C¹-regularity of solutions for p-Laplacian problems

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In this work, we study C¹-regularity of solutions for one-dimensional p-Laplacian problems and systems with a singular weight which may not be in L¹. On the basis of the regularity result, we give an example to show the multiplicity of positive (or negative) solutions as well as sign-changing solutions especially when the nonlinear term is p-superlinear.

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1. Introduction and main results

Let us consider the following one-dimensional singular p-Laplacian problem [1,3,4,9]:

\[
\begin{align*}
\phi_p(u'(t))' + f(t, u(t)) &= 0, & t \in (0, 1), \\
\phi_p(u(0)) &= u(1) = 0,
\end{align*}
\]

where \( \phi_p(x) = |x|^{p-2}x, p > 1 \). We assume that \( f \in C((0,1) \times \mathbb{R}, \mathbb{R}) \) satisfies

\( (F) \quad f(t, u)u > 0, \) for a.e. \( t \in (0, 1) \) and \( u \neq 0 \), and for each \( M > 0 \), there exists \( c_M > 0 \) such that \( |f(t, u)| \leq c_M h(t)|u|^{p-1}, \) for \( t \in (0, 1) \) and \( |u| \leq M \)

and \( h \) is a nonnegative continuous function on \( (0, 1) \) which may be singular at the boundary 0 and/or 1. Radial problems for partial differential equations on exterior domains can be converted to problem \( (P) \). We say that \( u \) is a solution of \( (P) \) if \( u \in C[0,1] \cap C^1(0,1) \) and \( \phi_p(u') \) is absolutely continuous in any compact subinterval of \( (0,1) \) and \( u \) satisfies the first equation in \( (P) \) in \( (0,1) \) and \( u(0) = 0 = u(1) \).

In general, \( C[0,1] \) or \( C^1[0,1] \) serves as a natural solution space for \( (P) \) depending on the conditions on \( h \). Nevertheless if we know better regularity of solutions at the boundary, we may flexibly apply many theories, for example, the fixed point theorem, cone index theory and so on. Thus, to know regularity of solutions for \( (P) \) is important.

First of all, let us assume \( 0 \leq h \in L^1(0,1) \). Then it is not hard to see that all solutions for \( (P) \) are of \( C^1[0,1] \). In fact, let \( u \) be a nontrivial solution; then since \( u \in C[0,1] \), there exists \( \sigma \) with \( 0 < \sigma < 1 \) such that \( u'(\sigma) = 0 \). Thus, we have

\[
\phi_p(u'(t)) = \int_0^t f(s, u(s))ds.
\]

The conclusion follows from the inequality

\[
|u'(t)| \leq \phi_p^{-1}\left(c_1 \int_t^\sigma h(s)ds\right) \leq \phi_p^{-1}(\epsilon_2),
\]

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where $c_1$, $c_2$ are positive constants. Next, we choose $f(t, u) = h(t)\varphi_p(u(t))$ where $h(t) = -\theta^{p-1}(\theta - 1)(p - 1)t^{-p}$, with $0 < \theta < 1$. Then it follows from easy computation that $u(t) = t^0$ satisfies
\[
\begin{cases}
\varphi_p(u'(t))' + h(t)\varphi_p(u(t)) = 0, & t \in (0, 1), \\
u(0) = 0.
\end{cases}
\]
This example shows lack of $C^1$-regularity at the boundary by choice of weight $h(t) = t^{-p}$. We note $h \not\in L^1(0, 1)$. So it is interesting to consider a suitable class of $h$ which is bigger than $L^1(0, 1)$ and excludes $t^{-p}$ type to guarantee $C^1$-regularity of all solutions. Throughout this work, we assume $h \neq 0$ for any compact subinterval $I$ in $(0, 1)$ and introduce the following class of weights:
\[
\mathcal{A} \triangleq \left\{ h \in C((0, 1), [0, \infty)) : \exists \alpha, \beta > 0 \text{ such that } \alpha, \beta < p - 1 \text{ with } \int_0^1 s^\alpha(1-s)^\beta h(s)ds < \infty \right\}.
\]
The main goal in this work is to show $C^1$-regularity of solutions for (P) at the boundary if $h$ is of class $\mathcal{A}$.

**Theorem 1.1.** Assume $h \in \mathcal{A}$ and also assume (F). If $u$ is a solution of (P), then $u \in C^1[0, 1]$.

Next, we shall make use of the main technique to show $C^1$-regularity of solutions for a couple of systems \cite{2,10}. First, let us consider a cycled system:
\[
\begin{cases}
\varphi_p(u_i'(t))' + f_i(t, u_i(t)) = 0, & t \in (0, 1), \\
\varphi_p(u_i'(t))' + f_i(t, u_i(t)) = 0, & t \in (0, 1), \\
\vdots \\
\varphi_p(u_n'(t))' + f_n(t, u_n(t)) = 0, & t \in (0, 1), \\
u_1(0) = \cdots = u_n(0) = 0 = u_1(1) = \cdots = u_n(1).
\end{cases}
\] (CS)

We assume that $h_i \in \mathcal{A}$ and $f_i \in C((0, 1) \times \mathbb{R}, \mathbb{R})$ satisfies
\[(F_1)\ f_i(t, u) > 0, \text{ for a.e. } t \in (0, 1) \text{ and } u \neq 0, \text{ and for each } M > 0, \text{ there exists } c_{i,M} > 0 \text{ such that } |f_i(t, u)| \leq c_{i,M} h_i(t)|u|^{p-1}, \text{ for } t \in (0, 1) \text{ and } |u| \leq M, 1 \leq i \leq n.
\]

We say that $(u_1, u_2, \ldots, u_n)$ is a solution of (CS) if $u_i \in C[0, 1] \cap C^1(0, 1)$ and $\varphi_p(u'_i)$ is absolutely continuous in any compact subinterval of $(0, 1)$ and each $u_i$ satisfies the equations in (CS) in $(0, 1)$ and $u_i(0) = \cdots = u_i(1) = 0 = u_1(1) = \cdots = u_n(1)$.

**Theorem 1.2.** Assume $h_i \in \mathcal{A}$ and also assume (F1). If $(u_1, u_2, \ldots, u_n)$ is a nontrivial solution of (CS), then $u_i \in C^1[0, 1], 1 \leq i \leq n$.

Second, we have $C^1$-regularity of positive solutions for a strongly coupled system:
\[
\begin{cases}
\varphi_p(u'_1(t))' + f_1(t, u_1(t), u_2(t), \ldots, u_n(t)) = 0, & t \in (0, 1), \\
\varphi_p(u'_2(t))' + f_2(t, u_1(t), u_2(t), \ldots, u_n(t)) = 0, & t \in (0, 1), \\
\vdots \\
\varphi_p(u'_n(t))' + f_n(t, u_1(t), u_2(t), \ldots, u_n(t)) = 0, & t \in (0, 1), \\
u_1(0) = \cdots = u_n(0) = 0 = u_1(1) = \cdots = u_n(1).
\end{cases}
\] (SCS)

We assume that $h_i \in \mathcal{A}$ and $f_i \in C((0, 1) \times \mathbb{R}_+ \times \cdots \times \mathbb{R}_+, \mathbb{R}_+)$ with $\mathbb{R}_+ := [0, \infty)$ satisfies
\[(F_2)\text{ for each } M > 0, \text{ there exists } d_{i,M} > 0 \text{ such that } f_i(t, u_1, u_2, \ldots, u_n) \leq d_{i,M} h_i(t) (u_1 + u_2 + \cdots + u_n)^{p-1}, \text{ for a.e. } t \in (0, 1) \text{ and } 0 \leq u_i \leq M, 1 \leq i \leq n.
\]

**Theorem 1.3.** Assume $h_i \in \mathcal{A}$ and also assume (F2). If $(u_1, u_2, \ldots, u_n)$ is a positive solution of (SCS), then $u_i \in C^1[0, 1], 1 \leq i \leq n$.

In Section 3, we illustrate an example as a simple application.

2. **Proofs**

Let us consider the initial value problem
\[
\begin{cases}
\varphi_p(u'(t))' + f(t, u(t)) = 0, \\
u(t_0) = 0, \quad u'(t_0) = 0.
\end{cases}
\] (IVP)
Lemma 2.1. Assume \( h \in \mathcal{A} \) and also assume (F). Then (IVP\(_{t_0}\)) has only a trivial solution.

Proof of Theorem 1.1. We only prove that \( u \in C^1[0, 1] \). The proof of \( u \in C^1(0, 1) \) can be treated similarly. Let \( u \) be a nontrivial solution for (P). We claim that there exists \( a > 0 \) with \( 0 < a < 1 \) such that \( u(t) > 0, u'(t) > 0 \) or \( u(t) < 0, u'(t) < 0 \) for \( 0 < t \leq a \). Indeed, suppose on the contrary that \( u \) has a sequence \( \{t_n\} \) of zeros converging to 0. Multiplying (P) by \( u \) and integrating over \( (t_n, t_{n-1}) \), from (F), we have
\[
\int_{t_n}^{t_{n-1}} |u'|^p dt = \int_{t_n}^{t_{n-1}} f(t, u) dt \leq c \int_{t_n}^{t_{n-1}} h|u|^p dt.
\]
By Hölder’s inequality, we get
\[
|u(t)| \leq (t - t_n)^{(p-1)/p} \left( \int_{t_n}^{t} |u'|^p ds \right)^{1/p} \leq t^{(p-1)/p} \left( \int_{t_n}^{t} |u'|^p ds \right)^{1/p},
\]
for \( t \in (t_n, t_{n-1}) \). Substituting (2.2) into the integrand on the right-hand side in (2.1), we get
\[
\int_{t_n}^{t_{n-1}} |u'|^p dt \leq c \int_{t_n}^{t_{n-1}} h(s)s^{p-1} ds \int_{t_n}^{t_{n-1}} |u'|^p dt.
\]
Fix \( n \) so large that \( c \int_{t_n}^{t_{n-1}} h(s)s^{p-1} ds < 1/2 \). Then it follows from (2.3) that \( u \equiv 0 \) in \( (t_n, t_{n-1}) \). Lemma 2.1 implies \( u \equiv 0 \). This contradicts \( u \neq 0 \). Thus, there exists a constant \( a_1 > 0 \) with \( 0 < a_1 < 1 \) such that \( u(t) > 0 \) or \( u(t) < 0 \) on \( 0 < t \leq a_1 \). Condition (F) with \( u(t) > 0 \) on \( 0 < t \leq a_1 \) implies the concavity of \( u \) on \( 0 < t \leq a_2 \), for \( a_2 \leq a_1 \). Thus, \( u'(t) > 0 \) on \( 0 < t \leq a_2 \). Similarly, condition (F) with \( u(t) < 0 \) on \( 0 < t \leq a_1 \) implies the convexity of \( u \) on \( 0 < t \leq a_3 \), for \( a_3 \leq a_1 \). Thus, \( u'(t) < 0 \) on \( 0 < t \leq a_3 \). Taking \( a = \min(a_2, a_3) \), we get \( u(t) > 0, u'(t) > 0 \) or \( u(t) < 0, u'(t) < 0 \) for \( 0 < t \leq a \). First, we consider the case \( u(t) > 0, u'(t) > 0 \) for \( 0 < t \leq a \). For \( 0 < s < a \), we have
\[
\varphi_p(u'(s)) = \varphi_p(u'(a)) + \int_{a}^{s} f(\tau, u(\tau)) d\tau \leq \varphi_p(u'(a)) + c \int_{s}^{\infty} h(\tau)(u(\tau))^{p-1} d\tau.
\]
Therefore, defining \( \varphi_p(u'(a)) = c_0 \) and \( \|u\|_{\infty} = M \), we have
\[
\varphi_p(u'(s)) = \varphi_p(u'(a)) + \int_{a}^{s} f(\tau, u(\tau)) d\tau \leq c_0 + c_1 \int_{a}^{\infty} h(\tau) d\tau \\
\leq c_0 + c_1 \int_{a}^{\infty} \frac{\tau^a}{s^a} h(\tau) d\tau \leq c_0 + c_2 s^{-a}.
\]
Hence, we get
\[
u'(s) \leq (c_0 + c_2 s^{-a})^{p-1} \leq c_1 + c_4 s^{-a}.\]
Integrating it over \((0, t)\), for \( t < s \), we have
\[
u(t) \leq c_3 t + c_4 \left( \frac{1}{p-1} + 1 \right) t^{-\frac{a}{p-1}} \leq c t^{-\frac{a}{p-1}}.\]
For the case \(-a + (p-1) \geq a\), that is, \( p-1 \geq 2a \), plugging this inequality into (2.4), we have
\[
\varphi_p(u'(s)) \leq c_0 + c_5 \int_{s}^{\infty} h(\tau) \tau^{-a+p-1} d\tau \leq c_0 + c_5 \int_{s}^{\infty} h(\tau) \tau^{-a} d\tau.
\]
Thus, \( u \in C^1[0, 1] \). For the case \( 2a > p-1 \), putting \( 0 < a_1 = \frac{a}{p-1} + 1 < 1 \) and plugging this inequality into (2.4), we have
\[
\varphi_p(u'(s)) \leq c_0 + c_5 \int_{s}^{\infty} h(\tau) \tau^{-a+p-1} d\tau \leq c_0 + c_5 \int_{s}^{\infty} \frac{\tau^{2a} - (p-1)}{s^{2a} - (p-1)} h(\tau) \tau^{-a+p-1} d\tau \\
\leq c_0 + c_6 s^{(p-1)-2a}.
\]
Therefore, we have
\[
u'(s) \leq c_7 + c_8 s^{\frac{2a+p-1}{p-1}}.
\]
Hence, noting that \( 2a_1 < 1 \), we have
\[
u(t) \leq c_7 t + c_8 \frac{1}{2a_1} t^{2a_1} \leq c_9 t^{2a_1}.\]
Consider the case $0 < t < t_0$. As in the proof of Theorem 1.3 in [6], we have

$$\varphi_p(|u_1(t)|) \leq c_1 \int_t^{t_0} \tau^\alpha h_1(\tau) \varphi_p(|u_2(\tau)|) \, d\tau \quad (2.9)$$

and

$$\varphi_p(|u_2(t)|) \leq c_2 \int_t^{t_0} \tau^\alpha h_2(\tau) \varphi_p(|u_1(\tau)|) \, d\tau. \quad (2.9)$$

By Gronwall's inequality, we get $u_1 \equiv 0$, on $[0, t_0]$. From (2.9), we have $u_2 \equiv 0$, on $[0, t_0]$. Similarly, we get $u_1 = u_2 \equiv 0$, on $[t_0, 1]$. The case $t_0 = 0$ and $t_0 = 1$ can be proved by a similar argument. \(\square\)
Proof of Theorem 1.2. We only prove that $u_1, u_2 \in C^1([0, 1])$. We claim that there exists $a > 0$ with $0 < a < 1$ such that $u_1(t) > 0, u_1'(t) > 0, u_2(t) > 0, u_2(t) > 0$ or $u_1(t) < 0, u_1'(t) < 0, u_2(t) < 0, u_2'(t) < 0$ for $0 < t \leq a$. If one of them has a sequence $\{t_n\}$ of zeros converging to 0, then this is a contradiction to concavity from the equations. Suppose on the contrary that $u_1$ has a sequence $\{t_n\}$ of zeros converging to 0 or $u_2$ has a sequence $\{s_n\}$ of zeros converging to 0. Then we should get $t_n = s_n$, from a convexity–concavity argument. Multiplying the first equation in (CS) by $u_1$ and integrating over $(t_n, t_{n-1})$, we have from (F1)

$$
\int_{t_n}^{t_{n-1}} |u'_1|^p dt = \int_{t_n}^{t_{n-1}} f_1(t, u_2)u_1 dt \leq c_1 \int_{t_n}^{t_{n-1}} h_1 |u_2|^{p-1} |u_1| dt. 
$$

(2.10)

Multiplying the second equation in (CS) by $u_2$ and integrating over $(t_n, t_{n-1})$, we have from (F1)

$$
\int_{t_n}^{t_{n-1}} |u'_2|^p dt = \int_{t_n}^{t_{n-1}} f_2(t, u_1)u_2 dt \leq c_2 \int_{t_n}^{t_{n-1}} h_2 |u_1|^{p-1} |u_2| dt.
$$

(2.11)

By Hölder’s inequality, as in (2.2), we get $|u_i(t)| \leq t^{(p-1)/p} \left( \int_{t_n}^{t_{n-1}} |u'_i|^p dt \right)^{1/p}$, for $t \in [t_n, t_{n-1}]$, $i = 1, 2$. Plugging this into (2.10), noting $\frac{1}{p} + \frac{1}{q} = 1$, we get

$$
\int_{t_n}^{t_{n-1}} |u'_1|^p dt \leq \int_{t_n}^{t_{n-1}} f_1(t, u_2)u_1 dt \leq c_1 \int_{t_n}^{t_{n-1}} h_1 |u_2|^{p-1} |u_1| dt
$$

\begin{align*}
&\leq c_1 \int_{t_n}^{t_{n-1}} h_1(s) s^{p-1} \left( \int_{t_n}^{s} |u'_2|^p dt \right)^{\frac{p-1}{p}} s^{\frac{q}{p}} \left( \int_{t_n}^{s} |u'|^p dt \right)^{\frac{1}{p}} dt \\
&\leq c_1 \int_{t_n}^{t_{n-1}} h_1(s) s^{p-1} dt \left( \int_{t_n}^{t_{n-1}} |u'_2|^p dt \right)^{\frac{p-1}{p}} \left( \int_{t_n}^{t_{n-1}} |u'_1|^p dt \right)^{\frac{1}{p}}.
\end{align*}

Thus,

$$
\left( \int_{t_n}^{t_{n-1}} |u'_1|^p dt \right)^{\frac{p-1}{p}} \leq c_1 \int_{t_n}^{t_{n-1}} h_1(s) s^{p-1} dt \left( \int_{t_n}^{t_{n-1}} |u'_2|^p dt \right)^{\frac{p-1}{p}}.
$$

(2.12)

Similarly, from (2.11), we get

$$
\left( \int_{t_n}^{t_{n-1}} |u'_2|^p dt \right)^{\frac{p-1}{p}} \leq c_2 \int_{t_n}^{t_{n-1}} h_2(s) s^{p-1} dt \left( \int_{t_n}^{t_{n-1}} |u'_1|^p dt \right)^{\frac{p-1}{p}}.
$$

(2.13)

Then, it follows from (2.12) and (2.13) that

$$
\left( \int_{t_n}^{t_{n-1}} |u'_1|^p dt \right)^{\frac{p-1}{p}} \leq c_1 c_2 \int_{t_n}^{t_{n-1}} h_1(s) s^{p-1} dt \int_{t_n}^{t_{n-1}} h_2(s) s^{p-1} dt \left( \int_{t_n}^{t_{n-1}} |u'_1|^p dt \right)^{\frac{p-1}{p}}.
$$

(2.14)

Fix $n$ so large that $c_1 c_2 \int_{t_n}^{t_{n-1}} h_1(s) s^{p-1} ds \int_{t_n}^{t_{n-1}} h_2(s) s^{p-1} ds < 1/2$. Then it follows from (2.14) that $u_1 \equiv 0$ on $(t_n, t_{n-1})$. Similarly, we have $u_2 \equiv 0$ on $(t_n, t_{n-1})$. Lemma 2.2 implies $u_1 = u_2 \equiv 0$ on $[0, 1]$. This contradicts $u_1 \neq 0$ and $u_2 \neq 0$. Therefore, there is a constant $a_1$ with $0 < a_1 < 1$ such that $u_1(t) > 0$ or $u_1(t) > 0$ or $u_2(t) > 0$ or $u_2(t) > 0$ on $0 < t \leq a_1$. If $u_1(t) > 0$, then $u_2$ must be concave on $0 < t \leq a_1$ from (F1). Lemma 2.2 (the case $t_0 = 0$) implies $u_2(t) > 0$ on $0 < t \leq a_2$, for some $a_2 \leq a_1$. Similarly, if $u_1(t) < 0$ on $0 < t \leq a_1$, then $u_2(t) < 0$ on $0 < t \leq a_2$, for some $a_2 \leq a_1$. Using a similar argument to the proof of Theorem 1.1, we have $a > 0$ with $0 < a < 1$ such that $u_1(t) > 0$, $u'_1(t) > 0$, $u_2(t) > 0$, $u_2(t) > 0$ or $u_1(t) < 0$, $u'_1(t) < 0$, $u_2(t) < 0$, $u_2(t) < 0$ for $0 < t \leq a$. To show $C^1$-regularity at the boundary, we follow the argument in the proof of Theorem 1.1. Since $u_1, u_2 \in C([0, 1])$, we have (2.5) for $u_1, u_2$. We plug (2.5) for $u_1$ into the second equation for (CS) and (2.5) for $u_2$ into the first equation for (CS). Then we obtain (2.7) for $u_1, u_2$. Repeating this process as in the proof of Theorem 1.1 and noting that if one of them is in $C^1([0, 1])$, then the other one is automatically in $C^1([0, 1])$, we get $u_1, u_2 \in C^1([0, 1])$ for the finite process. The proof is complete. 

Proof of Theorem 1.3. Since we only consider positive solutions, it is obvious that there exists $a > 0$ with $0 < a < 1$ such that $u_1(t) > 0, u'_1(t) > 0$ and $u_2(t) > 0, u'_2(t) > 0$ for $0 < t \leq a$. For $0 < s < a$, noting $\|u_i\|_\infty \leq M, i = 1, 2$, for some $M > 0$, we have
\[ \varphi_p(u'_1(s)) = \varphi_p(u'_1(a)) + \int_a^s f_1(t, u_1(t), u_2(t))\,dt \]
\[ \leq \varphi_p(u'_1(a)) + d_1 \int_a^s h_1(t)(u_1(t) + u_2(t))^{p-1}\,dt \]
\[ \leq c_0 + d_2 \int_a^s h_1(t)\,dt \leq c_0 + d_2 s^{p-1}. \quad (2.15) \]

Therefore, we have (2.5) for \( u_1 \). Similarly, we get (2.5) for \( u_2 \). Plugging these into the first and second equations in (SCS), by (F2), we obtain (2.7) for \( u_1, u_2 \). As in the last part of the proof of Theorem 1.2, repeating the process, we can show \( C^1 \)-regularity at the boundary. \( \square \)

3. Example

In this section, we illustrate an example which makes use of \( C^1 \)-regularity of solutions, e.g., Theorem 1.1, to show the existence of sign-changing solutions.

Let us consider
\[ \begin{cases} \varphi_p(u'(t))' + h(t)g(u(t)) = 0, & t \in (0, 1), \\ u(0) = 0 = u(1), \end{cases} \]
where \( h \) is a nonnegative continuous function and \( g \in C(\mathbb{R}, \mathbb{R}) \). Under assumptions on \( g \) such as

- (G1) \( \text{sg} \,(s) > 0 \) for \( s \neq 0 \)
- (G2) \( g_0 \equiv \lim_{|s| \to 0} g(s)/\varphi_p(s) = 0, \quad g_\infty \equiv \lim_{|s| \to \infty} g(s)/\varphi_p(s) = \infty, \)

we are looking for general conditions on \( h \) in which problem (Q) satisfies the following existence result.

**Result A** For each \( k \in \mathbb{N} \), problem (Q) has two solutions \( u_k^+ \) and \( u_k^- \) such that \( u_k^+ \) has exactly \( k - 1 \) zeros and is positive near \( t = 0 \), and \( u_k^- \) also has exactly \( k - 1 \) zeros and is negative near \( t = 0 \).

For the linear case \( (p = 2) \), Ma and Thompson [8], Ma [7] proved Result A for when \( h \in C^1[0, 1] \). Lee and Sim [5] proved it for the \( \Delta \)-Laplacian case for when \( h \in L^1(0, 1) \). It is natural to ask whether the result is valid or not when \( h \in \mathcal{A} \) and it is not an obvious extension from the previous studies mainly due to the following question:

**Q:** Is the corresponding operator equation for sign-changing solutions well-defined? The operator equation for the \( \Delta \)-Laplacian problem was first introduced by Manásevich and Mawhin [9]; for more details, consider
\[ \varphi_p(u'(t))' = v, \quad \text{a.e. in} \ (0, 1), \]
\[ u(0) = u(1) = 0. \quad (3.1) \]

If \( v \in L^1(0, 1) \), then (3.1) has a unique solution \( u \equiv G_p(v) \) which is explicitly written as
\[ G_p(v)(t) = \int_0^t \varphi_p^{-1} \left( \xi(v) + \int_0^s v(\tau)d\tau \right) \,ds \]
with
\[ \int_0^1 \varphi_p^{-1} \left( \xi(v) + \int_0^s v(\tau)d\tau \right) \,ds = 0, \]
where \( \xi(v) \) is uniquely determined up to a given \( v \). We note that if \( v \notin L^1(0, 1) \), then the uniqueness of \( \xi(v) \) is not guaranteed and this process cannot be applied. For our problem (Q), \( v \) in (3.1) is given by \( v(t) = -h(t)g(u(t)) \). Since \( h \) may not be in \( L^1(0, 1) \), it is not obvious whether \( v \in L^1(0, 1) \) or not. It is easy to check by (G1) and (G2) that for \( f(t, u) = h(t)g(u) \) condition (F) holds and, thus, all solutions for (Q) are of class \( C_0^0[0, 1] \) by Theorem 1.1. For \( u \in C_0^0[0, 1] \), we may show \( v \in L^1(0, 1) \); indeed, denoting \( C_0^0[0, 1] \) by \( \mathcal{E} \) with norm \( \|u\|_\mathcal{E} \equiv \text{max}_{0 \leq s \leq 1} |u'(s)| \), we see \( |u(t)| \leq 2t(1-t)\|u\|_\mathcal{E} \). Thus \( u \in \mathcal{E} \) implies
\[ |h(t)g(u)| \leq Ch(t)|u|^{p-1} \leq 2^{p-1}Ch(t)|u|^{p-1}(1-t)^{p-1}\|u\|_\mathcal{E}^{p-1}. \]
Since \( h \in \mathcal{A} \), the right term of the above inequalities is integrable and we have
\[ v = h(t)g(u) \in L^1(0, 1). \]

For \( (\lambda, u) \in \mathbb{R} \times \mathcal{E} \), we define
\[ F(\lambda, u) \equiv G_p(-\lambda h(t)f(u)). \]
Then solutions of \( u = F(\lambda, u) \) correspond to those of (Q). Checking the complete continuity of \( F \) on \( \mathbb{R} \times \mathcal{E} \) and following the lines of Lee and Sim [5], we can prove Result A. Because proofs are lengthy, we will introduce all the details in some other research article. This idea can also be applied to certain systems with weights of class \( \mathcal{A} \).
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