\dot{\sigma}_{0jk}(q)$ the initial stress field. $U_{ij}(p, Q)$ and $T_{ij}(p, Q)$ are the Kelvin fundamental solutions for displacements and tractions at the field point $Q$ in the $j$-th direction due to a unit load at the source point $p$ in the $i$-th direction. $E_{ijk}$ is the strain kernel which corresponds to the displacement field $U_{ij}$ and is obtained from

$$E_{ijk}(p, q) = \frac{1}{2}(U_{ij,k} + U_{ik,j}) \quad (8)$$

Eq. 7 is only applicable for internal points. For points on the boundary $\Gamma$, the limiting form is obtained by allowing the source point to approach the boundary ($p \to P$). This yields to the direct initial stress boundary integral equation

$$C_{ij}(P)\dot{u}_j(P) = \int_{\Gamma} U_{ij}(P, Q)\dot{u}_j(Q)\,d\Gamma - \int_{\Gamma} T_{ij}(P, Q)\dot{u}_j(Q)\,d\Gamma + \int_{\Omega} E_{ijk}(P, q)\dot{\sigma}_{0jk}(q)\,d\Omega \quad (9)$$

where $C_{ij}(P)$ is the so called jump term which is influenced by the geometry of the boundary. Apart from the additional domain integral, which considers the influence of the initial stresses, eq. 9 is the same as in linear elastic analysis. It should be mentioned that an alternative formulation is possible in which initial strains are employed rather than initial stresses. For more details see for example Cisilino and Aliabadi [4].

In order to solve eq. 9 the initial stresses $\dot{\sigma}_{0jk}(q)$ have to be determined. This requires an additional integral equation for stresses within the domain $\Omega$. The strains can be computed by differentiating eq. 7. Furthermore, the following stress integral equation can be obtained by using generalized Hooke’s law

$$\dot{\sigma}_{ij}(p) = \int_{\Gamma} D_{ijk}(p, Q)\dot{u}_k(Q)\,d\Gamma - \int_{\Gamma} S_{ijk}(p, Q)\dot{u}_k(Q)\,d\Gamma + \int_{\Omega} W_{ijkl}(p, q)\dot{\sigma}_{0kl}(q)\,d\Omega + F_{ijkl}\dot{\sigma}_{0kl}(p) \quad (10)$$

where $D_{ijk}, S_{ijk}$ and $W_{ijkl}$ are fundamental solutions and $F_{ijkl}$ are the free terms which arise from the singularity at $p$. This equation is valid for points located inside the domain $\Omega$. In case the point is located at the boundary $\Gamma$, the limit of the integral when the source point $p$ approaches the boundary can be taken. However, this procedure is computationally expensive due to the occurrence of hyper-singular integrands in the boundary integrals. To overcome this problem, an alternative and relatively simple method of evaluating boundary stresses from tractions and displacement tangential derivatives can be used (see Gao and Davies [5], Cisilino and Aliabadi [4] and Beer et al. [2]).
Error Controlled Domain Mesh Refinement

In the past, several strategies for self-adaptive linear-elastic boundary element analysis were developed. An overview of such methods for example can be found in Kita and Kamiya [8] and Liapis [9]. However, this has not been the case for non-linear analysis. Only a few contributions have been made in this area until today. For example, Astrinidis et al. [1] used an adaptive scheme based on a total strain smoothing error criterion for elasto-plasticity. Gaspari and Aristodemo [6] introduced a refinement criterion based on a check of the elastic stress level for general material non-linearities. Nevertheless, this work presents a new method which automatically controls the domain discretization error.

For evaluating the domain integrals in eq. 9 and 10, which consider the non-linear influence, either a mesh of area or volume cells is used. The internal cells used are of the type \( C_0 \), and therefore, they cannot reproduce the expected initial stress field in the non-linear zone. The initial stress field is approximated for each cell by

\[
\sigma_{ij}^0 = \sum_{n=1}^{c} N_n \sigma_{ij}^{n0},
\]

where \( N \) are the shape functions and \( c \) the number of nodes of the cell. The proposed self-correcting domain mesh algorithm uses the derivative of the approximated initial stress field by the cells as error indicator. The derivative is computed for all internal nodes by differentiating eq. 11 with respect to the Cartesian directions. This gives

\[
\sigma_{ij,i}^0 = \sum_{n=1}^{c} N_n, \sigma_{ij}^{n0}.
\]

Since each of the internal nodes has at least 2 adjacent internal cells, the derivative gives different values (see Fig. 2). The difference between such values can be taken to estimate the local error. In the algorithm this error is bounded by a certain value. If the computed value is higher then the given value the adjacent cells will be refined.

Non-Linear Solution Algorithm

In continuum mechanics, non-linear problems are commonly solved by adopting a load incremental procedure and by iterating on the strain-stress constitutive equation. In this work
a simple explicit procedure based on the classical initial stress approach with constant stiffness (see e.g. Gao and Davies [5]) was chosen. The general unknowns are boundary displacements, tractions and initial stresses. The advantage of these techniques is that the system matrices are kept constant and only the right hand side changes. However, their drawback is the increase in the number of iterations.

In the incremental procedure the load is divided into several increments. After computing the first trial for the initial stresses at each incremental step the quality of the domain mesh is checked and in the regions where the continuity of the initial stress field is not given the domain mesh automatically becomes finer.

### Numerical Results

To demonstrate the applicability and the accuracy of this new method an example which is well known in the literature is presented. A plate with a hole subject to constant tension is considered (see e.g. Astrinidis et al. [1]). Plane stress conditions are assumed. Due to the symmetry of the problem, only one half of the problem is analyzed. The geometry of the problem is shown in Fig. 3. To discretize the problem quadratic boundary elements and quadratic cells are used. The following material parameters are assumed (von Mises yield criterion):

- Elastic modulus $E = 7000 \text{kg/mm}^2$
- Plastic modulus $H' = 224 \text{kg/mm}^2$
- Yield stress $f_y = 24.3 \text{kg/mm}^2$
- Poisson’s ratio $\nu = 0.2$
- Loading $p = 0.91 \cdot f_y$

The following figures show the automatic non-conforming mesh refinement at different load increments, from the initial mesh (a) to the final mesh (e):

Fig. 4: Cell mesh refinement (the dark cells indicate the ones which have to be refined)

In Fig. 5 a comparison of the stress variation $\sigma_{yy}$ along the minimum section $A-A$ is made between some published results and the new method proposed in this paper. A good agreement is observed.
Fig. 5: $\sigma_{yy}$ variation along minimum section A-A

References


On the use of Boundary Element Methods for Inverse Problems of Damage Detection in Structures

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Keywords: Inverse Problems; Damage detection; Boundary Element Methods; Elastostatics; Acoustics; Uncertainties; Optimization.

Abstract. In this work, the inverse problem of identifying damages in a plate structure is modeled using optimization techniques. The numerical modeling consists in two problems: a direct problem, in which the Boundary Element Method (BEM) is used to obtain the potential, stress, or acoustical distribution throughout the damaged structure; and an inverse problem, in which an optimization model is used to locate the damage in the structure, given measured information on the quantity of interest at some interior points (sensor locations). This paper discusses issues related to the use of BEM as the direct method, in order to improve damage identification and also the confidence in the damage location and size results. The discussion includes: i) the use of quantities obtained from derivatives of the original densities instead of the densities themselves, as the main variables in the objective functions and constraint equations, for faster convergence of the optimization procedure; ii) the ability of the direct method to properly capture the proximity between the “numerical” damage and the “real” damage; iii) the use of independent scalar quantities (for example, invariants of the stress tensor for the elasticity problem) at the interior points of interest (where measured and numerical information are compared in the optimization procedure), in order to avoid comparing derivatives obtained in different planes and directions at a particular interior point; iv) the influence of the BEM discretization (mesh refinement) in the numerical results.

Introduction

In this work, the inverse problem of identifying the presence, location and size of damages, such as cracks and holes, in a plate structure is modeled using optimization techniques. The numerical modeling consists in two main parts. The first part on the modeling is the direct problem, in which a model is required to obtain information on the distribution of the quantity of interest throughout the structure, given the boundary conditions and the presence of the damage. The models investigated in this work include the elastostatics and acoustics (potential) formulations. For the potential (acoustics) problem, the quantities of interest are the potential and its gradient at interior points. Also, for the elastostatics problem, the quantities of interest are the interior point displacements, strains, and stresses. The modeling of the structure is carried out using the Boundary Element Method (BEM) in all cases.

The second part of the modeling consists in the inverse problem, in which a model is required for the procedure of locating the damage in the structure given some (partial) information on the quantity of interest at some particular locations (for example, where some sensors are placed). For this inverse problem, an optimization technique (genetic algorithm), will be used for all comparisons performed. This paper discusses four issues related to the use of BEM as the direct method, in order to increase damage identification and also to improve the confidence in the damage location and size results. Several runs were made for the various BEM models, for elastostatics and potential (acoustics). The first issue is the use of quantities obtained from derivatives of the original densities (such as strain or stress, for example) instead of the densities themselves (such as the displacement), as the main variables in the objective functions and constraint equations. The use of derivatives of the densities is expected to lead to faster convergence of the optimization procedure, as these derivative fields are expected to be more sensitive (local changes more evident) than the density fields, in the presence of damage. The second issue is the ability of the direct method to properly capture the proximity between the “numerical” damage (the damage included in the numerical model for a particular run of the BEM code) and the “real” damage (the damage to be detected in the real structure). In this work, the damage was simulated by a hole from which information on the density values was available at interior points corresponding to sensor locations. For this discussion, a comparison was done between the numerical results obtained with the “real” hole and results from several “numerical” holes approaching the “real” hole, to simulate the desired effect of proximity between holes. The third issue is the use of independent scalar fields, and not vector or tensor fields, as the variables of interest for the optimization procedure, in order to avoid comparing derivatives obtained in different planes and directions at a particular interior point. With this
procedure, all interior point information to be used in the optimization procedure is direction-independent. For example, the invariants of the stress tensor (mean stress and octahedral stress, for the elasticity problem) at the interior points of interest (where measured and numerical information are compared in the optimization procedure) can be used, instead of the original stresses obtained with the BEM formulation. In this work, the mean stress is used, as the scalar quantity of interest evaluated at interior points. The fourth issue is the influence of the numerical errors due to the BEM discretization of a particular problem. Different meshes and different approaches for mesh refinement may have an influence in the numerical results. Thus, the discretization is expected to interfere with the optimization results, and consequently to have an impact in the quality of the damage identification information.

**Inverse Problems of Damage Detection in Structures**

The detection of damage in structures is an important engineering issue, with applications in flight safety and aircraft maintenance. The development of damage detection techniques can contribute to a better structural integrity analysis of a structure. The analysis of a damaged structure must involve the numerical treatment of data gathered from sensors spread throughout critical points in the structure, and the comparison of this data with numerical results used as reference (results from the same structure, undamaged or with known damage). To analyze a damage detection problem in a structure, first the modeling of the direct problems is required, to obtain the behavior of this structure in the presence of one or more pre-established damages, with assumed format and size, and at given positions (see references [1] and [2] for damage detection problems). In this work, two methods of analysis need to be given particular attention: 1) the study of stress and strain distributions in damaged structural elements, performed with a displacement-BEM model for elastostatics (see references [3] to [5] for elastostatics); and 2) the study of the distribution of sound waves (emitted from a pre-established source) in the damaged element, performed with a BEM model for acoustics (see references [6] to [9] for acoustics). The next step is to study the inverse problem, which consists of two parts: 1) monitoring the structural integrity, with sensors spread throughout the structure, to obtain some knowledge about the distribution of the quantity of interest (for example, stresses or strains, strain-gages, or the acoustic potential or pressure using microphones, accelerometers, or other sensors); and 2) computation of a functional obtained from adding differences (evaluated at all measurement points) between the values evaluated using the numerical model from the first part (direct problem), for an assumed damage, and the experimental values measured in the same points for the structure with the real damage. This functional is a function of the damage location, either numerical or measured from the real structure. This functional is expected to increase in value when the assumed numerical damage is far away from the real damage, and to reach its minimum value when both damages (numerical and real) coincide.

**Direct Problem: Boundary Element Methods**

The boundary element method is a numerical procedure well adapted for the modeling of a structure with a damage. In this method, the distribution of the quantities of interest in the domain is obtained from the information of the distribution of certain quantities in the boundary. Thus, the problem is described based on what happens in its boundaries, reducing the dimension of the problem and simplifying numerically the treatment. When modeling the damage detection problem by means of an analysis of the elastic or acoustic response of the structure under excitation, perturbations in the expected response imply in the presence of damage. Thus, the damage in the structure shall characterize its behavior, static or dynamic.

**Inverse Problem: Optimization Techniques**

The inverse problem might be modeled by means of optimization techniques. For the discussion, a simple direct boundary element problem for the distribution of a potential field in a domain is considered. The damage is simulated by the presence of small holes in the domain, and the goal is to obtain size and location of the damage. The direct method (BEM) provides one piece of information (the potential) for any desired point in the domain. Without the hole, the distribution of the potential is known a priori. If a small hole is included, the potential distribution is unknown and must be obtained numerically from the BEM solution. In this problem, an inverse method (optimization) is implemented. The results obtained for the inverse method by means of this technique are used to find the location and size of the hole. Increasing the problem complexity, the BEM for the elasticity problem can be used. In this case, boundary conditions for the displacement and traction shall be provided. Differently from the BEM for the potential, the BEM for elasticity (in a 2D problem) provides two pieces of information at a single interior point – one normal stress and one shear stress. But this information cannot be used directly in the optimization problem, as it depends on the system of coordinates being used, or on the normal direction of the cutting plane that passes through the point of interest. Therefore, a choice is made to adopt the
stress invariants of the stress tensor at the point of interest – in 2D, the mean stress and the octahedral stress – as the vector field to be analyzed and used in the optimization problem. In order to increase the complexity of the damage detection problem using the BEM formulation, the goal could be to identify and locate the presence in the plate of one or more different types of damages, such as circular holes (number of holes, radius, and location of each hole), and cracks (number of cracks, orientation, size, and location of each crack). For all the different direct problems, the same optimization technique can be used to solve the inverse problem for damage detection. From the point of view of the inverse problem, the direct model is just a 'black box' to be supplied to give the numerical information needed to be used in the optimization procedure.

**Damage detection by means of optimization techniques.** The “measured” or “real” data is obtained at sensor locations spread throughout the structure, for the assumed size and location of the real damage, simulated using BEM. The potential values at interior points simulate the information collected by the sensors at these points. In order to solve the inverse problem, an optimization algorithm, such as the Genetic Algorithm (GA), is used. The evaluation (fitness) function is formulated as a functional defined as a difference between measured (simulated) values of the local difference in the potential (between the undamaged plate and the plate with the damage) and the values of the same differences in potential calculated at the same points by the code (assuming several different locations and sizes for the ‘numerical’ damage). The general form for the functional to be minimized is given by Eq. (1).

\[ J = \frac{1}{n} \sum_{i=1}^{n} (\text{measured}_i - \text{calculated}_i)^2 \]  

where:

- \( n \) - Number of internal points ("sensors" placed in the plate) where differences are evaluated;
- \( \text{measured}_i \) - Vector of simulated values for the differences obtained using BEM, for a given damage;
- \( \text{calculated}_i \) - Vector of differences in potential calculated by the code for each individual \( j \).

Figure 1(a) represents an undamaged thin plate with sensors indicating the points where the measurement of the quantities of interest (such as differences in potential or in stresses) is being performed. In order to solve the damage detection problem, an initial population is given to the GA. This initial population is formed by individuals constituting a possible solution for the problem. These individuals are chromosomes, which are themselves constituted by genes. Each gene in a chromosome represents one variable in the problem (such as position and size of a hole). As an example, Fig. 2(b) represents three possible configurations of chromosomes. While the location and size of the hole varies, the number and location of the sensors remains the same, for all chromosomes. The information on the quantity of interest is collected at these sensor locations for all cases.

![Figure 1](image1.png)  
(a) Undamaged plate with four representative sensors; (b) Plate with a hole: three possible configurations for the chromosomes

For a review of calculus-based optimization algorithms, see reference [10]. For heuristics based in the imitation of behaviors found in nature, such as genetic algorithms, see references [11] and [12].

**Numerical Results and Discussion**

Several runs were made for the various BEM models, for elastostatics and acoustics (potential), in order to discuss several aspects related to the use of BEM as the direct method, and the influence of these aspects in the ability of the optimization algorithm to identify the damage and also in the reliability in the damage location and size results. For the various runs a plate problem of a square domain was considered, where the defect is simulated by the presence of a circular hole. For the elasticity problem, a BEM model was built for the plate with a hole with the boundary conditions illustrated in Fig. 2(a). Two discretizations were implemented for the external contour, a coarse mesh with 12 elements and a fine mesh with 48 elements. Fig. 2(b) shows the discretization for the case of 48 elements in the outer boundary and 12 elements in the hole, as well as the position of the nine sensors.
For the acoustics problem, a model was built for the plate with a hole with the dimensions and location of sound source as illustrated in Fig. 3(a). Fig. 3(b) shows the boundary conditions and also the discretization for the case of 24 elements in the outer boundary and 24 elements in the hole.

The numerical results and discussion were obtained considering four issues. The first issue is the use of quantities obtained from derivatives of the original densities instead of the densities themselves, as the main variables in the objective functions and constraint equations. For the acoustic model of the plate, several interior point results were obtained, both for the potential field and for its derivative. These results were obtained as a function of time (where \( t = 0 \) represents the instant when the source has emitted a sound), for a plate with and without a hole. Fig. 4 presents comparisons through time for an illustrative interior-point, where the potential results are shown in Fig. 4(a), and the results for the derivative of the potential are presented in Fig. 4(b).

Figure 2 – Elasticity plate model: (a) dimensions, loading, boundary conditions. Insert shows a stress-free hole; (b) boundary discretization (fine mesh) and sensor locations. Insert shows hole discretization.

Figure 3 – Acoustic model for the plate with a hole: (a) dimensions and position of the sound source; (b) loading, boundary conditions and plate discretization. The insert shows the discretization of the hole.

Figure 4 – Results through time at an interior point, for the plate with and without hole (a) comparison of the distribution of the potential field; (b) comparison of the distribution of the derivative of the potential.
One can see from the numerical results in Fig. 5 that the derivative of the density is more sensitive to the presence of the hole than the density itself. Thus, one can expect that the use of the derivative of the potential may lead to a faster convergence for the optimization algorithm used for the detection of the presence of damage (such as the hole in this example), when comparing to the use of the potential in the model of the direct problem in this algorithm.

The second issue is the ability of the direct method to properly capture the proximity between the “numerical” hole (the hole included in the numerical model for a particular run of the BEM code) and the “real” hole (the hole to be detected in the real structure). A “real” hole was simulated in this work as a central hole, and the “measured” information on the sensor locations was obtained simply as numerical results for the interior points corresponding to these sensor locations. With this approach, a comparison was done between the numerical results obtained with the “real” hole and the numerical results obtained for several “numerical” holes approaching the center of the plate, to simulate the desired effect of proximity between holes. For this discussion, several runs were made using the acoustic model, with the “numerical” holes approaching the “real” hole, as illustrated in Fig. 5(a). Numerical results for the time distribution of the potential at interior points were obtained for the various cases. Fig. 5(b) illustrates the fact that, for a particular interior point, the differences in the potential (throughout a given time interval) decrease when the “numerical” hole approaches the “real” hole. Thus, the area between the “real” curve and the curve obtained from the “numerical” hole decreases in the vicinity of the real hole.

Figure 5 – Influence of the proximity between “numerical” and “real” holes for the acoustic model: (a) setup of “numerical” holes approaching the “real” hole; (b) comparison for potential results at an interior point; (c) comparison of areas between curves for the time distribution of the potential at these points.

In this work, a metric was established for a numerical measure of the approximate area between these curves of the potential for the various cases, evaluated as shown in Eq. (2):

$$A = \int \left( f_1(t) - f_2(t) \right)^2 dt = \sum_{i=1}^{12} \left( f_1(t_i) - f_2(t_i) \right)^2$$  \hspace{1cm} (2)

where: - $f_1$ - potential at a given interior point for the “real” hole;
- $f_2$ - potential at a given interior point for the “numerical” hole being simulated.

Fig. 5(c) shows the plot of this metric, used to approximate the areas between curves, where teh integrals were approximated by summations, in the form of Eq. (2). One can note that the metric used for the areas between curves of potential at the interior point considered has decreased when the “numerical” hole has approached the “real” hole, thus showing that this metric has the ability to properly capture the proximity between the “numerical” hole and the “real” hole.

For further developments and discussions, one can expect that a more adequate metric to indicate the proximity between the “numerical” hole and the “real” hole could be given by the summation of the areas between curves of potential for a greater set of interior points, where sensors would be placed in the real structure. Furthermore, following the above discussion of the first issue, one can expect also that the information of the areas between curves of derivatives of the potential would be more sensitive to the proximity between the “real” and “numerical” holes than the information on the differences of areas for the potential.

Either by using the potential or the derivative of the potential, a functional for the differences among areas between curves, for the various interior points corresponding to sensor locations, could be built in a form similar to Eq. (1), by replacing the potential information by the areas between curves. One such functional $J$ could be used in an optimization algorithm, in the form of a minimization problem ($\min J = \sum A_i$), as the inverse problem to detect the presence of the hole in the structure.

The optimization algorithm is currently being implemented for the acoustics problem, and is already implemented for the elastostatics problem. In what follows, the issues proposed were analyzed and discussed using numerical results for the inverse problem using genetic algorithm (GA) for the optimization, and using the elastostatics model as the direct problem.
The third issue is the use of independent scalar fields as the variables of interest for the optimization procedure, to avoid comparing derivatives obtained in different planes and directions at a particular interior point. The elastostatics problem shown in Fig. 2 was solved for the mean stress at interior points of the damaged plate, and also for the stress components in the \( x \) and \( y \) directions. The plots in Fig. 6(a) show a comparison between the good damage identification results (in this case, location of the \( y \)-coordinate of the center of the hole) using this scalar field, and the poor results (both in terms of mean value and of uncertainty) for the identification when some elements of the original stress tensor (say, normal or shear stresses in the given directions \( x \) and \( y \)) were used as variables in the optimization procedure. Fig. 6(b) and (c) show the plots of the location and size of the holes obtained from 10 different runs of the genetic algorithm (GA) used in the optimization procedure. The genetic algorithm, due to its own randomness, generates a different optimal solution every time it is run, but the GA results obtained from the 10 runs using the mean stress present a tendency to be more concentrated near the "real" hole than the GA results using other variables. This feature is clearly illustrated in Fig. 6(b) and (c), for locating the hole using the mean stress (\( \sigma_m \)) and the normal stress in \( x \)-direction (\( \sigma_x \)), respectively.

![Figure 6](image1.png)

\textbf{Figure 6} - comparison between results using a scalar quantity (\( \sigma_m \)) versus vector components (\( \sigma_x, \sigma_y, \tau_{xy} \)) as variables in the optimization procedure (a) location of \( y \)-coordinate of identified hole (mean values and uncertainty); (b) plot of holes located using \( \sigma_m \); (c) plot of holes located using \( \sigma_x \).

Also, one must note that the results obtained using the mean stress and/or the octahedral stress are direction-independent, while a simple rotation of axis could have led to completely different results for the identification procedure using the original components of the stress tensor as variables for the optimization problem.

The fourth issue is the influence of the numerical errors due to the BEM discretization in the optimization results. A comparison is shown in Fig. 7 for the optimization results (using 10 runs of a GA algorithm) and for the quality of the damage identification for two different holes, for two meshes (a coarse mesh and a fine mesh) using the elastostatics formulation for the plate shown in Fig. 2(a). Fig. 7(a) shows illustrative results for the mean values of the error in the location (\( x \) and \( y \) coordinates) and size (radius \( r \)) of a center hole, while Fig. 7(b) shows illustrative results for the values of the coefficient of variation (COV) of the error in the location and size of a hole located at a different position (\( x=2.5, y=3.0 \)).

![Figure 7](image2.png)

\textbf{Figure 7} - influence of BEM discretization in illustrative damage identification results: (a) mean values for the error in location and size of a center hole; (b) coefficient of variation (COV) of the error in the location and size of a hole located at a different position.

The results presented in the plots in Fig. 7 indicate the influence of the mesh refinement, showing a trend in which the fine mesh has led to better damage identification results, both in mean values and in terms of
uncertainty, as expected. These results are preliminary, as several different runs need yet to be made, for different damage types, sizes and locations, and for different degrees of freedom in the mesh refinement.

Conclusions

In this work, four issues were discussed concerning the use of the Boundary Element Method (BEM) as the direct problem in an inverse problem of identifying damages in a plate structure using optimization techniques. The BEM codes discussed include potential, elastostatics and acoustical distributions throughout the damaged structure. The focus in the discussion was the improvements in the damage identification and also in the confidence in the damage location and size results. The numerical results obtained led to some preliminary conclusions the issues discussed, as follows:

i) the use of quantities obtained from derivatives of the original densities instead of the densities themselves, as the main variables in the objective functions and constraint equations, is expected to lead to faster convergence of the optimization procedure. Results were presented in this work for the acoustics case, but available preliminary results (not shown) for the elasticity case indicate the same trend;

ii) the BEM codes used in the direct method have the ability properly capture the proximity between the “numerical” damage and the “real” damage. Results were presented in this work were for the acoustics case in time domain. One can expect results for the acoustics problem in frequency domain to present the same trend (ongoing research). Available preliminary results (not shown) for the elasticity case also indicate the same trend;

iii) the use direction-independent scalar quantities (in this work, invariants of the stress tensor for the elasticity problem) at the interior points of interest (where measured and numerical information are compared in the optimization procedure), has led to consistently accurate results for the damage identification, when comparing to the results obtained using direction-dependent quantities directly obtained from the BEM output in the optimization algorithm; and

iv) the BEM discretization (mesh refinement) has shown an influence in the numerical results, for the elastostatics case evaluated, as the damage identification results for the finer meshes tested were more accurate and presented less uncertainty than in the case of the coarse meshes tested.

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References

Algebraic preconditioning techniques for large-scale boundary integral equations in electromagnetism: a short survey

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Abstract. The Fast Multipole Method has been introduced by Greengard and Rokhlin to reduce the complexity of large particle simulations. Its use has been successively extended to the analysis of wave scattering problems. For Maxwell and Helmholtz equations, it can reduce significantly the algorithmic and memory cost of one matrix-vector product with matrices arising from the Galerkin discretization, avoiding the storage and computation of all the entries. However, for most integral equations of practical interest the number of iterations remains unacceptably large and the use of robust preconditioning is urged. In this paper we survey some recent advances in the design of algebraic preconditioners for this problem class. Nowadays realistic simulations involving many million unknowns can be carried out in only a few hours of CPU time on a moderate number of processors.

Introduction

Recent developments on boundary element techniques have contributed to increase significantly the popularity of integral equation methods for the solution of Helmholtz and Maxwell equations on 2D and 3D problems as an efficient alternative to differential equation methods. Scattering applications address the physical issue of detecting the diffraction pattern of the electromagnetic (EM) radiation that is scattered by a large and complex body illuminated by an incident incoming radiation. The relevant interest for EM scattering problems is due to the fact that their accurate numerical solution is required in the simulation of many industrial processes, such as the prediction of the Radar Cross Section (RCS) of arbitrarily shaped 3D objects like aircrafts, the analysis of EM compatibility of electrical devices with their environment, the design of antennas, absorbing materials, and many others. Objects of interest in industrial applications generally have large dimension in terms of the wavelength, and the computation of their scattering cross section can be very demanding in terms of computer resources. Classical discretization schemes like the finite-element method (FEM) or the finite-difference method (FDM) can be used to discretize the continuous model and give rise to a sparse linear system of equations. The domain outside the object is truncated and an artificial boundary is introduced to simulate an infinite volume (see e.g. [1,2]). In the last thirty years a growing research attention has been devoted to integral equation methods, because they require a simple surface description of the target by means of triangular facets; this means that a 3D volume problem is reduced to solving a 2D problem on the surface simplifying the mesh generation significantly. This feature is especially advantageous for modeling moving objects and avoiding grid dispersion errors which typically occur in the discretization of large 3D domains by differential equation solvers (see e.g. [3]). The Method of Moments discretization of integral equations [4] gives rise to dense and complex-valued linear systems. The number of unknowns N of the pertinent system grows linearly with the size of the scatterer and quadratically with the frequency of the incoming radiation. In fact, for physical consistency it is necessary to use at least ten discretization points per wavelength [5] to recover the oscillating behavior of the Green's function as well as geometric or physical singularities of the scatterer. An electromagnetic scattering simulation of a
realistic aircraft illuminated at one GHz of frequency may lead to linear systems with a few millions unknowns. In-core direct solvers, that have \( O(N^2) \) memory complexity and \( O(N^3) \) algorithmic complexity in a straightforward implementation, are not feasible even for solving medium-size problems. Indeed, the large dimension of these linear systems is often the main computational bottleneck limiting the viability of integral equations in real-life simulations. Iterative Krylov methods can solve the memory bottleneck of direct methods provided fast matrix-vector (M-V) product operations are employed. For boundary integral equations, efficient algorithms have been proposed to carry out fast and approximate M-V products using less than the straightforward \( O(N^3) \) arithmetic operations and memory locations. The Fast Multipole Method (FMM) by Greengard and Rokhlin is an algorithm in this class \([6,7]\). Originally proposed in the context of particles simulations to evaluate rapidly the potential and force fields in systems involving a large number of interactions, it was successively applied to develop fast Helmholtz and Maxwell solvers expressed in an integral formulation in acoustic and electromagnetic scattering applications (see e.g. \([8,9]\)).

The algorithm computes interactions amongst degrees of freedom in the mesh at different levels of accuracy, depending on their physical distance. A two-level implementation of FMM reduces both the memory and the complexity of a matrix-vector product operation from \( O(N^2) \) to \( O(N^{1.5}) \), a three level to \( O(N^{4/3}) \), and the Multilevel Fast Multipole Algorithm (MLFMA) to \( O(N \log N) \). From a linear algebra point of view, with the help of the addition theorem the FMM can be represented as a decomposition of the coefficient matrix \( A \) as the sum of three terms

\[
A = A_{\text{diag}} + A_{\text{near}} + A_{\text{far}}, \quad (1)
\]

where \( A_{\text{diag}} \) is the block diagonal part of \( A \) associated with interactions of basis functions belonging to the same box, \( A_{\text{near}} \) is the block near-diagonal part of \( A \) associated with interactions of basis functions belonging to one level of neighboring boxes (they are 9 in 2D and 26 in 3D), and \( A_{\text{far}} \) is the far-field part of \( A \). In the M-V product operation, the contributions \( A_{\text{diag}} \cdot x \) and \( A_{\text{near}} \cdot x \) can be derived from MoM and are computed exactly, while the product \( A_{\text{far}} \cdot x \) is computed approximately by MLFMA; note that MLFMA is applied to off-diagonal matrix elements only, which are two or three orders of magnitude less than diagonal elements for EM scattering problems. A matrix problem involving \( N \) unknowns can be solved using Krylov subspace solvers in \( \alpha \cdot n_{\text{iter}} \cdot O(Ax) \) flops where the constant \( \alpha \) depends on the implementation of the specific iterative method, \( n_{\text{iter}} \) is the number of required iterations to achieve a certain accuracy for the approximate solution and \( O(Ax) \) is the algorithmic complexity of each M-V product. The number of iterations \( n_{\text{iter}} \) can be unacceptably large for some integral operators. Analytical preconditioners are based on regularized techniques that may lead to integral formulations requiring less iterations, but they are more expensive and in general problem dependent. Another approach is to use algebraic preconditioners; a preconditioner \( M \) is a matrix that transforms the original linear system into an equivalent one which is more amenable to the iterative solution. Preconditioning is generally not required for solving smooth and compact integral operators \([10,11]\) while it is essential for non-compact operators associated with singular integral equations. Also, properties of the target like geometry and material may affect the speed of convergence. Problems with cavities or open surfaces are likely to require many more iterations than closed objects of the same physical size; nonuniform meshes are known to produce ill-conditioned MoM matrix equations as well. In this short paper, we survey recent results with algebraic preconditioners for this problem class.

**Preconditioning boundary integral equations**

For surface integral equations, three formulations are generally considered. For open targets, the so-called Electric Field Integral Equation (EFIE) is mandatory to use; for closed targets, the Magnetic Field Integral Equation (MFIE) can be used. Both formulations suffer from
nonuniqueness of the solution. This problem can be solved by combining linearly EFIE and MFIE. The resulting formulation, known as Combined Field Integral Equation (CFIE), is the formulation of choice for closed targets. The standard Galerkin method can be used to discretize the integral equation and generate a dense linear system of equations. The coefficient matrix of the linear system is symmetric for EFIE, nonsymmetric for CFIE and MFIE. The unknowns of the linear system are associated with the vectorial flux across an edge in the mesh, and the right-hand side depends on the frequency and the direction of the illuminating wave. CFIE leads to a Fredholm equation of the second kind and the coefficient matrix of the discretized system is well conditioned with most of the eigenvalues grouped in the right half-plane \cite{12}. On CFIE, unsymmetric Krylov solvers scale as $O(N^{0.25})$. For EFIE, most of the eigenvalues are scattered in the left half-plane and some are grouped close to zero; near-singularity of the discretized operator is also revealed by pseudospectra analysis \cite{13}. As a result, Krylov methods scale as $O(N^{0.5})$ and preconditioning is crucial to accelerate the convergence. Most of the preconditioners for dense matrices follow a common design pattern: given a decomposition of the system of the form

$$\begin{align*}
(S+B)x &= b \\
(I + S^{-1}B)x &= S^{-1}b
\end{align*}$$

(2)

where $S$ is sparse, it retains the most relevant contributions to the singular integrals and is easy to invert, while $B$ can be dense, we compute $M$ from $S$ using only local information. The transformed preconditioned system has the form

$$\begin{align*}
(I + S^{-1}B)x &= S^{-1}b
\end{align*}$$

(3)

The motivation to consider decompositions of the form (2) can be settled in the framework of splitted operators \cite{14}; of course, the choice of $S$ is important for performance. The simplest approach is to compute $S$ from $A$ by means of sparsification strategies that use either algebraic, graph or mesh information. Algebraic strategies compute $S$ by dropping all the entries lower than a prescribed threshold \cite{15,16}; graph-based strategies exploit information extracted from the connectivity graph of the underlying physical mesh \cite{16, 17} by performing a breadth-first search on the neighbors of each edge; mesh-based strategies use the spatial coordinates of the nodes in the mesh describing geometric neighborhoods amongst the edges \cite{18}. Comparative experiments reported in \cite{18} suggest that there is little to choose. All different approaches can provide good approximations to the dense coefficient matrix for very low sparsity ratios (up-to 2%). However, geometric information can take into account possible deformations of the geometry and are particularly suited to capture geometric singularities of non-smooth scatterers, like in the presence of breaks on the surface, cavities, disconnected parts. A splitted form of the discrete operator is directly available when fast integral solvers like the Fast Multipole Method are considered. We recall that fast methods partition the mesh of the object by recursive subdivision into disjoint aggregates or boxes of small size compared to the wavelength, and the coefficient matrix of the pertinent system can be written in the form (1). In this case, we may take $S = A_{\text{diag}} + A_{\text{near}}$ as sparse local matrix that is determined using information from the underlying physical mesh. In the next sections we examine the behavior of some popular algebraic preconditioning methods for this problem class.

**Incomplete factorization.** The Incomplete LU (ILU) preconditioner is one of the most robust and efficient algebraic method for preconditioning a linear system. It computes an approximate triangular factorization of $S$ by performing an incomplete Gaussian elimination; sparsity may be imposed on the triangular factors extracting information from either the graph of the matrix or the magnitude of the entries of the approximate factors \cite{19}. Early experiments with ILU on small systems arising from the discretization of boundary integral equations are found in \cite{20}. The most
basic preconditioner in this class is ILU(0) that uses the sparsity pattern of S for the triangular factors; it can be considered the method of choice for preconditioning CFIE as it significantly improves block Jacobi and often delivers rates of convergence very similar to LU [13]. The setup of ILU(0) is completely automatic so that the iterative solver maintains the same algorithmic and memory cost of FMM. More sophisticated methods that allow some fill-in in the factors may result in a slight reduction of the number of iterations but for CFIE their use does not pay off in general as the construction and the application are more expensive. On the other hand, on indefinite formulations like EFIE, ILU(0) is not robust enough, and it is necessary to introduce some fill in the factors. The reason is that EFIE gives rise to linear systems whose coefficient matrix is indefinite and thus more difficult to solve. To control memory storage, the maximum number of entries per column in the approximate factors can be taken to be of the same order as the number of nonzeros in the near-field matrix; this value is easy to determine from the problem after the initial setup. A threshold parameter can be also introduced during the factorization to drop small entries that contribute little to the quality of the preconditioner. For sequential runs, the incomplete factorization is computationally attractive and competitive because the construction is quite cheap. The parallelization is not straightforward but can be carried out efficiently using domain decomposition techniques with moderate computational overhead. At our knowledge, the largest reported experiment so far with ILU preconditioners on the EFIE has O(10^5) unknowns [13]; the scalability of ILU for very large problems remains a relevant open research issue to explore. Due to indefiniteness, on the EFIE the triangular factors may be ill-conditioned and the triangular solves numerically unstable so that the results may be unpredictable (see Table 1). This numerical issue may be determined by excessive dropping and/or occurrence of small pivots during the factorization. The true reason of failure can be detected using condition estimators. The conditioning should be checked before starting the iterations at the additional cost of one forward and backward substitution. If it reveals potential ill-conditioning of the factors, the reason of failure is mostly due to the presence of small pivots. In this case, the matrix needs to be preprocessed using either diagonal shifts [20], reordering schemes, or pivoting [13]. Pivoting seems the most robust stabilization approach.

Table 1: Experiments with incomplete factorization preconditioner with no pivoting on a model problem (size n=2048).

<table>
<thead>
<tr>
<th>Density of sparsification of A = 2%</th>
<th>Density of sparsification of A = 4%</th>
</tr>
</thead>
<tbody>
<tr>
<td>IC (level)</td>
<td>Density of L Cond (L)</td>
</tr>
<tr>
<td>IC(0)</td>
<td>2.0%</td>
</tr>
<tr>
<td>IC(1)</td>
<td>5.1%</td>
</tr>
<tr>
<td>IC(2)</td>
<td>9.1%</td>
</tr>
</tbody>
</table>

Approximate inverse methods. In the last ten years, a considerable amount of research work has been devoted to approximate inverse methods for solving boundary integral equations (see e.g. [18, 21, 22, 23]). The idea is to compute an explicit sparse approximation M of A^(-1) or of its factors and use it as preconditioner at each step of an iterative solver. This approach is clearly appealing for massively parallel implementations as the preconditioning operation reduces to carry out one or more sparse matrix-vector products. In the general case it is not obvious how to determine a good sparsity pattern for M that captures most of the largest entries of A^(-1); indeed, it is known that the
inverse of an irreducible matrix is structurally dense. However, for dense matrices arising from the discretization of boundary integral equations selecting a suitable structure of $M$ turns out not to be a critical issue if $M$ is computed in unfactorized form. Owing to the rapid decay of the discrete Green’s function, the entries of $A^{-1}$ tends to decay very rapidly far from the diagonal and many of them are very small. The discrete Green’s function can be considered as a row or as a column of the exact inverse depicted on the physical computational grid. Additionally, the pattern of $A^{-1}$ after dropping the small entries resembles very closely to the pattern of a sparsification of the coefficient matrix (see Figure 1).

![Figure 1: Nonzero pattern of $A$ (on the left) $A^{-1}$ (on the right) after dropping the small entries.](image)

This means that a very sparse matrix can effectively approximate the exact inverse. One natural option is to impose on $M$ the same pattern of the near-field matrix $S$; this strategy generally provides very good results. Thus the preconditioner can be efficiently combined with FMM techniques resulting in a purely gray-box method for this problem class. Pattern selection strategies based on graph or mesh or algebraic information for approximate inverse methods are thoroughly discussed in [18,21,22]. Frobenius-norm minimization methods have been successfully used by several authors on this problem class; they compute the approximate inverse as the matrix $M$ that minimizes $\|p - AM\|_F$, subject to certain sparsity constraints. The Frobenius-norm is generally chosen since it allows the decoupling of the constrained minimization problem into $N$ independent linear least-squares problems, one for each column (resp. row) of $M$ when preconditioning from the right (resp. left). The independence of the least-squares problems follows immediately from the identity

$$\|p - AM\|_F = \sum_{j=1}^{N} \|e_j - Am_j\|_F,$$

where $e_j$ is the $j$-th canonical unit vector and $m_j$ is the column vector representing the $j$-th column of $M$. Both the construction and the application of $M$ are embarrassingly parallel. The box-wise decomposition of the domain carried out by FMM naturally leads to an a priori pattern selection strategy for $M$ using geometric information, that is on the spatial distribution of its degrees of freedom. The nonzero structure of the column of the preconditioner associated with a given edge is defined by retaining all the edges within its leaf box and those in one level of neighboring boxes. The approximate inverse turns out to have a sparse block structure; each block is dense and is
associated with one leaf box. Indeed the least-squares problems corresponding to edges within the same box are identical because they are defined using the same nonzero structure and the same set of entries of $A$. It means that only one block QR factorization per leaf box is necessary and can be computed using BLAS-3 that enables to exploit data locality in the cache memory and reduce the setup cost [24]. It is evident that the construction of the preconditioner is straightforward to parallelize.

**Multilevel methods**

The lack of scalability with respect to the frequency of the problem is an important limit of local preconditioners. Indeed, the number of nonzeros in the MoM part of the FMM matrix tends to decrease with the frequency. When the matrix $S$ becomes very sparse, information related to the far-field is completely lost and some global mechanism has to be introduced to recover information on the numerical behavior of the discrete Green’s function. Multilevel mechanisms are designed to enhance the robustness and the scalability of the iterative solver, possibly preserving inherent parallelism. The multipole matrix is a suitable candidate to consider also in the design of global preconditioners. The two-level algorithm described in [24] performs a few steps of an inner Krylov method for the preconditioning operation (see Figure 2). The outer solver must be able to work with variable preconditioners; amongst various possibilities, we mention FGMRES [19]. From a numerical point of view, the efficiency of the algorithm relies on two main factors: the inner solver has to be preconditioned so that the residual in the inner iterations can be significantly reduced in a few steps, and the matrix-vector products within the inner and the outer solvers can be carried out with a different accuracy. The desirable feature of using different accuracies for the matrix-vector products is enabled by the use of FMM: a highly accurate FMM has to be used within the outer solver, as it governs the final accuracy of the computed solution. A less accurate FMM is used within the inner solver, as it is a preconditioner for the outer scheme. Inner-outer schemes can enable the solution of problems involving several millions unknowns on parallel computers [24,25].

---

**Iterative solver: flexible Krylov method**

\[
\text{Do} \quad \\
\quad \text{M-V product: } y = A_{\text{FMM}} \cdot x \\
\quad \text{Preconditioning: } z = M^{-1} y \quad (\text{GMRES, QMR, ...}) \\
\text{For } i=1,2,... \\
\quad \text{M-V product: } M \approx A_{\text{FMM}} \\
\quad \text{Preconditioning: Approximate inverse} \\
\text{until convergence}
\]

Figure 1: Inner-outer iterative schemes with different levels of accuracy for the M-V product operations.

**Other approaches.**

Efficient multigrid methods having the same memory and arithmetic cost of the M-V product operation have been proposed for integral equations problems, like the hypersingular and the single-layer potential integral operators arising from the Laplace equation (see e.g. [26]). Geometric multigrid can be tricky to implement as they need a hierarchy of grids; on the other hand, algebraic multigrid can offer the advantage to preserve good convergence properties using only single grid information. Combining multigrid with fast solvers makes it possible to maintain the cost of the preconditioning procedure of the same order of a M-V multiplication up to some constant factor.
Finally, we mention that another active research area is currently addressing the development of efficient preconditioners based on wavelet transformations for this problem class (see e.g. [28]). Nowadays the main bottleneck is rapidly moving from the design and simulation to the pre- and postprocessing of the results, as the tools are not yet available to easily manipulate large meshes with millions of degrees of freedom.

Table 2: Number of inner+outer iterations and CPU time for experiments on a realistic aircraft simulation on a Compaq machine (8 processors, 1.2 Gflops).

<table>
<thead>
<tr>
<th>dof</th>
<th>GMRES(∞)</th>
<th>FGMRES(30,60)</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>Iter</td>
<td>Time</td>
</tr>
<tr>
<td>213084</td>
<td>973</td>
<td>7h 19m</td>
</tr>
<tr>
<td>591900</td>
<td>1461</td>
<td>16h 42m •</td>
</tr>
<tr>
<td>1160124</td>
<td>•</td>
<td>•</td>
</tr>
</tbody>
</table>

Notation: - : memory or CPU time exceeded; • = run on 64 processors; d=day, h= hour, m=minutes.

References