Adaptive generalized function projective synchronization of uncertain chaotic systems

Yongguang Yu a,c,∗, Han-Xiong Li b,c

a Department of Mathematics, Beijing Jiaotong University, Beijing 100044, China
b School of Mechanical and Electrical Engineering, Central South University, Changsha, 410083, China
c Department of MEEM, City University of Hong Kong, Kowloon, Hong Kong, China

ABSTRACT

This paper mainly investigates adaptive generalized function projective synchronization of two different uncertain chaotic systems, which is a further extension of many existing projection synchronization schemes, such as modified projection synchronization, function projective synchronization and so on. On the basis of Lyapunov stability theory, an adaptive controller for the synchronization of two different chaotic systems is designed, and some parameter update laws for estimating the unknown parameters of the systems are also gained. This technique is applied to achieve synchronization between Lorenz and Rössler chaotic systems. The numerical simulations demonstrate the validity and feasibility of the proposed method.

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1. Introduction

Chaos synchronization has gained a lot of attention in a variety of research fields over the last two decades [1–3]. Because chaos synchronization can be applied in vast areas of physics and engineering science, and in particular in secure communication [3,4], various types of chaos synchronization have been investigated, such as complete synchronization [5–8], phase synchronization [9,10], anti-synchronization [11], partial synchronization [12], generalized synchronization [13], lag synchronization [14], Q-S synchronization [15] and projective synchronization (FPS), where the responses of the synchronized dynamical states can synchronize up to a scaling function factor. Up to now, there have only been a few papers investigating the FPS method [31–35]. In Refs. [31–34], the FPS of two identical chaotic systems was presented. Furthermore, the FPS of two different chaotic systems was also investigated in Ref. [35], but the corresponding scaling function factor is too simple, and the uncertainty can only arise in the drive system.

∗ Corresponding author at: Department of Mathematics, Beijing Jiaotong University, Beijing 100044, China. Tel.: +86(10) 51683792.
E-mail address: ygyu@bjtu.edu.cn (Y. Yu).

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At present, most of the theoretical results concerning synchronization of chaos are mainly focused on systems whose models are identical or similar, and parameters are exactly known in advance. But in many practical situations, the synchronization of chaos can be discussed for two strictly different systems, such as in social science and biological science. Furthermore, the parameters of many systems cannot be known entirely; the synchronization will be greatly affected by these uncertainties. Inspired by the above discussion, in this paper we will investigate a new synchronization scheme—generalized function projective synchronization (GFPS) of two entirely different systems with fully unknown parameters. The proposed synchronization scheme is an extension of many existing projective synchronization schemes, such as MPS, FPS and so forth. The responses of synchronized dynamical states can synchronize up to a function matrix in the GFPS schemes. Combining Lyapunov stability theory with adaptive control theory, we design an adaptive controller and parameter update law to ensure that the GFPS of two different uncertain chaotic systems is obtained and the unknown parameters are also estimated. The canonical Lorenz and Rössler chaotic systems are chosen as the drive and response systems to demonstrate the effectiveness of the results obtained.

The layout of this paper is as follows. In Section 2, the definition of generalized function projective synchronization is given. In Section 3, the GFPS between uncertain Lorenz and Rössler chaotic systems is derived, and a scheme of GFPS of uncertain chaotic systems is designed. Numerical simulations are performed to verify the effectiveness of proposed scheme in Section 4. Finally, the conclusion ends the paper.

2. The scheme of generalized function projective synchronization of chaotic systems

The chaotic (master and slave) systems can be given in the following form:

\[
\dot{X}(t) = F(X),
\]

\[
\dot{Y}(t) = G(Y) + U(X, Y, t)
\]

where \(X = (x_1, x_2, \ldots, x_n)^T, Y = (y_1, y_2, \ldots, y_m)^T \in \mathbb{R}^n\) are state vectors of the systems (1) and (2), respectively; \(F, G : \mathbb{R}^n \rightarrow \mathbb{R}^n\) are two continuous vector functions and \(U : [\mathbb{R}^n, \mathbb{R}^n, \mathbb{R}] \rightarrow \mathbb{R}^n\) is a controller which will be designed.

**Definition 1.** For the master system (1) and the slave system (2), there is said to be generalized function projective synchronization (GFPS) if there exists a vector function \(U(X, Y, t)\) such that

\[
\lim_{t \to +\infty} \|Y - \Lambda(X)X\| = 0,
\]

where \(\Lambda(X) = \text{diag}(h_1(X), h_2(X), \ldots, h_n(X)), h_i(X) : \mathbb{R}^n \rightarrow \mathbb{R}\) \((i = 1, 2, \ldots, n)\) are continuous functions, \(\| \cdot \|\) represents a vector norm induced by the matrix norm.

**Remark 1.** The function matrix \(\Lambda(X)\) is called a scaling matrix, and \(h_1, h_2, \ldots, h_n\) are called scaling function factors. In particular, if \(h_1(X) = h_2(X) = \cdots = h_n(X)\), the GFPS is simplified to the FPS.

**Remark 2.** We define \(E = Y - \Lambda(X)X\) which is called the error vector between systems (1) and (2) for GFPS, where \(E = (e_1, e_2, \ldots, e_n)^T\) and \(e_i = y_i - h_i(X)x_i, (i = 1, 2, \ldots, n)\).

**Remark 3.** If \(\Lambda = \sigma I, \sigma \in \mathbb{R}\), the GFPS problem will be reduced to projective synchronization, where \(I\) is an \(n \times n\) identity matrix. In particular, if \(\sigma = 1\) and \(-1\), the problem is further simplified to complete synchronization and anti-phase synchronization, respectively. And if \(\Lambda = \text{diag}(a_1, a_2, \ldots, a_k)\), the modified projective synchronization will appear, i.e. the MPS is also the special case of the proposed scheme.

**Remark 4.** If the scaling matrix \(\Lambda = 0\), the synchronization problem will be turned into a chaos control problem.

To investigate the GFPS of two different chaotic systems with unknown parameters, the drive and response systems (1), (2) can be rewritten as

\[
\dot{X}(t) = F_1(X) + F_2(X)\xi,
\]

and

\[
\dot{Y}(t) = G_1(Y) + G_2(Y)\eta + U(X, Y, t),
\]

respectively, where \(F_1, G_1 : \mathbb{R}^n \rightarrow \mathbb{R}^n\) are continuous vector functions, \(F_2 : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}, G_2 : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times l}\) are continuous matrix functions, and \(\xi \in \mathbb{R}^m, \eta \in \mathbb{R}^l\) are unknown parameter vectors of systems (3) and (4).

**Remark 5.** For a system with unknown parameters in this paper it is only considered that its parameters cannot be known in advance, but it has a certain structure.

How do we design an appropriate controller \(U(X, Y, t)\) and corresponding parameter estimate vectors \(\hat{\xi}, \hat{\eta}\) such that the GFPS of two different uncertain chaotic systems (3) and (4) can be gained, and the unknown parameters can be identified?
Theorem 1. For a given continuous differential scaling matrix function $\Lambda(X)$ and any initial values $X(0), Y(0)$, the GFPS between systems (3) and (4) will be obtained and the uncertain parameters $\xi, \eta$ will be estimated if the controller and the parameter update laws are designed as below:

$$U(X, Y, t) = J(F_2(X)\tilde{\xi} + F_1(X)) - G_1(Y) - G_2(Y)\tilde{\eta} - E$$

and

$$\begin{align*}
\dot{\tilde{\xi}} &= -\frac{\xi}{\xi} - \xi - F_1^T(X)J^TE\\
\dot{\tilde{\eta}} &= -\frac{\eta}{\eta} - \eta + G_2^T(Y)E,
\end{align*}$$

respectively, where $E = Y - \Lambda(X)X, \tilde{\xi} = \xi - \xi, \tilde{\eta} = \eta - \eta$ and $J = \frac{d(\Lambda(X)X)}{dx}$ represents the Jacobian matrix of vector $\Lambda(X)X$.

Proof. From Eqs. (3) and (4), the error system is

$$\dot{E} = \dot{Y} - \frac{d(\Lambda(X)X)}{dx}\dot{X} = G_1(Y) + G_2(Y)\eta - J(F_1(X) + F_2(X)\tilde{\xi}) + U(X, Y, t).$$

Choose the following Lyapunov function:

$$V(t) = \frac{1}{2}E^T E + \frac{1}{2}\xi^T\xi + \frac{1}{2}\eta^T\eta.$$  

According to the controller (5) and the parameter update law (6), and calculating the derivative of $V(t)$ along the solutions of error system (7), one has

$$\dot{V} = E^T \dot{E} + \xi^T \dot{\xi} + \eta^T \dot{\eta}$$

$$= E^T (G_1(Y) + G_2(Y)\tilde{\eta} - J(F_1(X) + F_2(X)\tilde{\xi}) + U(X, Y, t)) + \xi^T (\xi - F_2^T(X)J^TE) + \eta^T (\eta + G_2^T(Y)E)$$

$$= -E^T E - \xi^T \xi - \eta^T \eta$$

$$< 0.$$
On the basis of the Lyapunov stability theorem, the zero point of error system (7) is globally asymptotical stability, i.e. the GFPS between systems (3) and (4) is achieved, and the error parameters \( \tilde{\xi} \), \( \tilde{\eta} \) will converge to zero as the time \( t \) goes to infinity. The proof is complete. □

3. The GFPS between Lorenz and Rössler chaotic systems with unknown parameters

In this section, we will discuss the GFPS of two different specific chaotic systems with unknown parameters. For the sake of convenience, the Lorenz and Rössler chaotic systems are chosen as the master and slave systems for detailed description.

The Lorenz chaotic system can be described by the following ODEs:

\[
\begin{align*}
\dot{x}_1 &= a(y_1 - x_1) \\
\dot{y}_1 &= cx_1 - y_1 - x_1z_1 \\
\dot{z}_1 &= x_1y_1 - bz_1
\end{align*}
\]

(9)

where \( x_1, y_1, z_1 \) are state variables, and \( a, b, c \) are unknown parameters to be identified. In particular, when \( a = 10, b = 8/3 \) and \( c = 28 \), the system (9) can display a chaotic attractor.
The Rössler chaotic system, as the response system, is also given as below:

\[
\begin{align*}
\dot{x}_2 &= -(y_2 + z_2) + u_1 \\
\dot{y}_2 &= x_2 + \alpha y_2 + u_2 \\
\dot{z}_2 &= x_2 z_2 + \beta - \gamma z_2 + u_3,
\end{align*}
\]

(10)

where \(x_2, y_2, z_2\) are state variables, \(\alpha, \beta, \gamma\) are unknown parameters to be estimated, and \(u_1, u_2, u_3\) are the control laws to be designed.

According to the GFPS scheme presented in the previous section, without loss of generality, we choose the scaling function matrix \(\Lambda(X) = \text{diag}(d_{11}x_1 + d_{12}, d_{21}y_1 + d_{22}, d_{31}z_1 + d_{32})\), where \(d_{ij}(i = 1, 2, 3; j = 1, 2)\) are constant numbers. The error vector can be defined as

\[
\begin{align*}
e_1 &= x_2 - (d_{11}x_1 + d_{12})x_1 \\
e_2 &= y_2 - (d_{21}y_1 + d_{22})y_1 \\
e_3 &= z_2 - (d_{31}z_1 + d_{32})z_1.
\end{align*}
\]

(11)

Thus, the error system is

\[
\begin{align*}
\dot{e}_1 &= -(y_2 + z_2) - a(2d_{11}x_1 + d_{12})(y_1 - x_1) + u_1 \\
\dot{e}_2 &= x_2 + \alpha y_2 - (2d_{21}y_1 + d_{22})(c x_1 - y_1 - x_1 z_1) + u_2 \\
\dot{e}_3 &= x_2 z_2 + \beta - \gamma z_2 - (2d_{31}z_1 + d_{32})(x_1 y_1 - bz_1) + u_3.
\end{align*}
\]

(12)

From Eqs. (5) and (6) in Theorem 1, we can get the controller

\[
\begin{align*}
u_1 &= (y_2 + z_2) + \hat{a}(2d_{11}x_1 + d_{12})(y_1 - x_1) - e_1 \\
u_2 &= -x_2 - \hat{a}y_2 + (2d_{21}y_1 + d_{22})(c x_1 - y_1 - x_1 z_1) - e_2 \\
u_3 &= -x_2 z_2 - \hat{\beta} + \hat{\gamma} z_2 + (2d_{31}z_1 + d_{32})(x_1 y_1 - \hat{b}z_1) - e_3
\end{align*}
\]

(13)

and the parameter update laws

\[
\begin{align*}
\dot{\hat{a}} &= -\dot{\hat{a}} = -(2d_{11}x_1 + d_{12})(y_1 - x_1)e_1 + \hat{a} \\
\dot{\hat{b}} &= -\dot{\hat{b}} = z_1(2d_{31}z_1 + d_{32})e_3 + \hat{b} \\
\dot{\hat{c}} &= -\dot{\hat{c}} = -x_1(2d_{21}y_1 + d_{22})e_2 + \hat{c},
\end{align*}
\]

(14)
Fig. 4. The estimation of the unknown parameters of Lorenz and Rössler chaotic systems for anti-phase synchronization.

\[
\begin{align*}
\dot{\hat{\alpha}} &= -\hat{\alpha} = y_2 e_2 + \tilde{\alpha} \\
\dot{\hat{\beta}} &= -\hat{\beta} = e_3 + \tilde{\beta} \\
\dot{\hat{\gamma}} &= -\hat{\gamma} = -z e_3 + \tilde{\gamma},
\end{align*}
\]  

(15)

in which \(\hat{\alpha} = a - \tilde{a}, \hat{\beta} = b - \tilde{b}, \hat{\gamma} = c - \tilde{c}, \hat{\alpha} = \alpha - \tilde{\alpha}, \hat{\beta} = \beta - \tilde{\beta}, \hat{\gamma} = \gamma - \tilde{\gamma}\) are estimate variables of the uncertain parameters, and \(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}\) are the corresponding parameter errors. According to the results of Theorem 1, the GFPS between uncertain Lorenz and Rössler chaotic systems is obtained under the control of the controller (13), and the uncertain parameters are estimated using the parameter update laws (14) and (15).

Remark 6. During the GFPS design in this section, the scaling functions are only chosen in the form \(h_i(X) = d_{i1}x_i + d_{i2}\). In fact, they can be chosen as many other elementary functions arbitrarily.

Remark 7. The GFPS scheme for two different uncertain chaotic systems can also be applied to the synchronization of many hyperchaotic systems, even for discrete chaotic systems.
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Fig. 5. The control for the uncertain Rössler chaotic system; $d_i = 0$ ($i = 1, 2, 3; j = 1, 2$).

4. Simulations

To verify the effectiveness of the results obtained, some numerical simulations for the uncertain Lorenz and Rössler chaotic systems are performed. The true values of the “unknown” parameters of two uncertain systems are chosen as $a = 10$, $b = \frac{8}{3}$, $c = 28$ and $\alpha = 0.2$, $\beta = 0.2$, $\gamma = 5.7$.

Firstly, we choose the scaling function factors $h_1(X) = 2x_1 - 1$, $h_2(X) = 3y_1 + 2$ and $h_3(X) = -z_1 - 1$; the corresponding initial values are $(x_1(0), y_1(0), z_1(0))^T = (1, 3, 4)^T$, $(x_2(0), y_2(0), z_2(0))^T = (3, 43, -27)^T$, $(\hat{a}(0), \hat{b}(0), \hat{c}(0))^T = (-3, 6, 2)^T$ and $(\tilde{a}(0), \tilde{b}(0), \tilde{c}(0))^T = (4, -6, -2)^T$. As shown in Figs. 1 and 2, the GFPS between Lorenz and Rössler systems is gained, and the unknown parameters are also identified.

Furthermore, when the scaling factors are simplified as $h_i(X) = -1$ ($i = 1, 2, 3$), Figs. 3 and 4 show an anti-phase synchronization of the uncertain Lorenz and Rössler systems and the estimation of corresponding uncertain parameters, in which $d_{i1} = 0$, $d_{i2} = -1$ ($i = 1, 2, 3$), the initial values are given $(x_1(0), y_1(0), z_1(0))^T = (2, -5, -1)^T$, $(x_2(0), y_2(0), z_2(0))^T = (4, 15, -2)^T$, $(\hat{a}(0), \hat{b}(0), \hat{c}(0))^T = (3, -6, -2)^T$ and $(\tilde{a}(0), \tilde{b}(0), \tilde{c}(0))^T = (3, -4, 9)^T$.

And finally, we choose $d_{ij} = 0$ ($i = 1, 2, 3; j = 1, 2$); then the GFPS problem is simplified to a control problem for the uncertain Rössler chaotic system. The simulation results can be depicted as in Figs. 5 and 6, in which the initial values are $(x_2(0), y_2(0), z_2(0))^T = (-10, 10, -3)^T$ and $(\hat{a}(0), \hat{b}(0), \tilde{c}(0))^T = (3, -4, 9)^T$.

5. Conclusion

In this paper, a definition of generalized function projective synchronization of two different chaotic systems is first given. On the basis of the qualitative theory and adaptive control theory, a scheme of generalized function projective synchronization of two different uncertain chaotic systems is designed. Through this method, we not only achieve the GFPS of two chaotic systems, but also estimate the unknown parameters. The classical Lorenz and Rössler chaotic systems are chosen to illustrate the proposed technique. Finally, numerical simulations are provided to verify the effectiveness of the results obtained.

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Fig. 6. The estimation of the unknown parameters for the uncertain Rössler chaotic system; $d_{ij} = 0$ ($i = 1, 2, 3; j = 1, 2$).

References


