On restricted edge-connectivity of half-transitive multigraphs *

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\textbf{Abstract} Let $G = (V, E)$ be a multigraph (it has multiple edges, but no loops). We call $G$ maximally edge-connected if $\lambda(G) = \delta(G)$, and $G$ super edge-connected if every minimum edge-cut is a set of edges incident with some vertex. The restricted edge-connectivity $\lambda'(G)$ of $G$ is the minimum number of edges whose removal disconnects $G$ into non-trivial components. If $\lambda'(G)$ achieves the upper bound of restricted edge-connectivity, then $G$ is said to be $\lambda'$-optimal. A bipartite multigraph is said to be half-transitive if its automorphism group is transitive on the sets of its bipartition. In this paper, we will characterize maximally edge-connected half-transitive multigraphs, super edge-connected half-transitive multigraphs, and $\lambda'$-optimal half-transitive multigraphs.

\textbf{Keywords:} Multigraphs; Half-transitive multigraphs; Maximally edge-connected; Super edge-connected; Restricted edge-connectivity.

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\section{Introduction}

A graph $G$ consists of vertex set $V$ and edge set $E$, where $E$ is a multiset of unordered pairs of (not necessarily distinct) vertices. A \textit{loop} is an edge whose endpoints are the same vertex. An edge is \textit{multiple} if there is another edge with the same endvertices; otherwise it is simple. The \textit{multiplicity} of an edge $e$, denoted by $\mu(e)$, is the number of multiple...
edges sharing the same endvertices; the multiplicity of a graph $G$, denoted by $\mu(G)$, is the maximum multiplicity of its edges. A graph is a simple graph if it has no multiple edges or loops, a multigraph if it has multiple edges, but no loops, and a pseudograph if it contains both multiple edges and loops. The underlying graph of a multigraph $G$, denoted by $U(G)$, is a simple graph obtained from $G$ by destroying all multiple edges. It is clear that $\mu(G) = 1$ if the graph $G$ is simple.

Let $G = (V, E)$ be a multigraph. Denote by $\lambda(G)$ the edge-connectivity of $G$. Since $\lambda(G) \leq \delta(G)$, where $\delta(G)$ is the minimum degree of $G$, a multigraph $G$ with $\lambda(G) = \delta(G)$ is naturally said to be maximally edge-connected, or $\lambda$-optimal for simplicity. A multigraph $G$ is said to be vertex-transitive if for any two vertices $u$ and $v$ in $G$, there is an automorphism $\alpha$ of $G$ such that $v = \alpha(u)$, that is, $Aut(G)$ acts transitively on $V$. A bipartite multigraph $G$ with bipartition $V_1 \cup V_2$ is called half-transitive if $Aut(G)$ acts transitively both on $V_1$ and $V_2$. Mader [9] proved the following well-known result.

**Theorem 1.1.** [9] Every connected vertex-transitive simple graph $G$ is $\lambda$-optimal.

If $G$ is a vertex-transitive multigraph, then $G$ is not always maximally edge-connected. A simple example is the multigraph obtained from a 4-cycle $C_4$ by replacing each edge belongs to a pair of opposite edges in $C_4$ with $m$ ($m \geq 2$) multiple edges.

For half-transitive simple graphs, Liang and Meng [8] proved the following result:

**Theorem 1.2.** [8] Every connected half-transitive simple graph $G$ is $\lambda$-optimal.

The problem of exploring edge-connected properties stronger than the maximally edge-connectivity for simple graphs has been the theme of many research. The first candidate may be the so-called super edge-connectivity. We can generalize this definition to multigraphs. A multigraph $G$ is said to be super edge-connected, in short, super-$\lambda$, if each of its minimum edge-cut sets isolates a vertex, that is, every minimum edge-cut is a set of edges incident with a certain vertex in $G$. By the definitions, a super-$\lambda$ multigraph must be a $\lambda$-optimal multigraph. However, the converse is not true. For example, $K_m \times K_2$ is $\lambda$-optimal but not super-$\lambda$ since the set of edges between the two copies of $K_m$ is a minimum edge-cut which does not isolate any vertex.

The concept of super-$\lambda$ was originally introduced by Bauer et al. see [1], where combinatorial optimization problems in design of reliable probabilistic simple graphs were investigated. The following theorem is a nice result of Tindell [15], which characterized super edge-connected vertex-transitive simple graphs.

**Theorem 1.3.** [15] A connected vertex-transitive simple graph $G$ which is neither a cycle nor a complete graph is super-$\lambda$ if and only if it contains no clique $K_k$, where $k$ is the degree of $G$.

For further study, Esfahanian and Hakimi [4] introduced the concept of restricted edge-connectivity for simple graphs. The concept of restricted edge-connectivity is one
kind of conditional edge-connectivity proposed by Harary in [5], and has been successfully applied in the further study of tolerance and reliability of networks, see [2,3,7,11,12,18,20-22]. Let $F$ be a set of edges in $G$. Call $F$ a restricted edge-cut if $G - F$ is disconnected and contains no isolated vertices. The minimum cardinality over all restricted edge-cuts is called restricted edge-connectivity of $G$, and denoted by $\lambda'(G)$. It is shown by Wang and Li [17] that the larger $\lambda'(G)$ is, the more reliable the network is. In [4], it is proved that if a connected simple graph $G$ of order $|V(G)| \geq 4$ is not a star $K_{1,n-1}$, then $\lambda'(G)$ is well-defined and $\lambda'(G) \leq \xi(G)$, where $\xi(G) = \min \{d(u) + d(v) - 2 : uv \in E(G)\}$ is the minimum edge degree of $G$. A simple graph $G$ with $\lambda'(G) = \xi(G)$ is called a $\lambda'$-optimal graph. It should be pointed out that if $\delta(G) \geq 3$, then a $\lambda'$-optimal simple graph must be super-$\lambda$. In fact, a graph $G$ is super-$\lambda$ if and only if $\lambda(G) < \lambda'(G)$, see [6]. Thus, the concepts of $\lambda$-optimal graphs, super-$\lambda$ graphs and $\lambda'$-optimal graphs describe reliable interconnection structures for graphs at different levels.

In [10], Meng studied the parameter $\lambda'$ for connected vertex-transitive simple graphs. The main result may be restated as follows:

**Theorem 1.4.** [10] Let $G$ be a $k$-regular connected vertex-transitive simple graph which is neither a cycle nor a complete graph. Then $G$ is not $\lambda'$-optimal if and only if it contains a $(k - 1)$-regular subgraph $H$ satisfying $k \leq |V(H)| \leq 2k - 3$.

The authors in [13] proved the following result.

**Theorem 1.5.** [13] Let $G = (V_1 \cup V_2, E)$ be a connected half-transitive simple graph with $n = |V(G)| \geq 4$ and $G \not\cong K_{1,n-1}$. Then $G$ is $\lambda'$-optimal.

Since a graph $G$ is super-$\lambda$ if and only if $\lambda(G) < \lambda'(G)$, Theorem 1.5 implies the following corollary.

**Corollary 1.6.** The only connected half-transitive simple graphs which are not super-$\lambda$ are cycles $C_n (n \geq 4)$.

We can naturally generalize the concept of restricted edge-connectivity to multigraphs. The restricted edge-connectivity $\lambda'(G)$ of a multigraph $G$ is the minimum number of edges whose removal disconnects $G$ into non-trivial components. Similarly, define the minimum edge degree of $G$ as $\xi(G) = \min \{\xi(e) = d(u) + d(v) - 2\mu(e) : e = uv \in E(G)\}$, where $\xi(e) = d(u) + d(v) - 2\mu(e)$ is the edge degree of the edge $e = uv$ in $G$. But the inequality $\lambda'(G) \leq \xi(G)$ is not always correct. For example, the restricted edge-connectivity of the multigraph $G$ in Fig.1 is 6, but $\xi(G) = 4$.

![Fig.1](image-url)
In [14], we gave sufficient and necessary conditions for vertex-transitive multigraphs to be maximally edge-connected, super edge-connected and $\lambda'$-optimal. In the following, we will study maximally edge-connected half-transitive multigraphs, super edge-connected half-transitive multigraphs, and $\lambda'$-optimal half-transitive multigraphs.

2 Preliminary

Let $G = (V, E)$ be a multigraph. For two disjoint non-empty subsets $A$ and $B$ of $V$, let $[A, B] = \{e = uv \in E : u \in A$ and $v \in B\}$. For the sake of convenience, we write $u$ for the single vertex set $\{u\}$. If $\overline{A} = V \setminus A$, then we write $N(A)$ for $[A, \overline{A}]$ and $d(A)$ for $|N(A)|$. Thus $d(u)$ is just the degree of $u$ in $G$. Denote by $G[A]$ the subgraph of $G$ induced by $A$.

An edge-cut $F$ of $G$ is called a $\lambda$-cut if $|F| = \lambda(G)$. It is easy to see that for any $\lambda$-cut $F$, $G - F$ has exactly two components. If $N(A)$ is a $\lambda$-cut of $G$, then $A$ is called a $\lambda$-fragment of $G$. It is clear that if $A$ is a $\lambda$-fragment of $G$, then so is $\overline{A}$. Let $r(G) = \min\{|A| : A$ is a $\lambda$-fragment of $G\}$. Obviously, $1 \leq r(G) \leq \frac{1}{2}|V|$. A $\lambda$-fragment $B$ is called a $\lambda$-atom of $G$ if $|B| = r(G)$. A $\lambda$-fragment $C$ is called a strict $\lambda$-fragment if $2 \leq |C| < |V(G)| - 2$. If $G$ contains strict $\lambda$-fragments, then the ones with smallest cardinality are called $\lambda$-superatoms.

Similarly, we can give the definition of $\lambda'$-atom. A restricted edge-cut $F$ of $G$ is called a $\lambda'$-cut if $|F| = \lambda'(G)$. For any $\lambda'$-cut $F$, $G - F$ has exactly two components. Let $A$ be a proper subset of $V$. If $N(A)$ is a $\lambda'$-cut of $G$, then $A$ is called a $\lambda'$-fragment of $G$. It is clear that if $A$ is a $\lambda'$-fragment of $G$, then so is $\overline{A}$. Let $r'(G) = \min\{|A| : A$ is a $\lambda'$-fragment of $G\}$. Obviously, $2 \leq r'(G) \leq \frac{1}{2}|V|$. A $\lambda'$-fragment $B$ is called a $\lambda'$-atom of $G$ if $|B| = r'(G)$.

For a multigraph $G$, the inequality $\lambda'(G) \leq \xi(G)$ is not always correct. But if $G$ is a $k$-regular multigraph, we proved the following result.

Lemma 2.1. [14] Let $G$ be a connected $k$-regular multigraph. Then $\lambda'(G)$ is well-defined and $\lambda'(G) \leq \xi(G)$ if $|V(G)| \geq 4$.

We call a bipartite multigraph $G$ with bipartition $V_1 \cup V_2$ semi-regular if each vertex in $V_1$ has the same degree $d_1$ and each vertex in $V_2$ has the same degree $d_2$. For semi-regular bipartite multigraphs, a similar result can be obtained.

Lemma 2.2. Let $G$ be a connected semi-regular bipartite multigraph with bipartition $V_1 \cup V_2$. Then $\lambda'(G)$ is well-defined and $\lambda'(G) \leq \xi(G)$ if $|V(G)| \geq 4$ and $U(G) \not\cong K_{1,n-1}$.

Proof. Assume each vertex in $V_1$ has degree $d_1$ and each vertex in $V_2$ has degree $d_2$. Assume, without loss of generality, that $d_1 \leq d_2$. Let $e = uv$ be an edge such that $\xi(e) = \xi(G)$, where $u \in V_1$ and $v \in V_2$. If $G - \{u, v\}$ contains a non-trivial component, say $C$, then $N(V(C))$ is a restricted edge-cut and $|N(V(C))| \leq \xi(e) = \xi(G)$. 


Thus assume that \( G - \{u, v\} \) only contains isolated vertices. If there is a vertex \( w \) other than \( v \) in \( V_2 \), then \( d_1 + d_2 \leq |N(V\setminus\{u, v\})| = |N(\{u, v\})| = \xi(e) = d_1 + d_2 - \mu(e) < d_1 + d_2 \) by \( |V(G)| \geq 4 \), a contradiction. Thus \( V_2 = \{v\} \) and \( U(G) \cong K_{1,n-1} \), also a contradiction.

Because of Lemma 2.1 and Lemma 2.2, we call a regular multigraph (or a semi-regular bipartite multigraph) \( G \) \( \lambda' \)-optimal if \( \lambda'(G) = \xi(G) \). Since each vertex-transitive multigraph is regular and each half-transitive multigraph is semi-regular, thus a vertex-transitive multigraph (or a half-transitive multigraph) \( G \) is \( \lambda' \)-optimal if \( \lambda'(G) = \xi(G) \).

Recall that an imprimitive block for a permutation group \( \Phi \) on a set \( T \) is a proper, non-trivial subset \( A \) of \( T \) such that for every \( \phi \in \Phi \) either \( \phi(A) = A \) or \( \phi(A) \cap A = \emptyset \).

A subset \( A \) of \( V(G) \) is called an imprimitive block for \( G \) if it is an imprimitive block for the automorphism group \( Aut(G) \) on \( V(G) \). The following theorem shows the importance of imprimitive blocks:

**Theorem 2.3.** [16] Let \( G = (V, E) \) be a connected simple graph and \( A \) be an imprimitive block for \( G \). If \( G \) is vertex-transitive, then \( G[A] \) is also vertex-transitive.

By a similar argument as Theorem 2.3, we can obtain the following result for half-transitive multigraphs.

**Lemma 2.4.** Let \( G \) be a connected bipartite multigraph with bipartition \( V_1 \cup V_2 \). Assume \( A \) is an imprimitive block for \( G \) such that \( A \cap V_1 \neq \emptyset \) and \( A \cap V_2 \neq \emptyset \). If \( G \) is half-transitive, then \( G[A] \) is also half-transitive.

**Proof.** Since \( G \) is half-transitive, for any two vertices \( u, v \in A \cap V_i \ (i \in \{1, 2\}) \), there is \( \alpha \in Aut(G) \) such that \( \alpha(u) = v \). Because \( \alpha(A) \cap A \neq \emptyset \), we have \( \alpha(A) = A \) by \( A \) is an imprimitive block for \( G \). Thus the restriction of \( \alpha \) to \( A \) is an automorphism of \( G[A] \), which maps \( u \) to \( v \). It follows \( G[A] \) is a half-transitive multigraph.

### 3 Maximally edge-connected half-transitive multigraphs

In [9], Mader proved that any two distinct \( \lambda \)-atoms of a simple graph are disjoint. For multigraphs, this property still holds.

**Lemma 3.1.** Let \( G \) be a connected multigraph. Then any two distinct \( \lambda \)-atoms of \( G \) are disjoint.

**Proof.** Suppose to the contrary that there are two distinct \( \lambda \)-atoms \( A \) and \( B \) with \( A \cap B \neq \emptyset \). We have \( V(G) \setminus (A \cup B) \neq \emptyset \) by \( |A| \leq |V(G)|/2 \) and \( |B| \leq |V(G)|/2 \). Then \( N(A \cap B) \) and \( N(A \cup B) \) are edge-cuts of \( G \), thus \( d(A \cap B) = |N(A \cap B)| \geq \lambda(G) \) and
\[ d(A \cup B) = |N(A \cup B)| \geq \lambda(G). \] From the following well-known submodular inequality (see [16]),
\[ 2\lambda(G) \leq d(A \cup B) + d(A \cap B) \leq d(A) + d(B) = 2\lambda(G), \]
we conclude that both \( |d(A \cap B)| = \lambda(G) \) and \( |d(A \cup B)| = \lambda(G) \) hold. Thus \( A \cap B \) is a \( \lambda \)-fragment with \( |A \cap B| < |A| \), which contradicts to \( A \) is a \( \lambda \)-atom of \( G \). □

Theorem 3.2. Let \( G \) be a connected half-transitive multigraph with bipartition \( V_1 \cup V_2 \). Assume each vertex in \( V_1 \) has degree \( d_1 \) and each vertex in \( V_2 \) has degree \( d_2 \). Then \( G \) is not maximally edge-connected if and only if there is a proper induced connected half-transitive multi-subgraph \( H \) of \( G \) such that
\[ |A_1|(d_1 - d'_1) + |A_2|(d_2 - d'_2) \leq \min\{d_1, d_2\} - 1, \]
where \( A_1 = V_1 \cap V(H) \), \( A_2 = V_2 \cap V(H) \), \( d'_1 \) is the degree of each vertex of \( A_1 \) in \( H \) and \( d'_2 \) is the degree of each vertex of \( A_2 \) in \( H \).

Proof. Assume, without loss of generality, that \( d_1 \leq d_2 \). If \( G \) is not maximally edge-connected, then \( \lambda(G) \leq d_1 - 1 \). Let \( A \) be a \( \lambda \)-atom of \( G \) and \( H = G[A] \). By Lemma 3.1, we know \( A \) is an imprimitive block for \( G \). Thus \( H \) is a connected half-transitive multigraph by Lemma 2.4. Assume each vertex in \( A \cap V_1 \) has degree \( d'_1 \) in \( H \) and each vertex in \( A \cap V_2 \) has degree \( d'_2 \) in \( H \). Then
\[ |A \cap V_1|(d_1 - d'_1) + |A \cap V_2|(d_2 - d'_2) = d(A) = \lambda(G) \leq d_1 - 1. \]

Now we prove the sufficiency. Assume \( G \) contains a proper induced connected half-transitive multi-subgraph \( H \) such that \( |A_1|(d_1 - d'_1) + |A_2|(d_2 - d'_2) \leq \min\{d_1, d_2\} - 1 \), then \( \lambda(G) \leq d(V(H)) = |A_1|(d_1 - d'_1) + |A_2|(d_2 - d'_2) \leq \min\{d_1, d_2\} - 1 \), that is, \( G \) is not maximally edge-connected. □

4 Super edge-connected half-transitive multigraphs

In [16], Tindell studied the intersection property of \( \lambda \)-superatoms of vertex-transitive simple graphs. For half-transitive multigraphs, we have the following lemma.

Lemma 4.1. Let \( G \) be a connected half-transitive multigraph with bipartition \( V_1 \cup V_2 \). Assume \( G \) is not super edge-connected, \( A \) and \( B \) are two distinct \( \lambda \)-superatoms. If \( |A| = |B| \geq 3 \), then \( A \cap B = \emptyset \).

Proof. Assume each vertex in \( V_1 \) has degree \( d_1 \) and each vertex in \( V_2 \) has degree \( d_2 \). Without loss of generality, assume that \( d_1 \leq d_2 \). If \( A \cap B \neq \emptyset \), then by a similar argument as the proof of Lemma 3.1, we can conclude that \( |d(A \cap B)| = |d(A \cup B)| = \lambda(G) \). We claim that \( |A \cap B| = 1 \). Otherwise, if \( |A \cap B| \geq 2 \), then \( |V(G) \setminus (A \cup B)| \geq |A \cap B| \geq 2 \). Since \( G[A] \), \( G[V \setminus A] \), \( G[B] \) and \( G[V \setminus B] \) are connected, we have \( G[A \cup B] \) and \( G[V \setminus (A \cap B)] \) are connected. If \( G[A \cap B] \) is not connected, then we have \( d(A \cap B) \geq 2\lambda(G) \), a contradiction. If \( G[A \cap B] \) is connected, then \( A \cap B \) is a strict \( \lambda \)-fragment with \( |A \cap B| < |A| \), which contradicts to \( A \) is a \( \lambda \)-superatom. Hence \( |A \cap B| = 1 \).
Let \( C = V(G) \setminus B \). Then \(|A \cap C| = |A \setminus (A \cap B)| \geq 2\), and \( A, V(G) \setminus A, C \) and \( V(G) \setminus C \) are all strict \( \lambda \)-fragments. By a similar argument as above we can deduce that \( A \cap C \) is a strict \( \lambda \)-fragment with \(|A \cap C| < |A|\), which is impossible. \( \Box \)

**Theorem 4.2.** Let \( G \) be a connected half-transitive multigraph with bipartition \( V_1 \cup V_2 \). Assume each vertex in \( V_1 \) has degree \( d_1 \), each vertex in \( V_2 \) has degree \( d_2 \) and \(|V(G)| \geq 2 \min\{d_1, d_2\} + 2\). Then \( G \) is not super edge-connected if and only if there is a proper induced connected half-transitive multi-subgraph \( H \) of \( G \) such that

\[
|A_1|(d_1 - d'_1) + |A_2|(d_2 - d'_2) \leq \min\{d_1, d_2\},
\]

where \( A_1 = V_1 \cap V(H) \), \( A_2 = V_2 \cap V(H) \), \( d'_1 \) is the degree of each vertex of \( A_1 \) in \( H \) and \( d'_2 \) is the degree of each vertex of \( A_2 \) in \( H \).

**Proof.** Assume, without loss of generality, that \( d_1 \leq d_2 \). If \( G \) is not super edge-connected, then \( G \) contains \( \lambda \)-superatoms. Let \( A \) be a \( \lambda \)-superatom of \( G \) and \( H = G[A] \). If \(|A| = 2\), then \( H \) is isomorphic to a multigraph which contains two vertices and \( t \) edges between these two vertices. Thus \( H \) is an induced \( t \)-regular connected half-transitive multi-subgraph of \( G \). Therefore \(|A \cap V_1| (d_1 - t) + |A \cap V_2| (d_2 - t) = d(A) = \lambda(G) \leq d_1\).

In the following, we assume that \(|A| \geq 3\).

By Lemma 4.1, we know \( A \) is an imprimitive block for \( G \). Thus \( H \) is a connected half-transitive multigraph by Lemma 2.4. Assume each vertex in \( A \cap V_1 \) has degree \( d'_1 \) in \( H \) and each vertex in \( A \cap V_2 \) has degree \( d'_2 \) in \( H \). Thus \(|A \cap V_1| (d_1 - d'_1) + |A \cap V_2| (d_2 - d'_2) = d(A) = \lambda(G) \leq d_1\).

Now we prove the sufficiency. Assume \( G \) contains a proper induced connected half-transitive multi-subgraph \( H \) such that \(|A_1|(d_1 - d'_1) + |A_2|(d_2 - d'_2) \leq \min\{d_1, d_2\}\), then \( d(V(H)) = |A_1|(d_1 - d'_1) + |A_2|(d_2 - d'_2) \leq \min\{d_1, d_2\} \). If \( G - V(H) \) contains no isolated vertices, then \( V(H) \) is a strict \( \lambda \)-fragment. Thus \( G \) is not super edge-connected. Assume \( G - V(H) \) contains an isolated vertex \( w \), then \( N(w) = N(V(H)) \). Since \(|A_1| \leq \min\{d_1, d_2\}\) and \(|A_2| \leq \min\{d_1, d_2\}\) by \(|A_1|(d_1 - d'_1) + |A_2|(d_2 - d'_2) \leq \min\{d_1, d_2\}\), we see that \( G \) is not connected by \(|V(G)| \geq 2 \min\{d_1, d_2\} + 2\), a contradiction. \( \Box \)

## 5 \( \lambda' \)-optimal half-transitive multigraphs

In [19], the authors proved the following fundamental result for studying the restricted edge-connectivity of simple graphs.

**Theorem 5.1.** [19] Let \( G = (V, E) \) be a connected simple graph with at least four vertices and \( G \not\cong K_{1,n-1} \). If \( G \) is not \( \lambda' \)-optimal, then any two distinct \( \lambda' \)-atoms of \( G \) are disjoint.

For multigraphs, we cannot obtain a similar result as in Theorem 5.1. But for half-transitive multigraphs, the similar result holds.
Lemma 5.2. Let $G$ be a connected multigraph with $\delta(G) \geq 2\mu(G)$. If $G$ contains a $\lambda'$-atom $A$ with $|A| \geq 3$, then each vertex in $A$ has at least two neighbors in $A$.

Proof. By contradiction, assume there is a vertex $u \in A$ such that $u$ contains only one neighbor in $A$. Let $v$ be the only neighbor of $u$ in $A$. Set $A' = A \setminus \{u\}$. Then both $G[A']$ and $G[\overline{A}]$ are connected. We have $|A'| \geq 2$ by $|A| \geq 3$. Clearly, $|\overline{A}| = |A| + 1 \geq 4$. Thus $[A', \overline{A}]$ is a restricted edge-cut. Since $\delta(G) \geq 2\mu(G)$, we have

$$\lambda'(G) \leq |[A', \overline{A}]| = |[A, \overline{A}]| + \mu(uv) - (d(u) - \mu(uv)) \leq |[A, \overline{A}]| = \lambda'(G).$$

It follows that $A'$ is a $\lambda'$-fragment with $|A'| < |A|$, which contradicts to $A$ is a $\lambda'$-atom.

Lemma 5.3. Let $G$ be a connected half-transitive multigraph with bipartition $V_1 \cup V_2$ and $\delta(G) \geq 2\mu(G)$. Assume $G$ is not $\lambda'$-optimal, $A$ and $B$ are two distinct $\lambda'$-atoms. Then $|A| = |B| \geq 3$ and $A \cap B = \emptyset$.

Proof. Assume each vertex in $V_1$ has degree $d_1$ and each vertex in $V_2$ has degree $d_2$. Without loss of generality, assume that $d_1 \leq d_2$.

If $|A| = 2$, then $\lambda'(G) = d(A) = d_1 + d_2 - 2\mu(uv) \geq \xi(G)$ (where $A = \{u, v\}$), which contradicts that $G$ is not $\lambda'$-optimal. Thus $|A| \geq 3$.

Suppose to the contrary that $A \cap B \neq \emptyset$. Set $C = A \cap B$, $A_1 = A \cap \overline{B}$, $B_1 = B \cap \overline{A}$ and $D = \overline{A} \cap \overline{B} = \overline{A} \cup \overline{B}$. In the following, we will derive a contradiction by a series of claims.

Clearly, one of the following two inequalities must holds:

$$|[A_1, C]| \leq |[C, B_1]| + |[C, D]|,$$

(1)

$$|[B_1, C]| \leq |[C, A_1]| + |[C, D]|.$$  

(2)

In the following, we always assume, without loss of generality, that inequality (1) holds.

Claim 1. $A_1$ satisfies one of the following two conditions: (i) $A_1 = \{v_{21}\}$ ($v_{21} \in V_2$ and $d_1 > 2\mu(G)$, or (ii) $A_1 = \{v_{11}, \cdots, v_{1m}\}$ ($v_{1i} \in V_1$ for $1 \leq i \leq m$) and $d_2 > (m - 1)d_1 + 2\mu(G)$.

It follows from inequality (1) that

$$d(A_1) = |[A_1, D]| + |[A_1, C]| + |[A_1, B_1]| \leq d(A) = \lambda'(X).$$

Assume $G[A_1]$ has a component $\overline{G}$ with $|V(\overline{G})| \geq 2$. Set $F = V(\overline{G})$. Since $G[B]$ and $G[\overline{A}]$ are both connected, and $B \cap \overline{A} \neq \emptyset$, we see that $G[\overline{A}_1]$ is connected. Furthermore, since $G$ is connected, every component of $G[A_1]$ is joined to $G[\overline{A}_1]$, and thus $G[F]$ is connected. So $[F, \overline{F}]$ is a restricted edge-cut with $|d(F)| \leq \lambda'(G)$. Because $A$ is a $\lambda'$-atom and $F$ is a proper subset of $A$, we obtain $d(F) > d(A) = \lambda'(G)$, it is a contradiction. Thus, each component in $G[A_1]$ is an isolated vertex. By $d(A_1) \leq \lambda'(G) < d_1 + d_2 - 2\mu(G)$, we
can derive that $A_1$ satisfies one of the following two conditions: (i) $A_1 = \{v_{21}\}(v_{21} \in V_2)$ and $d_1 > 2\mu(G)$, or (ii) $A_1 = \{v_{11}, \cdots, v_{1m}\}(v_{1i} \in V_1$ for $1 \leq i \leq m)$ and $d_2 > (m - 1)d_1 + 2\mu(G)$.

Claim 2. $C \not\subseteq V_1$ and $C \not\subseteq V_2$.

By contradiction. Suppose $C \subseteq V_1$. Then $G[C]$ is an independent set. Since we have assumed that $|[A_1,C]| \leq |[C,B_1]| + |[C,D]|$, there exists a vertex $v$ in $C$ such that $|[v,A_1]| \leq |[v,D]| + |[v,B_1]|$. (3)

Set $F = A \setminus \{v\}$, then

$$d(F) = d(A) - |[v,D]| - |[v,B_1]| + |[v,A_1]| \leq d(A) = \lambda'(X).$$

Since $G[A]$ is connected and $C$ is an independent set, we have $|[v,A_1]| \geq 1$. It follows from inequality (3) that $|[v,A_1]| \geq 1$. So, $G[F]$ is connected. We claim that each component in $G[F]$ has at least 2 vertices. In fact, if there is an isolated vertex $u$ in $G[F]$, then $v$ is the only vertex adjacent to $u$ in $G[A]$, which contradicts to Lemma 5.2. Now, similarly as in the proof of Claim 1, a contradiction arises, since $F$ contains a smaller $\lambda'$-fragment than $A$. $C \not\subseteq V_2$ can be proved similarly.

Claim 3. $d(D) < \lambda'(G)$ and $D$ is an independent set contained in $V_1$.

By Claim 2, $|C| \geq 2$. We claim that $d(C) > \lambda'(G)$. In fact, if $G[C]$ contains a component of order at least 2, then similar to the proof of Claim 1, we can show that $[C,\overline{C}]$ contains a restricted edge-cut, and thus $d(C) > \lambda'(G)$. Otherwise, we assume that each component in $G[C]$ is an isolated vertex. Since not all vertices in $C$ are from the same bipartition, there must be at least one vertex in $V_2$. From $|C| \geq 2$, we have $d(C) \geq d_2 + d_1 > \xi(G) \geq \lambda'(X)$. Thus, we have that $d(C) > \lambda'(G)$.

From the well-known submodular inequality (see [16]), we have

$$d(C) + d(D) \leq d(A) + d(B) = 2\lambda'(G).$$

By (4) and $d(C) > \lambda'(G)$, we obtain $d(D) < \lambda'(G)$. Applying a similar argument as above, we can show that $D$ is an independent set contained in $V_1$.

Let $s = |D|$. Then $s \geq 2$ and

$$d(D) = sd_1.$$  

(5)

Denote by $e_1$ the number of edges in $G[\overline{C}]$. Clearly,

$$d(C) = d(\overline{C}) = \sum_{v \in \overline{C}} d(v) - 2e_1.$$  

(6)

Since $G[\overline{C}]$ is connected and $D$ is an independent set contained in $V_1$, Claim 1 (ii) can not hold, Thus, Claim 1 (i) is true. Since $G$ is a bipartite multigraph, we have

$$e_1 \leq 2s\mu(G).$$

(7)
Combining this with (4), (5) and (6), we see that
\[ 2d_1 + 2d_2 - 4\mu(G) - sd_1 > 2\lambda'(G) - d(D) \geq d(C) \geq sd_1 + 2d_2 - 4s\mu(G). \]
This implies \( d_1 < 2\mu(G) \), contradicting to the assumption that \( d_1 \geq 2\mu(G) \). □

**Theorem 5.4.** Let \( G \) be a connected half-transitive multigraph with bipartition \( V_1 \cup V_2 \) and \( \delta(G) \geq 2\mu(G) \). Assume each vertex in \( V_1 \) has degree \( d_1 \), each vertex in \( V_2 \) has degree \( d_2 \), \( |V_1| \geq \xi(G) \) and \( |V_2| \geq \xi(G) \). Then \( G \) is not \( \lambda' \)-optimal if and only if there is a proper induced connected half-transitive multi-subgraph \( H \) of \( G \) such that

\[ |A_1|(d_1 - d'_1) + |A_2|(d_2 - d'_2) \leq \xi(G) - 1, \]

where \( A_1 = V_1 \cap V(H) \), \( A_2 = V_2 \cap V(H) \), \( d'_1 \) is the degree of each vertex of \( A_1 \) in \( H \) and \( d'_2 \) is the degree of each vertex of \( A_2 \) in \( H \).

**Proof.** Assume, without loss of generality, that \( d_1 \leq d_2 \). If \( G \) is not \( \lambda' \)-optimal, then \( G \) contains \( \lambda' \)-atoms. Let \( A \) be a \( \lambda' \)-atom of \( G \) and \( H = G[A] \). By Lemma 5.3, we have \( |A| \geq 3 \) and \( A \) is an imprimitive block for \( G \). Thus \( H \) is a connected half-transitive multigraph by Lemma 2.4. Assume each vertex in \( A \cap V_1 \) has degree \( d'_1 \) in \( H \) and each vertex in \( A \cap V_2 \) has degree \( d'_2 \) in \( H \). Thus \( |A \cap V_1|(d_1 - d'_1) + |A \cap V_2|(d_2 - d'_2) = d(A) = \lambda'(G) \leq \xi(G) - 1. \)

Now we prove the sufficiency. Assume \( G \) contains a proper induced connected half-transitive multi-subgraph \( H \) such that \( |A_1|(d_1 - d'_1) + |A_2|(d_2 - d'_2) \leq \xi(G) - 1 \), then \( d(V(H)) = |A_1|(d_1 - d'_1) + |A_2|(d_2 - d'_2) \leq \xi(G) - 1 \), \( |A_1| \leq \xi(G) - 1 \) and \( |A_2| \leq \xi(G) - 1 \). If \( G - V(H) \) contains a non-trivial component, say \( B \), then \([B,B] \) is a restricted edge-cut and \( d(B) \leq d(V(H)) \leq \xi(G) - 1 \). Thus \( G \) is not \( \lambda' \)-optimal. Now we assume that each component of \( G - V(H) \) is an isolated vertex, then \( |N(V(G) \setminus V(H))| \geq d_1 + d_2 > \xi(G) \) by \( |V_1| \geq \xi(G) \) and \( |V_2| \geq \xi(G) \). On the other hand, \( |N(V(G) \setminus V(H))| = |N(V(H))| \leq \xi(G) - 1 \), it is a contradiction. □

**References**


