MULTIPLE STATIONARY SOLUTIONS OF EULER-POISSON EQUATIONS FOR NON-ISENTRIPIC GASEOUS STARS

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Abstract  The motion of the self-gravitational gaseous stars can be described by the Euler-Poisson equations. The main purpose of this paper is concerned with the existence of stationary solutions of Euler-Poisson equations for some velocity fields and entropy functions that solve the conservation of mass and energy. Under different restriction to the strength of velocity field, we get the existence and multiplicity of the stationary solutions of Euler-Poisson system.

Key words  Euler-Poisson equations; non-isentropic; stationary solutions

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1 Introduction

In three space dimensions, the time evolution of the gaseous stars can be described by the Euler-Poisson equations in astrophysics [2]. This system contains Euler equations for conservation of mass, momentum and energy:

\[
\begin{align*}
\rho_t + \text{div}(\rho v) &= 0, \\
(\rho v)_t + \text{div}(\rho v \otimes v) + \nabla P &= -\rho \nabla \Phi, \\
S_t + v \cdot \nabla S &= 0,
\end{align*}
\]

and Poisson equation through which the gravitational potential is determined by the density distribution of the gas itself:

\[
\Delta \Phi = 4\pi g \rho,
\]

where \( \rho = \rho(x, t), v = v(x, t), S = S(x, t), P, g \) and \( \Phi \) denote the density, velocity, entropy, pressure, gravitational constant and gravitational potential respectively. For simplicity, we assume the pressure satisfies the following equation of state:

\[
P = A\rho^\gamma e^S,
\]

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where $A > 0$ is a constant, $1 < \gamma < 2$ is the adiabatic constant. We have $\gamma = \frac{5}{3}$ for a monatomic gas, $\frac{7}{5}$ for a diatomic gas and so on.

For the Euler-Poisson system, stationary solutions crucially depend on the adiabatic constant and entropy function. The stationary solutions to non-moving gas ($v = 0$) have been extensively explored. For the isentropic case, given the gaseous stars mass $\int_{\mathbb{R}^3} \rho \, dx = M$, there are many results about stationary solutions in [1, 2, 8]. For the non-isentropic case, there are also some results about stationary solutions in [4–6]. With a given velocity field $v(x)$, Deng and Yang in [6] turned the steady equation into an elliptic equation by using a transformation, then they used variational methods to consider the existence of multiple stationary solutions and the exact multiplicity of solutions. Luo and Smoller in [9] studied the steady solutions of Euler-Poisson equations in bounded domains with prescribed angular velocity and bounded entropy function. If the entropy function is unbounded or singular at some points, their approach seems to be not directly applied.

With more general velocity field and the entropy function which has a singular point at origin, we consider the steady problem of Euler-Poisson system in a smooth bounded domain. In order to match the exterior of the star, we assume that the density of gaseous stars is zero on the boundary of the domain.

If $v(x)$ is a rotation around $x_3$ axis with prescribed time-independent angular velocity $\Theta(\eta)$, here $\eta(x) = \sqrt{x_1^2 + x_2^2}$. Then the functions

\[(\rho, v, S)(x, t) = (\rho, v, S)(\eta(x), x_3, t) \quad \text{with} \quad v(x) = (-x_2\Theta(\eta), x_1\Theta(\eta), 0),\]

thus $\text{div}(\rho v) = 0$ and $v \cdot \nabla S = 0$. By (1.1)_1 and (1.1)_3, we obtain $\rho_t = 0$, $S_t = 0$. Therefore the macroscopic entropy $S$ can be assumed as:

\[S(x) = \theta \ln |x|, \quad (1.4)\]

here $\theta$ is a constant. Obviously, the origin is a singular point of the function in (1.4) if $\theta \neq 0$, it brings us a lot of trouble in considering the steady problem.

For a stationary solution to Euler-Poisson equations with a given velocity field $v(x)$, we can obtain from equation (1.1) that

\[v \cdot \nabla v + \frac{1}{\rho} \nabla P = -\nabla \Phi.\]

Applying div on both sides of the above equation gives

\[\text{div}(v \cdot \nabla v) + \text{div}\left(\frac{1}{\rho} \nabla P\right) = -\Delta \Phi. \quad (1.5)\]

Taking a nonlinear transformation

\[\rho = C_\gamma u^q |x|^{-l}, \quad (1.6)\]

here $C_\gamma = \left(\frac{42r}{3(-\gamma - 1)}\right)^{-\frac{1}{r+1}}$, $q = \frac{1}{\gamma - 1}$, $l = \frac{q}{r}$, and noting (1.2) (1.3), then (1.5) becomes

\[-\text{div}(|x|^l \nabla u) = |x|^{-l} u^q - \sigma h(x), \quad (1.7)\]

where $\sigma$ is a positive constant which represents the strength of velocity field, and

\[\sigma h(x) = -\left(4\pi g C_\gamma\right)^{-1}\text{div}(v \cdot \nabla v). \quad (1.8)\]
From a physical point of view, we look for positive solutions to equation (1.7) in an open bounded domain $\Omega \ni 0$ in $\mathbb{R}^3$ with smooth boundary $\partial \Omega$ and Dirichlet boundary condition, i.e.

$$
\begin{aligned}
-\text{div}(|x|^l \nabla u) &= |x|^{-l}u^q - \sigma h(x), & \text{in } \Omega,
|u|_{\partial \Omega} &= 0, & u > 0 & \text{in } \Omega,
\end{aligned}
$$

where $l, q$ are given in (1.6), $h(x) \in C^1(\Omega)$ is allowed to change sign.

When the gaseous star has zero velocity, there are only two kinds of forces acting on the gas particles, i.e., forces from the pressure and the gravitational potential. When these two forces are balanced at each point in the support of the gas, we have a stationary solution, and (1.9) becomes

$$
\begin{aligned}
-\text{div}(|x|^l \nabla u) &= |x|^{-l}u^q, & \text{in } \Omega,
|u|_{\partial \Omega} &= 0, & u > 0 & \text{in } \Omega.
\end{aligned}
$$

Problem (1.9) can be viewed as a perturbation of equation (1.10).

Let $H^1_0(\Omega, |x|^l)$ be the completion of $C_0^\infty(\Omega)$ under the inner product

$$
\langle u, \varphi \rangle = \int_\Omega |x|^l \nabla u \nabla \varphi dx,
$$

and $L^p(\Omega, |x|^{-l})$ be the weighted $L^p$ space with the weight $|x|^{-l}$. By Mountain pass lemma and compactness of embedding $H^1_0(\Omega, |x|^l) \hookrightarrow L^p(\Omega, |x|^{-l})$ for $1 < p < 2(3-l)/(1+l)$, equation (1.10) has a positive solution in $H^1_0(\Omega, |x|^l)$ if $1 < q < (5-3l)/(1+l)$.

On the other hand, the existence of positive solutions to (1.9) is also closely related to the solvability of the linear problem:

$$
\begin{aligned}
\text{div}(|x|^l \nabla \varphi) &= h(x), & \text{in } \Omega,
\varphi|_{\partial \Omega} &= 0, & \varphi > 0 & \text{in } \Omega.
\end{aligned}
$$

Set

$$
\mathcal{A} = \left\{ h(x) \in C^1(\Omega) \mid \text{equation (1.11) is solvable} \right\}.
$$

The main results of this paper are given as follows.

**Theorem 1.1** Assume $-\gamma < \theta < (5\gamma^2 - 6\gamma)/(3\gamma - 2)$, $1 < \gamma < 2$ and $h(x) \in \mathcal{A}$, $h|_{\partial \Omega} = 0$. Then there exists a constant $\sigma_h > 0$, such that problem (1.9) has a minimal solution $u_\sigma$ for any $\sigma \in (0, \sigma_h)$. Furthermore, $u_\sigma$ is increasing with respect to $\sigma$ and $u_\sigma \geq \sigma \varphi(x)$, where $\varphi(x)$ is the solution of (1.11).

**Theorem 1.2** Under the same assumptions in Theorem 1.1. Problem (1.9) has at least two solutions for $\sigma \in (0, \sigma_h)$, only one solution for $\sigma = \sigma_h$, and no solution for $\sigma > \sigma_h$.

This paper proceeds as follows: In the next section, we give some preliminaries and verify Theorem 1.1 by the method of subsolutions and supersolutions. In Section 3, if the minimal solution of (1.9) is $u_\sigma$, we consider the corresponding eigenvalue problem firstly. Then, by using Mountain Pass Lemma, we get the second solution of (1.9).

Throughout the paper, we assume that $1 < \gamma < 2$, $q(l) = 2(3-l)/(1+l)$, and $C, C_i$ stand for universal constants. We use

$$
\|u\| = \left( \int_\Omega |x|^l \nabla u^2 dx \right)^{\frac{1}{2}}, \quad |u|_{L^p(\Omega, |x|^\alpha)} = \left( \int_\Omega |x|^{\alpha} u^p dx \right)^{\frac{1}{p}}
$$
to denote the norm in $H^1_0(\Omega, |x|^l)$ and $L^p(\Omega, |x|^\alpha)$, respectively.
2 Preliminaries and the Proof of Theorem 1.1

In this section, we first give some useful lemmas, then by super-subsolution method, we prove the Theorem 1.1. To this end, we introduce the following eigenvalue problem

\[
\begin{cases}
-\text{div}(|x|^l \nabla u) = \lambda |x|^{-l} u, & \text{in } \Omega, \\
u|_{\partial \Omega} = 0.
\end{cases}
\]  
(2.1)

Lemma 2.1 Assume $-\gamma < \theta < \gamma$, then problem (2.1) has a principle eigenvalue, that is

\[
\lambda_1 := \left\{ \int_\Omega |x|^l |\nabla u|^2 \, dx : u \in H^1_0(\Omega, |x|^l) \text{ and } \int_\Omega |x|^{-l} u^2 \, dx = 1 \right\}
\]
is achieved by some positive $u_0 \in H^1_0(\Omega, |x|^l)$, and $\lambda_1 > 0$.

Proof Let \( \{u_n\} \in H^1_0(\Omega, |x|^l) \) satisfying

\[
\int_\Omega |x|^{-l} u_n^2 \, dx = 1, \quad \text{and} \quad \int_\Omega |x|^l |\nabla u_n|^2 \, dx \to \lambda_1 \quad \text{as } n \to +\infty.
\]

Obviously, \( \{u_n\} \) is bounded in $H^1_0(\Omega, |x|^l)$, we may extract a subsequence still denoted by \( \{u_n\} \) such that for some $u_0 \in H^1_0(\Omega, |x|^l)$,

\[
\begin{align*}
& u_n \rightharpoonup u_0 \text{ weakly in } H^1_0(\Omega, |x|^l), \\
& u_n \to u_0 \text{ strongly in } L^q(\Omega, |x|^{-l}) \quad \text{for } 1 < q < q(l).
\end{align*}
\]

Since $-\gamma < \theta < \gamma$, \( l = \frac{q}{2} \), then $q(l) > 2$. Therefore

\[
\int_\Omega |x|^{-l}|u_0|^2 \, dx = 1, \quad \int_\Omega |x|^l |\nabla u_0|^2 \, dx \geq \lambda_1.
\]  
(2.2)

Put \( v_n = u_n - u_0 \), we derive

\[
\lambda_1 + o(1) = \int_\Omega |x|^l |\nabla u_0|^2 \, dx = \int_\Omega |x|^l |\nabla u_n|^2 \, dx + \int_\Omega |x|^l |\nabla u_0|^2 \, dx + o(1).
\]

Using (2.2) and the above equality, we have

\[
\lambda_1 + \int_\Omega |x|^l |\nabla v_n|^2 \, dx \leq \int_\Omega |x|^l |\nabla v_n|^2 \, dx + \int_\Omega |x|^l |\nabla u_0|^2 \, dx = \lambda_1 + o(1),
\]

that is \( v_n \rightharpoonup 0 \) in $H^1_0(\Omega, |x|^l)$. So $\lambda_1$ is achieved by some $u_0 \in H^1_0(\Omega, |x|^l)$. Since $|u_0|$ is also a minimizer, we may assume that $u_0 \geq 0$. Strong maximum principle (\cite{3}) implies that $u_0 > 0$ in $\Omega$, and $\lambda_1 = \int_\Omega |x|^l |\nabla u_0|^2 \, dx > 0$.

Lemma 2.2 Assume $-\gamma < \theta < (5\gamma^2 - 6\gamma)/(3\gamma - 2)$. Let \( \underline{u}, \overline{u} \) be the weak subsolution and supsolution of equation (1.9), respectively, and $0 \leq \underline{u} \leq \overline{u}$ a.e., then there exists a weak solution $u$ of (1.9) such that

\[
u \leq u \leq \overline{u} \quad \text{a.e. in } \Omega.
\]

Furthermore, $u$ is a minimal solution of (1.9) in the interval \([\underline{u}, \overline{u}]\).
Hence, let $h$ in $H^1_0(\Omega, |x|^l)$, and hence we use (2.4),

$$u_{k+1} = \frac{1}{\sigma h(x)}.$$  

Similar to the proof of Theorem 9.3.1 in [7], we obtain

$$0 \leq u = u_1 \leq u_2 \leq \cdots \leq u_k \leq u_{k+1} \leq \cdots \leq \overline{u} \text{ a.e in } \Omega.$$  

Therefore $u(x) := \lim_{k \to +\infty} u_k(x)$ exists for a.e $x \in \Omega$, and we have $0 \leq u(x) \leq u(x) \leq \overline{u}(x)$ a.e in $\Omega$. Since $-\gamma > \theta < (5\gamma^2 - 6\gamma)/(3\gamma - 2)$, $\overline{u}$ is a weak supersolution of (1.9), then

$$\|\overline{u}(x)\| \leq C. \tag{2.4}$$

Multiplying (2.3) by $u_{k+1}$ and integrating it over $\Omega$ yield

$$\|u_{k+1}\|^2 = \int_\Omega |x|^{-l} u^q_k u_{k+1} dx - \sigma \int_\Omega h u_{k+1} dx \leq |\overline{u}|_{L^{q+1}(\Omega, |x|^{-l})} + \|h\|_{L^2(\Omega, |x|^l)} \|u_{k+1}\|_{L^2(\Omega, |x|^{-l})} \leq C + C\|u_{k+1}\|,$$

here we use (2.4), $h \in C^1$, Hölder inequality and the compact imbedding $H^1_0(\Omega, |x|^l) \hookrightarrow L^q(\Omega, |x|^{-l})$ for $1 < q < q(l)$. It follows from Young’s inequality that $\{u_{k+1}\}$ is bounded in $H^1_0(\Omega, |x|^l)$, and hence we have $u_k \rightarrow u$ weakly in $H^1_0(\Omega, |x|^l)$.

For any $v \in H^1_0(\Omega, |x|^l)$, from (2.3) we find that

$$\int_\Omega |x|^{-l} u_{k+1} \nabla v dx = \int_\Omega |x|^{-l} u^q_k v dx - \sigma \int_\Omega h v dx.$$

Let $k \to +\infty$, we derive that

$$\int_\Omega |x|^{-l} u \nabla v dx = \int_\Omega |x|^{-l} u^q v dx - \sigma \int_\Omega h v dx,$$

and hence $u$ is a weak solution of problem (1.9).

If $u^*$ is a solution of (1.9) in the interval $[u, \overline{u}]$, then it enjoys the same properties as $\overline{u}$, so the above arguments show that $u \leq u^*$ a.e. in $\Omega$.

**Lemma 2.3** For $-\gamma < \theta < (5\gamma^2 - 6\gamma)/(3\gamma - 2)$ and $h(x) \in A$, $h|_{\partial \Omega} = 0$, there exists a positive constant $\sigma_*$ such that problem (1.9) has a minimal solution $u_{\sigma} > \sigma \varphi$ for all $\sigma \in (0, \sigma_*)$, here $\varphi(x)$ is the solution of (1.11).

**Proof** For any $\sigma > 0$, let $\varphi(x)$ be the nonnegative solution of (1.11) and set $w(x) = \sigma \varphi(x)$. Then we have

$$-\text{div}(|x|^l \nabla w) - |x|^{-l} w^q + \sigma h(x) = \sigma (-\text{div}(|x|^l \nabla \varphi) + h(x)) - \sigma^q |x|^{-l} \varphi^q = -\sigma^q |x|^{-l} \varphi^q \leq 0.$$  

Hence, $w(x)$ is a subsolution of (1.9) for all $\sigma > 0$. 

**Proof** Set $u_1 = \overline{u}$, for given $u_k (k = 1, 2, \cdots)$, we define $u_{k+1} \in H^1_0(\Omega, |x|^l)$ as the unique solution to the following equation

$$\begin{cases}
-\text{div}(|x|^l \nabla u_{k+1}) = |x|^{-l} u^q_k - \sigma h(x), & \text{in } \Omega, \\
u_{k+1}|_{\partial \Omega} = 0.
\end{cases} \tag{2.3}$$
Let $u_0(x) > 0$ be the eigenfunction in Lemma 2.1. Define $W(x) = ku_0(x)$, it satisfies
\[-\text{div}(\|x\|^l\nabla W) - |x|^{-l}W^q + \sigma h(x) = k\lambda_1 |x|^{-l}u_0 - k^q |x|^{-l}u_0^q + \sigma h(x)\]

Choose the positive constant $k$ small enough such that
\[k\lambda_1 |x|^{-l}u_0 \geq k^q (|x|^{-l}u_0^q + |h(x)|),\]
when $\sigma \leq k^q$, we have
\[-\text{div}(\|x\|^l\nabla W) - |x|^{-l}W^q + \sigma h(x) \geq k^q |h(x)| + \sigma h(x) \geq 0.\]

Hence, $W(x)$ is a supersolution of (1.9) for all $0 < \sigma \leq k^q$.

Moreover, if we choose $\sigma_0$ small enough such that $\sigma_0 \varphi(x) \leq ku_0(x)$, then Lemma 2.2 implies that problem (1.9) has a minimal solution $u_\sigma(x)$ satisfying $w(x) \leq u_\sigma(x) \leq W(x)$ for all $\sigma \in (0, \sigma_0]$. Let $v(x) = u_\sigma(x) - \sigma \varphi(x) \neq 0$, it satisfies
\[\begin{cases} 
-\text{div}(\|x\|^l\nabla v) = |x|^{-l}u_\sigma^q \geq 0, & \text{in } \Omega, \\
v|_{\partial\Omega} = 0.
\end{cases}\]

The strong maximum principle guarantees $v(x) > 0$ in $\Omega$, that is $u_\sigma(x) > \sigma \varphi(x) \geq 0$. This completes the proof of the lemma by defining
\[\sigma_* = \sup \{\sigma_0 \in \mathbb{R}^+ : (1.9) \text{ has at least one weak solution for each } \sigma \in (0, \sigma_0)\}. \quad (2.5)\]

The monotone iteration method enables us to obtain the following lemma:

**Lemma 2.4** Under the conditions of Theorem 1.1. If (1.9) has a solution when $\sigma = \hat{\sigma}$, then it has a positive solution for every $\sigma \in (0, \hat{\sigma})$.

**Proof** Let $u_1(x) > 0$ be a solution of (1.9) for $\sigma = \hat{\sigma}$. Set $u_1 = \hat{\sigma}w_1$, then $w_1$ satisfies
\[\begin{cases} 
-\text{div}(\|x\|^l\nabla w_1) = \hat{\sigma}q^{-1}|x|^{-l}w_1^q - h(x), & \text{in } \Omega, \\
w_1|_{\partial\Omega} = 0, \quad w_1 > 0 & \text{in } \Omega.
\end{cases}\]

Hence, for any $0 < \sigma \leq \hat{\sigma}$, $w_1$ is a supersolution of the equation
\[\begin{cases} 
-\text{div}(\|x\|^l\nabla w) = \sigma q^{-1}|x|^{-l}w^q - h(x), & \text{in } \Omega, \\
w|_{\partial\Omega} = 0, \quad w > 0 & \text{in } \Omega.
\end{cases}\]

Since $h(x) \in \mathcal{A}$, equation (1.11) has a solution $\varphi(x) > 0$ satisfying
\[\begin{cases} 
-\text{div}(\|x\|^l\nabla \varphi) \leq \sigma q^{-1}|x|^{-l}w^q - h(x), & x \in \Omega, \\
\varphi|_{\partial\Omega} = 0, \quad \varphi > 0 & \text{in } \Omega.
\end{cases}\]

Thus $\varphi(x)$ is a subsolution of (2.7). Comparison principle implies $0 < \varphi(x) \leq w_1(x)$. By Lemma 2.2, we obtain a weak positive solution $w_\sigma(x)$ of (2.7) for every $\sigma \in (0, \hat{\sigma})$. Obviously, $u_\sigma = \sigma w_\sigma$ is a positive solution of equation (1.9).

**Lemma 2.5** For $-\gamma < \theta < (5\gamma^2 - 6\gamma)/(3\gamma - 2)$ and $h(x) \in \mathcal{A}$, there exists a positive constant $C_1$ such that problem (1.9) has no positive solution for all $\sigma > C_1$. 

Proof If problem (1.9) has a positive solution \( u(x) \), then
\[
\int_{\Omega} -u_0 \text{div}(|x|^{\gamma} \nabla u) \, dx = \int_{\Omega} u_0 (|x|^{-\gamma} u^q - \sigma h(x)) \, dx,
\]  
(2.9)
where \( u_0 > 0 \) is given in Lemma 2.1. Thus
\[
\int_{\Omega} \lambda_1 |x|^{-\gamma} u_0 dx = \int_{\Omega} |x|^{-\gamma} u_0 dx - \int_{\Omega} hu_0 dx,
\]  
(2.10)
that is
\[
\sigma \int_{\Omega} hu_0 dx = \int_{\Omega} |x|^{-\gamma} u_0 (u^q - \lambda_1 u) dx \geq (1 - q) \left( \frac{\lambda_1}{q} \right)^{\frac{q}{q-1}} \int_{\Omega} |x|^{-\gamma} u_0 dx. 
\]  
(2.11)
Let \( \varphi(x) \) be a positive solution of (1.11), we derive
\[
\int_{\Omega} hu_0 dx = \int_{\Omega} u_0 \text{div}(|x|^{\gamma} \nabla \varphi) \, dx = \int_{\Omega} \varphi \text{div}(|x|^{\gamma} \nabla u_0) \, dx = - \int_{\Omega} \lambda_1 |x|^{-\gamma} u_0 \varphi \, dx \leq 0.
\]  
(2.12)
From (2.11) and (2.12) we obtain
\[
\sigma \leq (1 - q) \left( \frac{\lambda_1}{q} \right)^{\frac{q}{q-1}} \int_{\Omega} |x|^{-\gamma} u_0 dx / \int_{\Omega} hu_0 dx := C_1.
\]
The proof is complete.

Define
\[
\sigma_h = \sup \{ \sigma \in \mathbb{R}^+: \text{ problem (1.9) has a positive weak solution} \}. 
\]  
(2.13)

Proof of Theorem 1.1 Under the conditions of Theorem 1.1. By Lemma 2.3–2.5, we obtain that problem (1.9) has a minimal solution \( u_\sigma \) when \( \sigma \in (0, \sigma_h) \), and \( u_\sigma \geq \sigma \varphi(x) \), where \( \varphi(x) \) is the solution of problem (1.11).

Let \( 0 < \sigma_1 < \sigma_2 < \sigma_h \), the corresponding minimal solutions of (1.9) are \( u_1 \) and \( u_2 \). Set \( u_1 = \sigma_1 w_1, \ u_2 = \sigma_2 w_2 \). Then when \( \sigma = \sigma_1, \ w_2 \) is a supersolution and \( \varphi(x) \) is a subsolution of the following equation
\[
\begin{cases}
-\text{div}(|x|^{\gamma} \nabla w) = \sigma^{\alpha-1} |x|^{-\gamma} u^q - h(x), & \text{in } \Omega, \\
w|_{\partial \Omega} = 0.
\end{cases}
\]
Lemma 2.2 implies \( \varphi \leq w_1 \leq w_2 \), that is \( \sigma \varphi \leq u_1 \leq u_2 \). Thus the minimal solution \( u_\sigma(x) \) is increasing with respect to \( \sigma \) for all \( \sigma \in (0, \sigma_h) \).

3 Proof of Theorem 1.2

In this section, we first give some lemmas which will be used in the proof of Theorem 1.2, then we get the second solution of (1.9) when \( \sigma \in (0, \sigma_h) \) and \(-\gamma < \theta < (5\gamma^2 - 6\gamma)/(3\gamma - 2)\).

Lemma 3.1 Under the conditions of Theorem 1.1, problem (1.9) has a minimal positive solution \( u_\sigma(x) \) when \( \sigma \in (0, \sigma_h) \). The corresponding eigenvalue problem
\[
\begin{cases}
-\text{div}(|x|^{\gamma} \nabla \xi) = \Lambda q |x|^{-\gamma} u^{q-1}_\sigma \xi, & \text{in } \Omega, \\
\xi|_{\partial \Omega} = 0,
\end{cases}
\]  
(3.1)
has the first eigenvalue $\Lambda_1 > 1$, and the corresponding eigenfunction $\xi_1(x) > 0$ in $\Omega$.

\textbf{Proof} Denote

$$\Lambda_1 := \inf \left\{ \int_{\Omega} |x|^k |\nabla u|^2 dx : u \in H_0^1(\Omega, |x|^l), \int_{\Omega} q|x|^{-l}u^q - 1 u^2 dx = 1 \right\}. \quad (3.2)$$

Similar to the proof of Lemma 2.1, we get that $\Lambda_1$ is achieved by some $0 < \xi_1(x) \in H_0^1(\Omega, |x|^l)$.

Next we verify $\Lambda_1 > 1$. In fact, for any $\bar{\sigma} \in (\sigma, \sigma_h)$, let $u_\sigma$, $u_{\bar{\sigma}}$ be the minimal solutions of (1.9)$_\sigma$ and (1.9)$_{\bar{\sigma}}$ with $u_\sigma < u_{\bar{\sigma}}$. Denote $v = u_{\bar{\sigma}} - u_\sigma$, we have

$$-\text{div}(|x|^l \nabla v) = |x|^{-l}(u_{\bar{\sigma}}^q - u_\sigma^q) - (\bar{\sigma} - \sigma)h(x). \quad (3.3)$$

Multiplying (3.3) by $\xi_1(x)$ and integrating the product over $\Omega$ give that

$$\int_{\Omega} -\xi_1 \text{div}(|x|^l \nabla v) dx = \int_{\Omega} |x|^{-l}(u_{\bar{\sigma}}^q - u_\sigma^q)\xi_1 dx - \int_{\Omega} (\bar{\sigma} - \sigma)h\xi_1 dx. \quad (3.4)$$

Using integration by parts formula and (3.1) yield

$$\Lambda_1 q \int_{\Omega} |x|^{-l} v u_{\bar{\sigma}}^q - 1 \xi_1 dx = \int_{\Omega} |x|^{-l}(u_{\bar{\sigma}}^q - u_\sigma^q)\xi_1 dx - \int_{\Omega} (\bar{\sigma} - \sigma)h\xi_1 dx. \quad (3.5)$$

Since $h \in \mathcal{A}$, we obtain that

$$-\int_{\Omega} \bar{h}\xi_1 dx = \int_{\Omega} -\text{div}(|x|^l \nabla \varphi)\xi_1 dx = \int_{\Omega} -\text{div}(|x|^l \nabla \xi_1)\varphi dx$$

$$= \Lambda_1 q \int_{\Omega} |x|^{-l} v u_{\bar{\sigma}}^q - 1 \xi_1 \varphi dx > 0.$$ 

Therefore, (3.5) becomes

$$\Lambda_1 q \int_{\Omega} |x|^{-l} v u_{\bar{\sigma}}^q - 1 \xi_1 dx > q \int_{\Omega} |x|^{-l} v u_{\bar{\sigma}}^q - 1 \xi_1 dx,$$ 

which gives $\Lambda_1 > 1$. The proof is complete.

\textbf{Lemma 3.2} Under the conditions of Theorem 1.1. Then problem (1.9) has a unique solution when $\sigma = \sigma_h$, where $\sigma_h$ is defined in (2.13).

\textbf{Proof} Set

$$D = \left\{ u_\sigma : \sigma \in (0, \sigma_h), u_\sigma \text{ is the minimal solution of problem (1.9)} \right\}. \quad (3.7)$$

We claim that there exists a positive constant $C$ independent of $\sigma \in (0, \sigma_h)$ such that

$$\|u_\sigma\|^2 \leq C \text{ for all } u_\sigma \in D. \quad (3.8)$$

In fact, for any $u_\sigma \in D$, Lemma 3.1 implies that

$$\int_{\Omega} |x|^{-l} u_{\bar{\sigma}}^{q+1} dx - \sigma \int_{\Omega} h u_\sigma dx = \int_{\Omega} |x|^l |\nabla u_\sigma|^2 dx \geq \Lambda_1 q \int_{\Omega} |x|^{-l} u_{\bar{\sigma}}^{q+1} dx. \quad (3.9)$$

That is

$$(\Lambda_1 q - 1) \int_{\Omega} |x|^{-l} u_{\bar{\sigma}}^{q+1} dx \leq -\sigma \int_{\Omega} h u_\sigma dx. \quad (3.10)$$
Since \( q \Lambda_1 > 1 \), so (3.10) becomes
\[
\int_{\Omega} |x|^{-l} u_\sigma^{q+1} dx \leq -\frac{\sigma}{q \Lambda_1 - 1} \int_{\Omega} h(x) u_\sigma dx. \tag{3.11}
\]
For any \( \varepsilon > 0 \), Lemma 2.1 and the above inequality imply that
\[
\lambda_1 \int_{\Omega} |x|^{-l} u_\sigma^2 dx \leq \int_{\Omega} |x|^l \nabla u_\sigma^2 dx = \int_{\Omega} |x|^{-l} u_\sigma^{q+1} dx - \sigma \int_{\Omega} h u_\sigma dx
\[
\leq \left( \frac{\sigma}{q \Lambda_1 - 1} + \sigma \right) \int_{\Omega} |h u_\sigma| dx
\[
\leq \frac{q \Lambda_1 \sigma h}{q \Lambda_1 - 1} \left( \frac{\varepsilon}{2} \right) \left( \int_{\Omega} |x|^{-l} u_\sigma^2 dx + \frac{1}{2\varepsilon} \int_{\Omega} |x|^l h^2 dx \right). \tag{3.12}
\]
Let \( \varepsilon = \lambda_1 (q \Lambda_1 - 1)/(q \Lambda_1 \sigma h) \), we deduce that
\[
\int_{\Omega} |x|^{-l} u_\sigma^2 dx \leq \left( \frac{q \sigma h \Lambda_1}{\lambda_1 (q \Lambda_1 - 1)} \right)^2 \int_{\Omega} |x|^l h^2 dx. \tag{3.13}
\]
Since \( h \in C^1 \) and \( \theta > -\gamma, l = \frac{a}{\gamma} > -1 \), therefore
\[
\int_{\Omega} |x|^l h^2 dx \leq C_2. \tag{3.14}
\]
From (3.13) and (3.14), we derive that
\[
||u_\sigma||^2 = |u_\sigma|_{L^{q+1}(\Omega, |x|^{-l})}^2 - \sigma |h u_\sigma|_{L^1(\Omega)} \leq \frac{q \Lambda_1 \sigma h}{2(q \Lambda_1 - 1)} \left( \int_{\Omega} |x|^{-l} u_\sigma^2 dx + \int_{\Omega} |x|^l h^2 dx \right) \leq C.
\]
Thus (3.8) is obtained.

Next, we consider the solution for (1.9) when \( \sigma = \sigma_h \). Suppose \( \{\sigma_n\} \) is an increasing sequence in \( (0, \sigma_h) \) satisfying \( \sigma_n \to \sigma_h \) as \( n \to \infty \). The corresponding sequence of solutions is denoted by \( \{u_n\} \). By (3.8), we can choose a subsequence, still denoted by \( \{u_n\} \), such that \( u_n \rightharpoonup \bar{u} \) weakly in \( H^1_0(\Omega, |x|^l) \), here \( \bar{u} \) is a nonnegative function and it is easy to prove that \( \bar{u} \) is a weak solution of (1.9) with \( \sigma = \sigma_h \).

Define a mapping \( F : \mathbb{R} \times H^1_0(B_R, |x|^l) \to H^1_0(B_R, |x|^l) \) by
\[
F(\sigma, u) = -\text{div}(|x|^l \nabla u) - |x|^{-l} u^q + \sigma h(x).
\]
Then the first eigenvalue of (3.1) is 1 when \( \sigma = \sigma_h \). In fact, suppose \( \Lambda_1(\sigma_h) \neq 1 \), then \( F_u(\sigma_h, u_{\sigma_h}) = 0 \) has no nontrivial solution. Applying the implicit function theorem to \( F \), we can find a neighborhood \( (\sigma_h - \varepsilon, \sigma_h + \varepsilon) \) of \( \sigma_h \) such that (1.9) possesses a solution if \( \sigma \in (\sigma_h - \varepsilon, \sigma_h + \varepsilon) \), this contradict to the definition of \( \sigma_h \).

Finally, we prove that \( u_{\sigma_h} \) is unique. Assume there are two different solutions \( u_1 \) and \( u_2 \) of (1.9) when \( \sigma = \sigma_h \), let \( u_1 \) be the minimal solution. Set \( w = u_2 - u_1 \geq 0 \), we have
\[
-\text{div}(|x|^l \nabla w) = |x|^{-l} (u_2^q - u_1^q). \tag{3.15}
\]
By \( \Lambda_1(\sigma_h) = 1 \) we know that the problem
\[
-\text{div}(|x|^l \nabla \xi) = q |x|^{-l} u_1^{q-1} \xi \tag{3.16}
\]
possesses a positive solution \( \xi_1(x) \). Multiplying (3.15) by \( \xi_1(x) \) and (3.16) by \( w(x) \), integrating and subtracting we deduce that
\[
0 = \int_{\Omega} |x|^{-\ell} [(w + u_1)^q - u_1^q - qu_1^{q-1} w] \xi_1 \, dx = \int_{\Omega} |x|^{-\ell} q(q - 1) \delta^\ell w^2 \xi_1 \, dx,
\]
where \( \delta \in (u_1, u_2) \). Thus \( w \equiv 0 \). The proof is complete.

Let \( u_\sigma \) be the minimal solution of (1.9) for \( \sigma \in (0, \sigma_h) \). In order to find the second solution of (1.9), we consider the following problem
\[
\begin{aligned}
\delta & \in \{ 0, 1, \} \quad \sigma \in (0, \sigma_h), \\
\end{aligned}
\]
where \( \Omega \) is the minimal solution of (1.9) for \( \sigma = 0 \). We prove that (3.17) has a positive solution by using a variational method. Define
\[
J(v) = \frac{1}{2} \int_{\Omega} |x|^\ell |\nabla v|^2 - q |x|^{-\ell} u_\sigma^{q-1} v^2 \, dx - \int_{\Omega} G(v, u_\sigma) \, dx,
\]
where \( G(v, u_\sigma) = \int_0^1 \frac{q(q + 1)}{2(2-q)} z^2 y^{q-1} \, ds \).

The proof is complete.

Clearly, if (3.17) has a solution \( v_\sigma \), then \( U_\sigma = v_\sigma + u_\sigma \) is the second solution of (1.9).

We prove that (3.17) has a positive solution by using a variational method. Define
\[
g(v, u_\sigma) = |x|^{-\ell} [(v + u_\sigma)^q - u_\sigma^q - qu_\sigma^{q-1} v],
\]
then equation (3.17) equivalent to the following problem
\[
\begin{aligned}
\end{aligned}
\]

The corresponding variational functional of (3.17) is
\[
J(v) = \frac{1}{2} \int_{\Omega} |x|^\ell |\nabla v|^2 - q |x|^{-\ell} u_\sigma^{q-1} v^2 \, dx - \int_{\Omega} G(v, u_\sigma) \, dx,
\]
where \( G(v, u_\sigma) = \int_0^1 g(s, u_\sigma) \, ds \).

In the following lemmas, we will verify that \( J(v) \) satisfies the conditions in Mountain Pass Lemma. It is easy to verify that \( J \in C^1(\mathcal{C}_0^1(\Omega, |x|^{\ell}), \mathbb{R}) \) and \( J(0) = 0 \).

**Lemma 3.3** Assume \( -\gamma < \theta < (5\gamma^2 - 6\gamma)/(3\gamma - 2) \). There exists positive constants \( a \) and \( r \) such that \( J(v)|_{\partial B_r} > a > 0 \).

**Proof** Fixed \( y > 0 \), by the definition of \( G \), we derive
\[
G(z, y) := \frac{1}{q + 1} |x|^{-\ell} \left( (z + y)^{q+1} - y^{q+1} - (q + 1) \frac{q(q + 1)}{2} z^2 y^{q-1} \right).
\]
Using Taylor’s formula, we obtain \( G(z, y) = \frac{q}{2} |x|^{-\ell} z^2 [\theta z + y]^{q-1} - y^{q-1} \) for some \( \theta \in (0, 1) \).

Let \( \Omega_1 = \{ x \in \Omega : u_\sigma \geq M \} \), so
\[
0 \leq \int_{\Omega_1} G(v^+, u_\sigma) \, dx = \frac{q}{2} \int_{\Omega_1} |x|^{-\ell} (v^+) \left( (v^+) + u_\sigma \right)^{q-1} \, dx \leq C \int_{\Omega_1} |x|^{-\ell} (v^+) \left( (v^+) + u_\sigma \right)^{q-1} \, dx \leq C |v|_{L^{q+1}(\Omega_1, |x|^{-\ell})} + C |u_\sigma|_{L^{q+1}(\Omega_1, |x|^{-\ell})} |v|_{L^{q+1}(\Omega_1, |x|^{-\ell})}.
\]
For any \( \varepsilon > 0, M \) large enough such that \( \text{meas}(\Omega_1) < \varepsilon \). By the absolute continuity of integration, we obtain \( |u_\sigma|_{L^{q+1}(\Omega_1, |x|^{-\ell})} < \varepsilon \). Sobolev embedding theorem yields
\[
0 \leq \int_{\Omega_1} G(v^+, u_\sigma) \, dx \leq C |v|_{L^{q+1}(\Omega_1, |x|^{-\ell})} + \varepsilon |v|_{L^{q+1}(\Omega_1, |x|^{-\ell})} \leq C |v|^{q+1} + \varepsilon |v|^2.
\]
Let \( \Omega_2 = \{ x \in \Omega : u_\sigma < M \} \), then

\[
0 \leq \frac{G(v^+, u_\sigma)}{|x|^{-i}(v^+)^2} \to 0 \quad \text{uniformly for } u_\sigma \in [0, M] \text{ as } v^+ \to 0,
\]

\[
0 \leq \frac{G(v^+, u_\sigma)}{|x|^{-i}(v^+)^{q+1}} \leq \left( 1 + \frac{M}{v^+} \right)^{q+1} \leq C \quad \text{uniformly for } u_\sigma \in [0, M] \text{ as } v^+ \to +\infty.
\]

From the above facts, we deduce

\[
0 \leq \int_{\Omega_2} G(v^+, u_\sigma) dx \leq \varepsilon |v^+|^2_{L^2(\Omega_2, |x|^{-i})} + C(\varepsilon) |v^+|^{q+1}_{L^{q+1}(\Omega_2, |x|^{-i})} \leq C(\varepsilon) \|v\|^{q+1} + \varepsilon \|v\|^2.
\]

Therefore

\[
0 \leq \int_{\Omega} G(v^+, u_\sigma) dx \leq C(\varepsilon) \|v\|^{q+1} + \varepsilon \|v\|^2.
\]

The variational functional of (3.17) is

\[
J(v) = \frac{1}{2} \int_{\Omega} \left( |x|^i |\nabla v|^2 - q |x|^{-i} u_\sigma^{q-1} v^2 \right) dx - \int_{\Omega} G(v^+, u_\sigma) dx
\]

\[
\geq \frac{1}{2} \left( 1 - \frac{1}{\Lambda_1} \right) \|v\|^2 - \varepsilon \|v\|^2 - C(\varepsilon) \|v\|^{q+1},
\]

where we have used Lemma 3.1. We choose \( \varepsilon = (\Lambda_1 - 1)/4\Lambda_1 \), then

\[
J(v) \geq \frac{1}{4} \left( 1 - \frac{1}{\Lambda_1} \right) \|v\|^2 - C \|v\|^{q+1}.
\]

Thus there exist positive constants \( a \) and \( r \) such that \( J(v)_{|\partial B_r} \geq a > 0 \).

**Lemma 3.4** There exists an element \( v_0 \in H^1_0(\Omega, |x|^i) \) with \( \|v_0\| > r \) and \( J(v_0) \leq 0 \).

**Proof** Since \((x + y)^{q+1} - x^{q+1} - y^{q+1} \geq (q + 1)xy^q\), where \( x, y \geq 0 \). Then for any \( 0 < v \in H^1_0(\Omega, |x|^i) \),

\[
J(tv) = \frac{t^2}{2} \|v\|^2 - \frac{1}{q + 1} \int_{\Omega} |x|^{-i} \left( (tv^+ + u_\sigma)^q v - u_\sigma^{q+1} + (q + 1)tv^q v^+ \right) dx
\]

\[
\leq \frac{t^2}{2} \|v\|^2 - \frac{q^{q+1}}{q + 1} \int_{\Omega} |x|^{-i} (v^+)^{q+1} dx.
\]

We obtain that \( J(tv) \to -\infty \) as \( t \to +\infty \). So there exists \( v_0 \in H^1_0(\Omega, |x|^i) \) such that \( \|v_0\| > r \) and \( J(v_0) \leq 0 \).

**Lemma 3.5** Any sequence \( \{v_n\} \in H^1_0(\Omega, |x|^i) \) such that \( c = \sup_n J(v_n) < +\infty \), \( J'(v_n) \to 0 \) as \( n \to +\infty \), contains a convergent subsequence.

**Proof** For any \( 0 < v \in H^1_0(\Omega, |x|^i) \), by the definition of \( g(v, u_\sigma) \) and \( G(v, u_\sigma) \), we derive

\[
\frac{1}{2 + \tau} g(v, u_\sigma) > G(v, u_\sigma), \quad \tau \in (0, 1).
\]

(3.19)

For \( n \) large enough, we have

\[
c + 1 + \|v_n\| \geq J(v_n) - \frac{1}{2 + \tau} J'(v_n), v_n
\]

\[
= \left( \frac{1}{2} - \frac{1}{2 + \tau} \right) \left( \|v_n\|^2 - q \int_{\Omega} |x|^{-i} u_\sigma^{q-1} v_n^2 dx \right)
\]

\[
+ \int_{\Omega} \frac{1}{2 + \tau} g(v, u_\sigma) - G(v, u_\sigma) dx
\]

\[
> \left( \frac{1}{2} - \frac{1}{2 + \tau} \right) \left( 1 - \frac{1}{\Lambda_1} \right) \|v_n\|^2,
\]
here we have used (3.19) and Lemma 3.1. Thus \( \{ v_n \} \) is bounded in \( H_0^1(\Omega, |x|^l) \).

There exists a subsequence still denoted by \( \{ v_n \} \) and \( v \in H_0^1(\Omega, |x|^l) \), such that \( v_n \to v \) weakly in \( H_0^1(\Omega, |x|^l) \) and \( v_n \to v \) strongly in \( L^p(\Omega, |x|^{-l}) \) for \( 1 < p < q(l) \). We observe that

\[
\|v_n - v\|^2 = \langle J'(v_n) - J'(v), v_n - v \rangle + \int_\Omega |x|^{-l}(v_n + u_\sigma)^q - (v_n + u_\sigma)^q)(v_n - v)\,dx
\]

\[
\leq \int_\Omega q|x|^{-l}(\zeta_n + u_\sigma)^{q-1}(v_n - v)^2\,dx + o(1)
\]

\[
\leq q\|v_n - v\|^2_{L^{q+1}(\Omega, |x|^{-l})} \zeta_n + u_\sigma^q_{L^{q+1}(\Omega, |x|^{-l})} \to 0 \quad \text{as} \quad n \to +\infty,
\]

where \( \zeta_n \) is between \( v \) and \( v_n \). Therefore \( v_n \to v \) strongly in \( H_0^1(\Omega, |x|^l) \) as \( n \to +\infty \).

**Lemma 3.6**  Assume \( h(x) \in A \) and \( -\gamma < \theta < (5\gamma^2 - 6\gamma)/(3\gamma - 2) \). Then problem (3.17) has at least one solution for \( \sigma \in (0, \sigma_h) \).

**Proof**  By Lemma 3.3–3.5 we know that the functional \( J(v) \) satisfies conditions in the Mountain Pass Lemma (see [7]). Define

\[
\Gamma := \{ g \in C([0, 1]; H_0^1(\Omega, |x|^l)) \mid g(0) = 0, \quad g(1) = v_0 \}.
\]

Then

\[
c = \inf_{g \in \Gamma} \max_{0 \leq \tau \leq 1} J[g(\tau)]
\]

is a critical value of the functional \( J \). Thus equation (3.17) has at least one solution.

**Proof of Theorem 1.2**  Under the conditions of Theorem 1.2, by Lemma 3.6, problem (3.17) has at least one solution \( v_\sigma > 0 \) in \( \Omega \). Then \( U_\sigma = v_\sigma + u_\sigma \) is the second solution of (1.9) with \( U_\sigma > u_\sigma \). From Lemma 3.2 and the definition of \( \sigma_h \), we can complete the proof of Theorem 1.2.

**References**


