The Problem of the Crack in the "Generally" Anisotropic Disk

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[1]Translated and re-edited from the Greek published book into English LaTeX, by I.N. Galidakis. Final typography partially verified courtesy of N.I. Rigos, Civil Engineer, graduate NTUA.

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To the Memory of My Father

Contents

Introduction

The usually unsurmountable mathematical difficulties which arise from trying to solve even the simplest of problems, are well known to those who have involved themselves with the subject of theoretical Elasticity.

The already known methods of approach to problems of Mathematical theory of Elasticity, belong to two categories.

In the first category belong the methods which concern themselves with the problem of boundary values of functions on a given contour of the elastic medium, whereas in the second category belong methods which derive from certain given general laws of “minima”, which characterize the balanced state of the ideal elastic body.

In the current work the methods used are from the theory of Complex functions belonging according to the method of study to the first category.

The use of the Complex variable in the theory of plane Elasticity of an Isotropic body was first used by G.V. Kolossoff in 1909 [11]. Despite this, a third of a century passed until the theory being extended beyond the initial stage of the “typical” use of the complex variable could enter into more substantial offers in the area of theoretical Elasticity [17] [21].

Today, the nature of the results of the plane theory of Elasticity cannot be considered solely a conclusion of the work of Kolossof [11] or Filon [6], who, even though have found certain complex expressions for stress, did not think to use them further.

First N.I. Muskhelisvili [17], by applying the above results on the “fundamental” boundary problems by using a conformal mapping and the properties of the Cauchy integral, managed to give completeness and unity to the approximating methods to problems relating to two dimensional Elasticity.

The reduction of the above problems into Fredholm integral equations of the second type [14], [15], simplifies generally the Mathematical description of the solution, the latter succeeded by completely elementary methods in the case where the “nucleus” is a rational function [17] [21].

The important contribution of the method is not so much the research by new means of known or almost known problems, rather the push that was given to the study of new problems, like for example, the cases of contact problems, by reducing them to the general problem of “Linear Correlation” or otherwise to the Riemann-Hilbert problem [14] [15] [17].

The extension of the theory of Muskhelisvili in the area of the two dimensional anisotropic elastic body appears (completely independently and approximately the same time period) in the West on one hand, and in Russia on the other, around 1950.

There are indications that the Russian works came first, but they appear to have been inaccessible not only for those outside Russia but also for many Russian Research Centers [8].

A.E. Green and W. Zerna [7] give form to the Mathematical theory of Elasticity on a more general basis so that it includes equally (at least in two dimensions) the isotropic and the anisotropic medium. The generality of the appearance of the whole subject however, fails to cover areas like those of investigating or even applying the austerely general extracted solutions.

The research of general Anisotropy (covering 80 pages out of a total of 450) contains the study of Complex Functional Potential and the specification of such in very special cases like that of the orthotropic half-space, the general similar charge of circular or elliptical contours, the concentrated charge at infinity plane, the influence of the uncharged crack as well as a general introduction to the problem of contact (without friction) of two elastic bodies.

A summary of the work of Russian researchers on the anisotropic elastic body was published in 1963 by S.G. Lekhnitskii [12]. In the latter work the problems mentioned in Green and Zerna are solved with additional material, such as the study of the half-container having a parabolic limit, the expression of the equations for the Somigliana problem (cylindrical body charged by external non-variable forces along the generational) and extraction of conclusions from their investigation, the general problem of bending with turning of beams of specific cross section, the turning of beams of double tenacity cross section, as well as the presentation of the general equations of balance of an elastic body having non-linear anisotropy.

S.G. Lemhniskii introduces essentially three independent Complex variables generalizing Muskhelishvili's method for anisotropic bodies having at least one level of symmetry (elastic or charging) but he doesn't give the general integral equations which describe the solutions for problems of this kind, neither does he prove most of the conclusions from the application of the above theory, referring usually to obscure because of non-circulation in the West and because of language Russian Monographs.[2]

The present work refers to the problem of the crack in the generally anisotropic disk, that is the elastic plane which has no elements of symmetry [13].

The crack considered as a simple Jordan curve, has the great advantage that it can be mapped onto the unit circle by a rational function, succeeding thus in finding the Complex Potential [7] [12] for cases of non-self-balanced charge.

The plane of the anisotropic disk (the regular Complex plane z) transforms by two homoparallel transformations to the planes z_1 and z_2 which are considered "projections" of a four-dimensional continuous bicylinder [3], "inside" which the functions of stress and displacement are defined.

When the homoparallel transformations are equal ($K_1 = K_2 = K$) or equivalently if the planes z_1 and z_2 coincide, then the case reduces to "Pseudoanisotropy" and the general two dimensional problem transforms itself properly to an isotropic problem [24].

The solution to the problem of the crack in the case of General Anisotropy is succeeded by determination of the functions of Complex potential $\Phi_1(z_1)$ and $\Phi_2(z_2)$, which are holomorphic functions on the planes z_1 and z_2 respectively.

By mapping conformally the above two planes into ζ_1 and ζ_2 respectively, such that the limits of the crack image contour are mapped into the unit circles $|\zeta_1| = 1$ and $|\zeta_2| = 1$ and so that for all z_i we have $|\zeta_i| \geq 1$ we conclude that $\Phi_i(z_i)$ will be holomorphic on the planes ζ_i for all $|\zeta_i| \geq 1$, with the possible exception the points $\zeta_i = \infty$.

This property, which expresses mathematically the "regularity" of the intensity state for the whole disk (with the possible exception of points on the contour), along with the known values of the derivatives of the Φ_i (on the contour) are sufficient for the determination of the above functions by application of simple properties of Cauchy integrals.

In the current work, the aforementioned problem is solved almost elementarily, by finding the functions of Complex potential in closed form for all the cases having a charge

[2] Some of them have been written in special Russian dialects, such as Georgian, etc.

upon the boundary of the crack.

Using appropriate limiting accessing methods we can find from the general solution the solutions to the problems referring to the half-space in the form of special cases, (since for $R \to \infty$ the plane separates into the two half-planes) as well as extract conclusions relating to the form and structure of the anisotropic medium.

It is mentioned lastly, that by this current work a new boundary (that of the crack) appears, which along with the infinite line of the half-plane contain the only known up to now half-spaces upon which the two fundamental Problems of the Mathematical theory of anisotropic Elasticity are solved for all cases of charging by closed form functions.

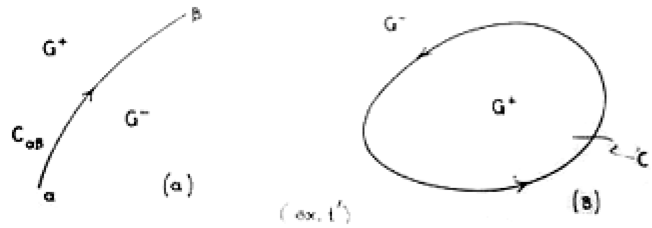

Figure 1: Jordan Curve

Mathematical Appendix:

Basic Notions and Theorems from the Theory of Functions of a Complex Variable

1. By the term "Simple Jordan Curve" we mean a continuous normal curve which has no double points and which can be thought of as a bijective image of a circular arc (Fig. 1a).

2. If the aforementioned curve is closed then we get the "Jordan curve" C, which has the characteristic property of separating the plane into two profound domains G_1 and G_2, which, along with C cover the entire Riemann sphere (Fig. 1b).

3. On the Jordan curve (Cab or C respectively) we can define a positive direction, therefore on its left we define G^+ and on its right G^- (Fig. 1).

4. Let there be a function $\Phi(z)$ given and continuous on the "neighborhood" of C (but not necessarily on C). We consider a point $t \in c$. We say that $\Phi(z)$ is continuous left of t (or right of t) if the function tends to a limit, as z tends to t along any line which lies left (or right) of C. The values of $\Phi(z)$ are called limiting values of the function and are represented as $\Phi(t)^*$ or $\Phi^-(t)$.

5. Let there be the function $F(t)$, where $t = x + yi$ a point of C. We say that $F(t)$ satisfies the Hölder condition on C, if for every pair t_1 and t_2 the inequality $|F(t_1) - F(t_2)| \leq A\,|t_1 - t_2|^\mu$, holds, for A and μ positive constants with $0 < \mu \leq 1$.

6. The integral $F(z) = \frac{1}{2\pi i} \int_C f \frac{f(t)}{t-z} dt$ (where $f(t)$ is finite and integrable in the sense of Riemann[3] on the plane z) taken over C, with z being a point of G^+ (or G^-) is called the Cauchy integral.

$F(z)$ is a holomorphic function[4] on the entire plane, except at the points of C.

7. The function $F(z)$ for $z \in C$ has meaning provided $f(t)$ satisfies the Hölder condition on the "neighborhood" of the point z.

[3] A function $f(z)$ is called integrable in the usual sense or in the sense of Riemann, if it is defined and continuous on the domain G, and if the value of the curvilinear integral $\int_{\gamma 1}^{\gamma 2} f(\zeta) d\zeta$ is independent of the curve joining the points $\gamma 1$ and $\gamma 2$.

[4] A single valued function $f(z)$ of a complex variable z, whose derivative has a single value on every point of the domain G is called regular analytic or holomorphic function on G.
An equivalent definition can be gotten from the decomposition of $f(z)$ into a series of powers of z in the domain G.

8. If we define the positive direction on C in a way such that G^+ is the finite domain and G^- is the infinite domain of the plane which C encloses, then:

a) If $f(z)$ is a holomorphic function on G^+ and continuous on $G^+ \cup C$, we have

$$\frac{1}{2\pi i}\int_C \frac{f(t)}{t-z}dt = f(z), \text{ for } z \in G^+$$
$$\frac{1}{2\pi i}\int_C \frac{f(t)}{t-z}dt = 0, \text{ for } z \in G^-$$

b) If $f(z)$ is a holomorphic function on G^- and continuous on $G^- \cup C$, we have

$$\frac{1}{2\pi i}\int_C \frac{f(t)}{t-z}dt = -f(z) + f(\infty), \text{ for } z \in G^-$$
$$\frac{1}{2\pi i}\int_C \frac{f(t)}{t-z}dt = f(\infty), \text{ for } z \in G^+$$

9. According to the above follow the two basic propositions of the theory of limiting values of functions:

a) A sufficient and necessary condition for a given $f(t)$ on C to be the limiting value of a holomorphic function on G^+ is

$$\frac{1}{2\pi i}\int_C \frac{f(t)}{t-z}dt = 0, \text{ for all } z \in G^-$$

b) A sufficient and necessary condition for a given $f(t)$ on C to be the limiting value of a holomorphic function on G^- is

$$\frac{1}{2\pi i}\int_C \frac{f(t)}{t-z}dt = \alpha, \text{ for all } z \in G^+$$

Expressions for the Previous Propositions when C is the Unit Circle

1. Let Γ be the unit circle $|\zeta| = 1$ and σ its points, where $\sigma = e^{i\theta}$, for $0 \leq \theta \leq 2\pi$.

We call Σ^+ and Σ^- the interior and exterior of Γ, or the same, for the domains where $|\zeta| < 1$ and $|\zeta| > 1$ respectively. Let also $\Phi(\zeta)$ be a function defined in Σ^+ (or Σ^-).

We define the function

$$\Phi_*(\zeta) = \overline{\Phi}\left(\frac{1}{\zeta}\right) = \overline{\Phi\left(\frac{1}{\overline{\zeta}}\right)}$$

From the definition it follows that if $\Phi(\zeta)$ is holomorphic on Σ^+ (or Σ^-), $\Phi_*(\zeta)$ will be holomorphic on Σ^+ (or Σ^-) and conversely.

If we assume that $\Phi(\zeta)$ defined on Σ^+ has the limiting value $\Phi(\sigma)$ for $\zeta \to \sigma$ then $\Phi_*(\zeta)$ has the limiting value $\Phi_*^-(\sigma)$ defined on Σ^- and such that

$$\Phi_*^-(\sigma) = \overline{\Phi^+(\sigma)}$$
$$\text{or, } \Phi_*^+(\sigma) = \overline{\Phi^-(\sigma)}$$

2. According to the above, conditions 1.9 a),b) can be expressed as follows:

a) A sufficient and necessary condition for the function $f(\sigma)$ (continuous on the circle Γ) to be the limiting value of a holomorphic function inside of Γ is:

$$\frac{1}{2\pi i}\int_\Gamma \frac{f(\sigma)}{\sigma - \zeta} d\sigma = \overline{\alpha}, \text{ for all } |\zeta| < 1 \text{ and where } \alpha = \overline{f(0)}$$

b) A sufficient and necessary condition for the function $f(\sigma)$, which is continuous on the circle Γ to be the limiting value of a holomorphic function outside of Γ is:

$$\frac{1}{2\pi i}\int_\Gamma \frac{\overline{f(\sigma)}}{\sigma - \zeta} d\sigma = 0, \text{ for all } |\zeta| > 1$$

A Plane Stress On the Anisotropic Disk - The Crack As a Container

A.1 General

1. As is well known the static intensity state at any point of the continuous disk is determined completely if the stress components are given on any two linear elements Δs_1 and Δs_2 passing through the aforementioned point.

If the linear elements are parallel to the axes x and y respectively, we say that the stress tensor T on the given point has components the linear magnitudes σ_x, σ_y, τ_{xy} and τ_{yx}. (Fig. 2).

If we "project" the above tensor against the linear element Δs (passing through the initial point), which is signed with respect to direction as $\vec{n}$, we find by using the Cauchy relations the two components X_n and Y_n of T according to $\vec{n}$ as follows[5]: (Fig. 2)[6]

$$\begin{aligned} X_n &= \sigma_x \cos(n, x) + \tau_{xy} \cos(n, y) \\ Y_n &= \tau_{yx} \cos(n, x) + \sigma_y \cos(n, y) \end{aligned} \tag{1}$$

We accept the magnitudes of σ_x, σ_y, τ_{xy} and τ_{yx} as continuous functions of the coordinates of the plane x and y, hence we also accept the existence of their first (at least) partial derivatives.

2. We consider now one element F of the balanced plane Disk. The corresponding components of all acting forces on it must be zero[7], in the x and y axes, Hence,

$$\int_\Gamma X_n ds = 0, \text{ and } \int_\Gamma Y_n ds = 0$$

(where Γ is the contour of F) or because of (1) and by applying Green's Theorem:

$$\int_\Gamma \left[\sigma_x \cos(n, x) + \tau_{xy} \cos(n, y)\right] ds = \iint_F \left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}\right) dF = 0$$

$$\int_\Gamma \left[\tau_{yx} \cos(n, x) + \sigma_y \cos(n, y)\right] ds = \iint_F \left(\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y}\right) dF = 0$$

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[5]Translator's Note: The book the dissertation was published in has the equations $X_n = \sigma_x \sigma\upsilon\nu(n, x) + \tau_{xy} \sigma\upsilon\nu(n, y)$ and $Y_n = \tau_{yx} \sigma\upsilon\nu(n, x) + \sigma_y \sigma\upsilon\nu(n, y)$ in Greek, for which the operator $\sigma\upsilon\nu$ translates as cos, which is of course a one-argument function. It appears that the author has n to be the unit normal to the linear element Δs and T to be a two-tensor. The translator thinks that the author means $X_n = T(\hat{n}, \hat{x})$ and $Y_n = T(\hat{n}, \hat{y})$. In other words, treating $T(\hat{n}, *)$ as a vector, X_n and Y_n are the "projected" components in the directions of the axes x and y. In that case, since T has a matrix expression in the standard basis given by the σ's and τ's, we see that this is consistent with the interpretation which has $\sigma\upsilon\nu(n, *) = P_*(n)$, or the "projection" in the direction of $*$, namely that the vector n when expressed in the standard basis is $(\sigma\upsilon\nu(n, x), \sigma\upsilon\nu(n, y))$. Furthermore, if n is assumed to be a unit vector, then the component of n in the x direction is precisely the cosine of the angle between them. So this also matches the interpretation which has $\sigma\upsilon\nu(n, *) = \cos(n, *)$ in the case that n is a unit, in which case the more austere notation $\cos\langle \hat{n}, \hat{*} \rangle$ could have been used.

[6]Stresses are linear magnitudes and hence Fig. 2 depicts vectors having components equal to the stresses.

[7]In the above as well as in what follows, we accept that there are no forces due to Mass.

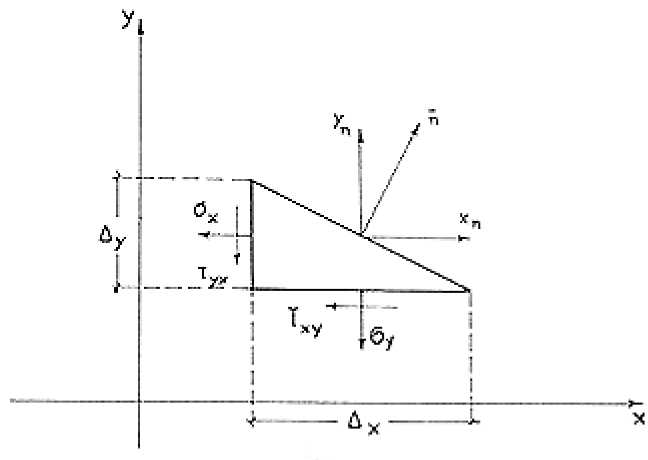

Figure 2: Component Stresses on the Linear Element Δs

and because the container F is random the following relationships must hold:

$$\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \\ \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0 \end{aligned} \tag{2}$$

Equations (2) in combination with the resultant from the vanishing of torque in the elementary rectangle $\tau_{xy} = \tau_{yx}$ comprise the balance system, known as Cauchy system, which holds for any continuous body irrespectively of its elastic behavior.

3. From the geometry of the deformation [9] [17] [21] we have[8]

$$\epsilon_x = \frac{\partial u}{\partial x},\ \epsilon_y = \frac{\partial v}{\partial y},\ \gamma_{xy} = \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \tag{3}$$

where $u(x,y)$ and $v(x,y)$ are continuous functions (since we preclude breaking of the body), which implies the existence of their first (at least) partial derivatives.

Relations (3) as is known connect the three characteristics of deformation with the two components of the vector of displacement. Consequently the first cannot be independent between themselves.

Accepting the existence of derivatives of u and v up to third order, we can eliminate these functions, so we get the following relationship:

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} \tag{4}$$

[8] It is not important that we usually consider $\gamma_{xy} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$ to make calculations for the expression of stress deformation easier.

This equation is called relation of compromise of B. De Saint-Venant (or relation of continuity) and is the necessary and sufficient condition so that the disk retains its consistency even after its deformation.

4. After the above we form the following relations which connect the characteristics of deformation with the components of the stress tensor, in the general case where there are no elements of elastic symmetry in the homogeneous anisotropic elastic body:

$$\begin{aligned} \epsilon_x &= \alpha_{11}\sigma_x + \alpha_{12}\sigma_y + \alpha_{13}\tau_{xy} \\ \epsilon_y &= \alpha_{21}\sigma_x + \alpha_{22}\sigma_y + \alpha_{23}\tau_{xy} \\ \gamma_{xy} &= \alpha_{31}\sigma_x + \alpha_{32}\sigma_y + \alpha_{33}\tau_{xy} \end{aligned} \text{ or } \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} \tag{5}$$

The relations (5) express the "Generalized Hook's Law" and can also be written conversely:

$$\begin{aligned} \sigma_x &= A_{11}\epsilon_x + A_{12}\epsilon_y + A_{13}\gamma_{xy} \\ \sigma_y &= A_{21}\epsilon_x + A_{22}\epsilon_y + A_{23}\gamma_{xy} \\ \tau_{xy} &= A_{31}\epsilon_x + A_{32}\epsilon_y + A_{33}\gamma_{xy} \end{aligned} \text{ or } \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} \tag{6}$$

where as usual, $(\alpha_{ij})(A_{kl}) = I$ (identity matrix).

If we call V Green's (hypothetically in the beginning) inserted function of elastic Potential [9] [13] [22], and we assume that the deformation of the absolutely elastic body takes place adiabatically or isothermally[9], then we know[10] that V indeed exists, and we will have (Castigliano):

$$\sigma_x = \frac{\partial V}{\partial \epsilon_x},\ \sigma_y = \frac{\partial V}{\partial \epsilon_y},\ \tau_{xy} = \frac{\partial V}{\partial \gamma_{yx}}$$

V is continuous and well-defined, which implies:

$$\frac{\partial \sigma_x}{\partial \epsilon_y} = \frac{\partial \sigma_y}{\partial \epsilon_x},\ \frac{\partial \sigma_x}{\partial \gamma_{xy}} = \frac{\partial \tau_{xy}}{\partial \epsilon_x},\ \frac{\partial \sigma_y}{\partial \gamma_{xy}} = \frac{\partial \tau_{xy}}{\partial \epsilon_y}$$

or because of (6): $A_{12} = A_{21}$, $A_{13} = A_{31}$ and $A_{23} = A_{32}$.

Easily from the above we conclude also that $\alpha_{12} = \alpha_{21}$, $\alpha_{13} = \alpha_{31}$ and $\alpha_{23} = \alpha_{32}$

5. Going back to the equations of balance (2) we find that the first relation $\frac{\partial \sigma_y}{\partial x} = -\frac{\partial \tau_{xy}}{\partial y}$ expresses the necessary and sufficient condition for the existence of a function $X(x, y)$ such that:

$$\frac{\partial X}{\partial y} = \sigma_x, \text{ and } \frac{\partial X}{\partial x} = -\tau_{xy} \tag{7}$$

Similarly, from the second equation of (2) we conclude that there exists a function $\Psi(x, y)$ such that:

[9] With generally different constants of elasticity.

[10] Sir W. Thomson. On the Thermoelastic Properties of Mater. Math. and phys. papers 1. 1882.

$$\frac{\partial \Psi}{\partial x} = \sigma_x, \text{ and } \frac{\partial \Psi}{\partial y} = -\tau_{xy}$$

From the second relations of the above we conclude $\frac{\partial X}{\partial x} = \frac{\partial \Psi}{\partial y}$, hence the following must hold:

$$X = \frac{\partial \Phi}{\partial y}, \text{ and } \Psi = \frac{\partial \Phi}{\partial x} \tag{8}$$

and finally because of (7) and the subsequent equations,

$$\sigma_x = \frac{\partial^2 \Phi}{\partial y^2}, \ \sigma_y = \frac{\partial^2 \Phi}{\partial x^2}, \ \tau_{xy} = -\frac{\partial^2 \Phi}{\partial x \partial y} \tag{9}$$

If we substitute equations (9) into (5) and then the resultants in the condition of compromise and we perform the necessary differentiations, we find, that the wanted function Φ will satisfy the following differential equation:

$$\begin{aligned} \alpha_{22}\frac{\partial^4 \Psi}{\partial x^4} - 2\alpha_{23}\frac{\partial^4 \Psi}{\partial x^3 \partial y} + (2\alpha_{12} + \alpha_{33})\frac{\partial^4 \Psi}{\partial x^2 \partial y^2} \\ -2\alpha_{13}\frac{\partial^4 \Psi}{\partial x \partial y^3} + \alpha_{11}\frac{\partial^4 \Psi}{\partial y^4} = 0 \end{aligned} \tag{10}$$

If we represent the fourth order differential operator with D:

$$\begin{aligned} \alpha_{22}\frac{\partial^4}{\partial x^4} - 2\alpha_{23}\frac{\partial^4}{\partial x^3 \partial y} + (2\alpha_{12} + \alpha_{33})\frac{\partial^4}{\partial x^2 \partial y^2} \\ -2\alpha_{13}\frac{\partial^4}{\partial x \partial y^3} + \alpha_{11}\frac{\partial^4}{\partial y^4} \end{aligned}$$

we can decompose the above into a product of four first order linear differential operators of the form:

$D_i = \frac{\partial}{\partial y} - \rho_i \frac{\partial}{\partial x}$, where ρ_i $(i = 1, 2, 3, 4)$ are the roots of the algebraic equation:

$$\alpha_{11}\rho^4 - 2\alpha_{13}\rho^3 + (2\alpha_{12} + \alpha_{33})\rho^2 - 2\alpha_{23}\rho + \alpha_{22} = 0 \tag{11}$$

We will prove that (11) has always imaginary roots[11].

The elastic potential is determined by the following relation:

$$2V = (\sigma_x \sigma_y \tau_{xy}) \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix}$$

where (α_{ij}) is the Matrix of coefficients of equations (5), hence:

$$2V = \alpha_{11}\sigma_x^2 + \alpha_{22}\sigma_y^2 + \alpha_{33}\tau_{xy}^2 + 2\alpha_{12}\sigma_x\sigma_y + 2\alpha_{13}\sigma_x\tau_{xy} + 2\alpha_{23}\sigma_y\tau_{xy} \tag{12}$$

[11] The proof of Green-Zerna [7] does not refer to the above roots ρ and is considerably more complex, whereas that of S.G. Lekhnitskii [12], based on the lemma of arbitrariness of stresses, lacks elegance.

Expression (12) is a homogeneous quadratic form of the variables σ_x, σ_y and τ_{xy}, which however, must be "positively defined" [1], given that $V > 0$ always.

Relation (12) by virtue of (9) can be written:

$$\begin{aligned} 2V =&\alpha_{11}\left(\frac{\partial^2\Phi}{\partial y^2}\right)^2 + \alpha_{22}\left(\frac{\partial^2\Phi}{\partial x^2}\right)^2 + \alpha_{33}\left(\frac{\partial^2\Phi}{\partial x\partial y}\right)^2 \\ &+2\alpha_{12}\frac{\partial^2\Phi\partial^2\Phi}{\partial x^2\partial y^2} - 2\alpha_{23}\frac{\partial^2\Phi\partial^2\Phi}{\partial x^2\partial x\partial y} - 2\alpha_{13}\frac{\partial^2\Phi\partial^2\Phi}{\partial y^2\partial x\partial y} \end{aligned} \tag{13}$$

If we search for the constants λ_i such that for $\frac{\partial\Phi}{\partial y} - \lambda_i\frac{\partial\Phi}{\partial x}$ we have $V = 0$, we find that the λ_i satisfy (11). Therefore since (13) has imaginary roots, the same must hold for (11).

The roots ρ_1, ρ_2, $\overline{\rho_1}$ and $\overline{\rho_2}$ are called complex parameters and depend on the constants of elasticity of the disk.

A.2 General Integral of (10). Form of stress functions - Boundary conditions

1. Let ρ_1, ρ_2, $\overline{\rho_1}$ and $\overline{\rho_2}$ be the roots of the algebraic equation

$$\alpha_{11}\rho^4 - 2\alpha_{13}\rho^3 + (2\alpha_{12} + \alpha_{33})\rho^2 - 2\alpha_{23}\rho + \alpha_{22} = 0 \tag{14}$$

We also assume that $\rho_1 \neq \rho_2$ (also $\rho_1 \neq \overline{\rho_2}$)[12].

The general solution of (10) therefore is [12] [10] the following:

$$\Phi = \Phi_1(z_1) + \Phi_2(z_2) + \Phi_3(\overline{z_1}) + \Phi_4(\overline{z_2})$$

where

$$\begin{aligned} z_1 &= x + \rho_1 y,\ \overline{z_1} = x + \overline{\rho_1} y \\ z_2 &= x + \rho_2 y,\ \overline{z_2} = x + \overline{\rho_2} y \end{aligned}$$

Because Φ must be a real function, it follows:

$$\Phi_3 = \overline{\Phi_1} \text{ and } \Phi_4 = \overline{\Phi_2}$$

Hence the used general solution can be written as:

$$\Phi = \Phi_1(z_1) + \Phi_2(z_2) + \overline{\Phi_1(z_1)} + \overline{\Phi_2(z_2)} = 2\Re\left\{\Phi_1(z_1) + \Phi_2(z_2)\right\}$$

hence:

$$\begin{aligned} \sigma_x &= \rho_1^2\Phi_1''(z_1) + \rho_2^2\Phi_2''(z_2) + \overline{\rho_1^2\Phi_1''(z_1)} + \overline{\rho_2^2\Phi_2''(z_2)} \\ \sigma_y &= \Phi_1''(z_1) + \Phi_2''(z_2) + \overline{\Phi_1''(z_1)} + \overline{\Phi_2''(z_2)} \\ \tau_{xy} &= -\rho_1\Phi_1''(z_1) - \rho_2\Phi_2''(z_2) - \overline{\rho_1\Phi_1''(z_1)} - \overline{\rho_2\Phi_2''(z_2)} \end{aligned} \tag{15}$$

If we substitute equations (15) into (5) and integrate, after performing all steps we will have:

$$\begin{aligned} u &= 2\Re\left\{\kappa_1\Phi_1'(z_1) + \kappa_2\Phi_2'(z_2)\right\} + u_0 \\ v &= 2\Re\left\{\lambda_1\Phi_1'(z_1) + \lambda_2\Phi_2'(z_2)\right\} + v_0, \text{ with} \\ \kappa_i &= \alpha_{11}\rho_i^2 - \alpha_{13}\rho_i + \alpha_{12} \\ \lambda_i &= \alpha_{12}\rho_i - \alpha_{23} + \frac{\alpha_{22}}{\rho_i},\ (i = 1, 2) \end{aligned} \tag{16}$$

According to the previous then, we see that the functions of stress and displacement are not defined on the initial plane z, but on a new, four-dimensional region W (bi-cylinder [3]), which has as projections the planes z_1 and z_2.

[12] The problem of the plane tensile situation in the case where $\rho_1 = \rho_2$ is solved by reducing it to an "iconic" isotropic case, based on geometrical transformations [24].

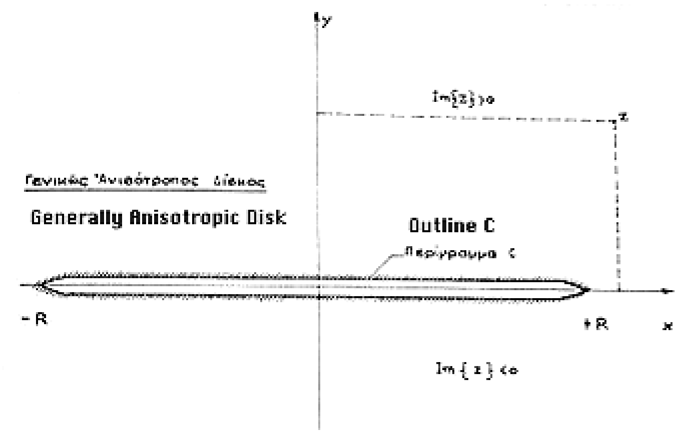

Figure 3: Generally Anisotropic Disk

The common domain of the functions, belongs both to W and to the planes z_1 and z_2, and is the intersection of the aforementioned planes.

The plane z_1, results from the homoparallel transformation L_1 such that $L_1(x+yi) = x + \rho_1 y$, or the same, $L_1(x+yi) = x + \rho_1 y = (x + \gamma_1 y) + i\delta_1 y$.

Similarly for z_2, $L_2(x+yi) = x + \rho_2 y = (x + \gamma_2 y) + i\delta_2 y$.

Consequently, the intersection of the planes z_1 and z_2 is the real axis.

The fact that the crack contour is found on W as well as on the intersection of the new complex planes, is the reason the fundamental problems 1 and 2 [17] [21], which are related to the aforementioned contour, admit a closed form solution.

2. In the given problem, the region which defines the elastic anisotropic disk is the infinite plane, excluding the points of a linear segment of length $2R$.

We choose the coordinate axes as in Fig. 3 and assume known the constants of elasticity of the plane with respect to the directions x and y, as well as those which depend on the ordered tuple xy.

We transform the plane z into the planes z_1 and z_2 according to the homoparallel transformations L_1 and L_2 such that $L_1(z) = z + \rho_1 y$ and $L_2(z) = z + \rho_2 y$.

Fig. 3 therefore transforms into Fig. 4 and Fig. 5 respectively[13].

The contour C, because of the special nature of its curve remains intact with respect to the transformations.

As far as the stress functions are concerned, defined on the bi-cylinder (which projects onto the planes z_1 and z_2), for any charge on the contour they must be holomorphic everywhere (including ∞) with the possible exception of a countable number (of points) which project against the part of the axis $|x| \leq R$ [17] [21] [7] [19].

Every holomorphic function of such properties is expressed as follows:

[13] If the complex parameters are pure imaginary then the points ρ_1 and ρ_2 will lie on the y axis, preserving thus the main directions also on the planes z_1 and z_2 (Orthotropy case).

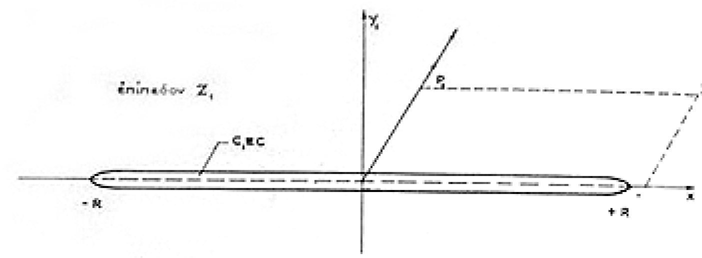

Figure 4: Plane z_1

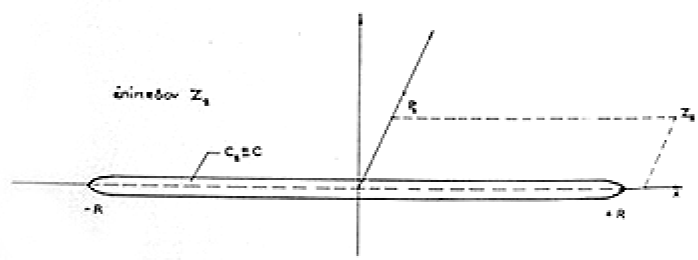

Figure 5: Plane z_2

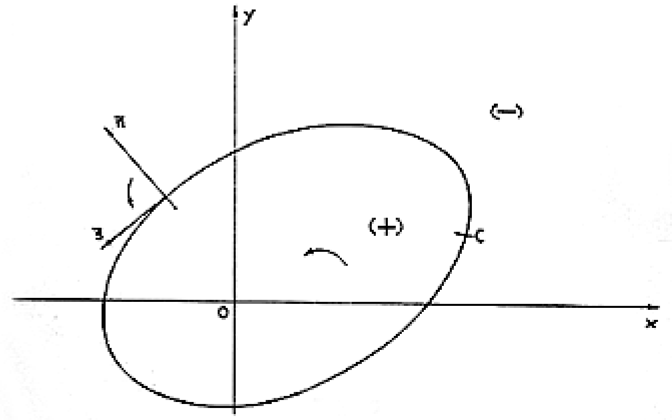

Figure 6: Contour C on plane z

$$F=\sum_{n=0}^{\infty}\alpha_n z_1^{-n}+\sum_{n=0}^{\infty}\beta_n z_2^{-n} \text{ (where } \alpha_n,\beta_n \text{ complex constants)}$$

Going back to equations (15) we conclude that

$$\Phi_1''(z_1)=\sum_{n=0}^{\infty}\alpha_n z_1^{-n} \text{ and } \Phi_2''(z_2)\sum_{n=0}^{\infty}\beta_n z_2^{-n}$$

and by integrating:

$$\begin{aligned}\Phi_1'&=\alpha_0 z_1+\alpha_1\ln(z_1)+\Phi_1^0\\ \Phi_2'&=\beta_0 z_2+\beta_1\ln(z_2)+\Phi_2^0\end{aligned} \tag{17}$$

where Φ_1^0 and Φ_2^0 are holomorphic functions everywhere on the planes z_1 and z_2 (including ∞) with the possible exception of a countable number of points on the boundary of C.

3. If we consider a contour C on the complex plane z (Fig. 6) and we define as usual the positive direction for its traversal to be that direction which leaves the interior to the left, we determine the vector $\vec{n}$ such that the system of n, s has the same orientation as that of x and y.

If we accept that there is a charge on the contour (in this case on the crack), the stress must result as a limiting value of the functions which describe the stresses (9).

The components X_n, Y_n of that stress on the contour however are given by (1) as follows:

$$\begin{aligned}X_n&=\sigma_x\cos(n,x)+\tau_{xy}\cos(n,y)\\ Y_n&=\tau_{xy}\cos(n,x)+\sigma_y\cos(n,y)\end{aligned}$$

Because we have[14]

$$\cos(n,x) = \frac{dy}{ds} \text{ and } \cos(n,y) = -\frac{dx}{ds}$$

we find:

$$X_n = \frac{\partial^2\Phi}{\partial y^2}\frac{dy}{ds} + \frac{\partial^2\Phi}{\partial x\partial y}\frac{dx}{ds} = \frac{d}{ds}\left(\frac{\partial\Phi}{\partial y}\right)$$
$$Y_n = -\frac{\partial^2\Phi}{\partial x\partial y}\frac{dy}{ds} - \frac{\partial^2\Phi}{\partial x^2}\frac{dx}{ds} = \frac{d}{ds}\left(\frac{\partial\Phi}{\partial x}\right)$$

and finally

$$\frac{\partial\Phi}{\partial y} = \int_O^s X_n ds \text{ and } \frac{\partial\Phi}{\partial x} = -\int_O^s Y_n ds \tag{18}$$

with the point O chosen arbitrarily along the contour C.

Relations (18), which are the equations which describe the boundary conditions in the case of the crack, can also be written as follows:

$$\begin{aligned} \rho_1\Phi_1'(z) + \rho_2\Phi_2'(z) + \overline{\rho_1\Phi_1'(z)} + \overline{\rho_2\Phi_2'(z)} &= \int_O^s X_n ds \\ \Phi_1'(z) + \Phi_2'(z) + \overline{\Phi_1'(z)} + \overline{\Phi_2'(z)} &= -\int_O^s Y_n ds \end{aligned} \tag{19}$$

where z is a real variable such that $0 \le |z| \le R$ and $ds = -dz$ because of $\cos(n,x) = 0$ and $\cos(n,y) = 1$.

The components X_n, Y_n are found as follows for the three basic cases of charge:

a) For continuous, uniformly distributed, normal charge on a section of the contour (Fig. 7a). $X_n = 0$, $Y_n = -p$ for z between $z'z''$.

b) For continuous, uniformly distributed, tangential charge q on a section of the contour (Fig. 7b). $X_n = q$, $Y_n = 0$ for z between $z'z''$.

c) For given stresses σ_x^∞, σ_y^∞ and τ_{xy}^∞ where $X_n = Y_n = 0$.

4. Examining the structure of the functions $\Phi_1'(z_1)$, $\Phi_2'(z_2)$ in equations (17) we observe:

a) The stresses resulting from substitution into equations (15) are exhausted at infinity.

b) If the stresses at infinity are non-zero, then at least one of the complex coefficients α_0, β_0 must be non-zero.

Otherwise ($\sigma_x^\infty = \sigma_y^\infty = \tau_{xy}^\infty = 0$), we must have $\alpha_0 = \beta_0 = 0$.

c) Based on the boundary relations (19) and for a full traversal of the contour, the integrals $\int X_n ds$ and $\int Y_n ds$ are incremented (for every full traversal) by $-X_0$ and $+Y_0$, where X_0, Y_0 are the components of the total stress charge in the direction of the axes x and y. Consequently, these functions are multi-valued, for the case of a non-self-balanced system with external forces.

[14]Translator's Note: See footnote 5 on page 11.

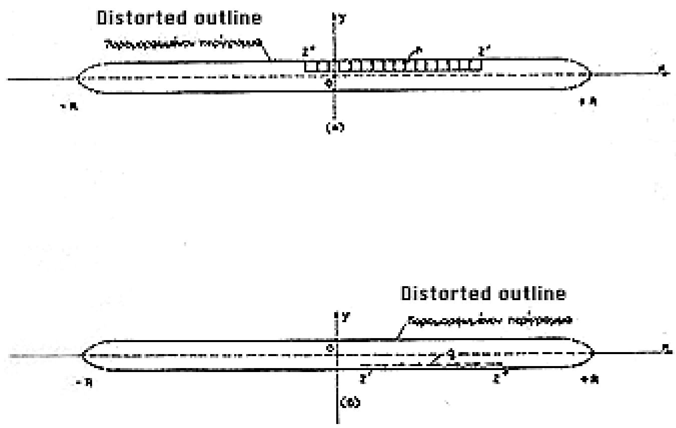

Figure 7: Cases a) and b)

Therefore, in order for equations (19) to hold, the corresponding left-hand-side functions must also be multi-valued functions.

Thus for $X_0+Y_0 i \neq 0$, we conclude: $\alpha_1, \beta_1 \neq 0$ whereas for $X_0+Y_0 i = 0$, $\alpha_1 = \beta_1 = 0$.

A.3 The Muskhelishvili Transformation

1. The search for a function $z = \omega(\zeta)$, which maps the complex plane z onto the plane ζ and such that the contour C of a normal closed curve is mapped onto the perimeter of the unit circle $|\zeta| = 1$, despite the known Theorem of Existence by Riemann [3] [2], is obstructed in most cases by such difficulties which allow $\omega(\zeta)$ to be given in closed form only in very few cases.

The above problem which we face during the solution of problems of the Mathematical Theory of Elasticity of an Isotropic Body is multiply compounded in the case of a generally anisotropic body.

In fact, in the second case one looks for three functions $\omega(\zeta)$, $\omega_1(\zeta_1)$ and $\omega_2(\zeta_2)$, which map the outlines C (on the z plane) and the transformed C_1 (on the z_1 plane) and C_2 (on the z_2 plane), onto the unit circles $|\zeta| = 1$, $|\zeta_1| = 1$ and $|\zeta_2| = 1$, on the planes ζ, ζ_1 and ζ_2 respectively.

In order to finally have closed-form expressions for the functions of stress - displacement, not only the functions $\omega(\zeta)$, $\omega_1(\zeta_1)$ should be rational[15][17], but also for all $\zeta = \zeta_1 = \zeta_2 = e^{i\theta}$ the found $z_1 = \omega_1(\zeta)$ and $z_2 = \omega_2(\zeta)$ must be "projections" of one and the same z.

Expressing the above, we have:

$$z_1 = \frac{z + \overline{z}}{2} + \rho_1 \frac{z - \overline{z}}{2i}, \; z_2 = \frac{z + \overline{z}}{2} + \rho_2 \frac{z - \overline{z}}{2i}$$

and by eliminating z we find the relation

$$(\rho_1 - \rho_2)\omega(\zeta_0) = (\rho_2 - i)\omega_1(\zeta_0) - (\rho_1 - i)\omega_2(\zeta_0), \text{ where } \zeta_0 = e^{i\theta} \tag{20}$$

which must hold (along with ω_1, ω_2, ω being rational) so that we can have a solution by functions in closed form.

2. Following the previous, we enter the investigation of finding a mapping of the planes z, z_1 and z_2 onto ζ, ζ_1 and ζ_2 such that the outlines C, C_1 and C_2 are mapped respectively onto the unit circles $|\zeta| = 1$, $|\zeta_1| = 1$ and $|\zeta_2| = 1$.

The mapping function has the same form in all three cases because of the invariance of the initial contour C under the homoparallel transformations $z \to z_1$ and $z \to z_2$ (the relation $\omega \equiv \omega_1 \equiv \omega_2$ is also sufficient for the validity of (20)), and there is:

$$\begin{aligned} z &= \frac{R}{2}\left(\zeta + \frac{1}{\zeta}\right), \text{ consequently} \\ z_1 &= \frac{R}{2}\left(\zeta_1 + \frac{1}{\zeta_1}\right) \\ z_2 &= \frac{R}{2}\left(\zeta_2 + \frac{1}{\zeta_2}\right), \text{ with} \\ &|\zeta| \geq 1, |\zeta_1| \geq 1, |\zeta_2| \geq 1 \end{aligned} \tag{21}$$

The coincidence of the points z, z_1 and z_2 on the boundary C implies the same for ζ, ζ_1 and ζ_2 on the unit circles (and conversely), a conclusion which allows us to refer to one unit circle(c) the one found on the "intersection" of the planes ζ_1 and ζ_2 (Fig. 8).

[15] Muskhelishvili Theorem for the case of an isotropic elastic body.

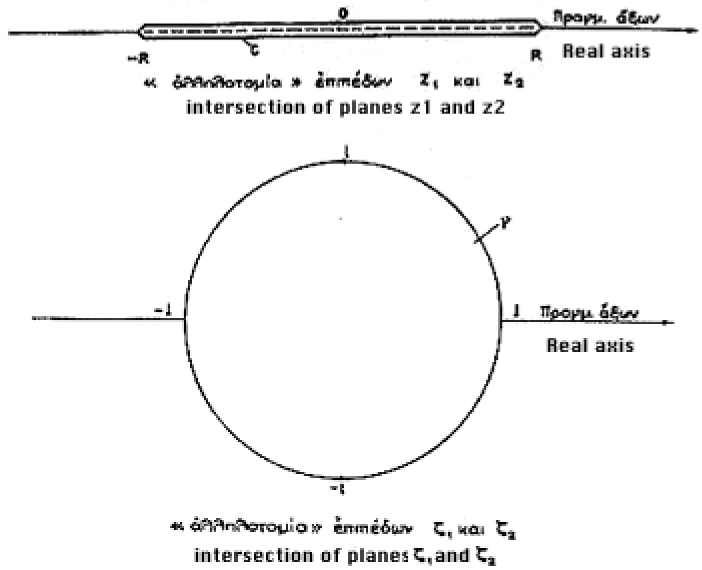

Figure 8: Intersection of planes z_1, z_2 and ζ_1, ζ_2

In order for transformation (21) to be invertible it must have a non-vanishing first derivative therefore it must have $\zeta_i^2 - 1 \neq 0$ or $\zeta_i \neq \pm 1$, a result which was expected because of the existence of a cusp[16] in the above positions. The transformations (21) substituted into relations (17) give:

$$
\begin{aligned}
\Phi_1'(z_1) = f_1(z_1) =& \alpha_0 \frac{R}{2}\left(\zeta_1 + \frac{1}{\zeta_1}\right) + \alpha_1 \ln\left[\frac{R}{2}\left(\zeta_1 + \frac{1}{\zeta_1}\right)\right] + \Phi_1^0\left[\frac{R}{2}\left(\zeta_1 + \frac{1}{\zeta_1}\right)\right] \\
=& \alpha_0 \frac{R}{2}\zeta_1 + \alpha_1 \ln(\zeta_1) + f_1^0(\zeta_1) \\
\Phi_2'(z_2) = f_2(z_2) =& \beta_0 \frac{R}{2}\left(\zeta_2 + \frac{1}{\zeta_2}\right) + \beta_1 \ln\left[\frac{R}{2}\left(\zeta_2 + \frac{1}{\zeta_2}\right)\right] + \Phi_2^0\left[\frac{R}{2}\left(\zeta_2 + \frac{1}{\zeta_2}\right)\right] \\
=& \beta_0 \frac{R}{2}\zeta_2 + \beta_1 \ln(\zeta_2) + f_2^0(\zeta_2)
\end{aligned}
\tag{22}
$$

where $f_1^0(\zeta_1)$ and $f_2^0(\zeta_2)$ are holomorphic functions on the planes ζ_1 and ζ_2 for $|\zeta_1| > 1$ and $|\zeta_2| > 1$ respectively.

One can conclude easily that $f_1^0(\zeta_1)$ (and $f_2^0(\zeta_2)$ respectively) can be expressed via the following relation.

[16] Translator's Note: "Cusp" in modern terminology would be the equivalent of a Branch Point.

$$f_1^0(\zeta_1) = \Phi_1^0\left[\frac{R}{2}\left(\zeta_1 + \frac{1}{\zeta_1}\right)\right] + \alpha_0 \frac{R}{2}\frac{1}{\zeta_1} + \alpha_1 \ln(R)$$
$$+ \alpha_1 \left\{\ln\left(\zeta_1 + \frac{1}{\zeta_1}\right) - \ln(\zeta_1)\right\}, \text{ for } |\zeta_1| > 1$$

as a sum of holomorphic functions therefore it ($f_1^0(\zeta_1)$) is also holomorphic on the entire plane ζ_1 (including ∞) with the exception of the points inside the unit circle.

B Solution of the "First Fundamental Problem" For the Three Basic Charge Cases

B.1 Uncharged boundary - Given stresses very far away from the boundary

[17]

1. All the previous take the following form:
$X_n = Y_n = 0$. All σ_x^∞, σ_y^∞, τ_{xy}^∞ are known.

Therefore according to section A.2.3 for $\alpha_1 = \beta_1 = 0$ equations (22) take the following form:

$$
\begin{aligned}
\Phi_1'(z_1) = f_1(\zeta_1) = \alpha_0 \frac{R}{2}\left(\zeta_1 + \frac{1}{\zeta_1}\right) + \Psi_1^0(\zeta_1) \\
\Phi_2'(z_2) = f_2(\zeta_2) = \beta_0 \frac{R}{2}\left(\zeta_2 + \frac{1}{\zeta_2}\right) + \Psi_1^0(\zeta_2)
\end{aligned} \tag{23}
$$

where $\Psi_1^0(\zeta_1) = \Phi_1^0\left[\frac{R}{2}\left(\zeta_1 + \frac{1}{\zeta_1}\right)\right]$ and $\Psi_2^0(\zeta_2) = \Phi_2^0\left[\frac{R}{2}\left(\zeta_2 + \frac{1}{\zeta_2}\right)\right]$ are holomorphic functions on the entire planes ζ_1 and ζ_2 respectively, for $|\zeta_1| \geq 1$, $|\zeta_2| \geq 1$.

The boundary conditions (19) on the unit circle, given that for $\sigma = e^{i\theta}$ the relation $\overline{\sigma} = \frac{1}{\sigma}$ holds, are written:

$$
\begin{aligned}
&\alpha_0 \frac{R}{2}\left(\sigma + \frac{1}{\sigma}\right) + \Psi_1^0(\sigma) + \beta_0 \frac{R}{2}\left(\sigma + \frac{1}{\sigma}\right) + \Psi_2^0(\sigma) \\
&+\overline{\alpha_0} \frac{R}{2}\left(\frac{1}{\sigma} + \sigma\right) + \overline{\Psi_1^0(\sigma)} + \overline{\beta_0} \frac{R}{2}\left(\frac{1}{\sigma} + \sigma\right) + \overline{\Psi_2^0(\sigma)} = 0 \\
&\rho_1 \alpha_0 \frac{R}{2}\left(\sigma + \frac{1}{\sigma}\right) + \rho_1 \Psi_1^0(\sigma) + \rho_2 \beta_0 \frac{R}{2}\left(\sigma + \frac{1}{\sigma}\right) + \rho_2 \Psi_2^0(\sigma) \\
&+\overline{\rho_1 \alpha_0} \frac{R}{2}\left(\frac{1}{\sigma} + \sigma\right) + \overline{\rho_1 \Psi_1^0(\sigma)} + \overline{\rho_2 \beta_0} \frac{R}{2}\left(\frac{1}{\sigma} + \sigma\right) + \overline{\rho_2 \Psi_2^0(\sigma)} = 0
\end{aligned}
$$

or after performing the calculations:

$$
\begin{aligned}
&\Psi_1^0(\sigma) + \Psi_2^0(\sigma) + \overline{\Psi_1^0(\sigma)} + \overline{\Psi_2^0(\sigma)} = \\
&-\frac{R}{2}\left(\sigma + \frac{1}{\sigma}\right)\left[\alpha_0 + \beta_0 + \overline{\alpha_0} + \overline{\beta_0}\right] \\
&\rho_1 \Psi_1^0(\sigma) + \rho_2 \Psi_2^0(\sigma) + \overline{\rho_1 \Psi_1^0(\sigma)} + \overline{\rho_2 \Psi_2^0(\sigma)} = \\
&-\frac{R}{2}\left(\sigma + \frac{1}{\sigma}\right)\left[\rho_1 \alpha_0 + \rho_2 \beta_0 + \overline{\rho_1 \alpha_0} + \overline{\rho_2 \beta_0}\right]
\end{aligned} \tag{24}
$$

[17] The solution for the first charge case is drafted roughly in general terms and using a different method in [7] page 366. See also [13], where the Russian Method is presented, but the solution of problems with sectional charge of the contour is avoided. The solution 1 in the current work concerns on one hand the unity of charge cases, on the other the extraction of conclusions in the case of Gen. Anisotropy.

We now multiply both sides of equations (24) by $\frac{1}{2\pi i}\frac{d\sigma}{\sigma-\zeta}$ and we integrate on the unit circle.

If we consider[18] that:

$$\frac{1}{2\pi i}\int_\gamma \frac{\sigma d\sigma}{\sigma-\zeta} = 0 \text{ and } \frac{1}{2\pi i}\int_\gamma \frac{d\sigma}{\sigma(\sigma-\zeta)} = -\frac{1}{\zeta}$$

and

$$\frac{1}{2\pi i}\int_\gamma \frac{\Psi_1^0(\sigma)}{\sigma-\zeta}d\sigma = -\Psi_1^0(\zeta) \text{ and } \frac{1}{2\pi i}\int_\gamma \frac{\Psi_2^0(\sigma)}{\sigma-\zeta}d\sigma = -\Psi_2^0(\zeta)$$

and also that:

$$\frac{1}{2\pi i}\int_\gamma \frac{\overline{\Psi_1^0(\sigma)}}{\sigma-\zeta}d\sigma = \frac{1}{2\pi i}\int_\gamma \frac{\overline{\Psi_2^0(\sigma)}}{\sigma-\zeta}d\sigma = 0$$

We finally find

$$\begin{aligned} \Psi_1^0(\zeta) + \Psi_2^0(\zeta) &= -\frac{R}{2}\left(\alpha_0 + \beta_0 + \overline{\alpha_0} + \overline{\beta_0}\right)\frac{1}{\zeta} \\ \rho_1\Psi_1^0(\zeta) + \rho_2\Psi_2^0(\zeta) &= -\frac{R}{2}\left(\rho_1\alpha_0 + \rho_2\beta_0 + \overline{\rho_1\alpha_0} + \overline{\rho_2\beta_0}\right)\frac{1}{\zeta} \end{aligned} \tag{25}$$

We observe that in equations (23) that for $z \to \infty$ (in which case z_1, z_2, ζ_1 and ζ_2 also go to infinity) $\Phi_1'(z_1)$ can be expressed as:

$$\alpha_0\left[\frac{R}{2}\left(\zeta_1 + \frac{1}{\zeta_1}\right)\right]$$

or the same:

$$\Phi_1'(z_1) = O(\alpha_0 z_1)$$

The same holds for $\Phi_2'(z_2)$ which is expressed by $\beta_0 z_2$ or the same, $\Phi_2'(z_2) = O(\beta_0 z_2)$[19]

Therefore differentiating with respect to z_1 and to z_2 respectively, we have:

$$\lim_{z_1\to\infty}\Phi_1''(z_1) = \alpha_0 \text{ and } \lim_{z_1\to\infty}\Phi_2''(z_2) = \beta_0 \tag{26}$$

We substitute the above results into (15) and we get: (given that we know the stress at infinity)

$$\begin{aligned} \alpha_0 + \beta_0 + \overline{\alpha_0} + \overline{\beta_0} &= \sigma_y^\infty \\ \rho_1\alpha_0 + \rho_2\beta_0 + \overline{\rho_1\alpha_0} + \overline{\rho_2\beta_0} &= -\tau_{xy}^\infty \\ \rho_1^2\alpha_0 + \rho_2^2\beta_0 + \overline{\rho_1^2\alpha_0} + \overline{\rho_2^2\beta_0} &= \sigma_x^\infty \end{aligned} \tag{27}$$

[18]See Mathemat. Appendix in the following order: 1.8.a2, 1.8.b1, 2.2.b, 1.8.b1, 1.8.b1. It is also known that $\Psi_1^0(\infty) = \Psi_2^0(\overline{\infty}) = 0$.

[19]The functions $\Psi_1^0(\zeta_1)$ and $\Psi_1^0(\zeta_1)$, being holomorphic for $|\zeta_1| > 1$, $|\zeta_2| > 1$, can be expanded as usual into series of the form $A_0 + \frac{A_1}{\zeta_i} + \frac{A_2}{\zeta_i^2} + \cdots$, hence for $\zeta_i \to \infty$, we have $\lim_{\zeta_i\to\infty}\Psi_i^0(\zeta_i) = A_0$.

Using the first two equations from the above in system (25) we have

$$\Psi_1^0(\zeta) + \Psi_2^0(\zeta) = -R\frac{\sigma_y^\infty}{2}\frac{1}{\zeta}$$
$$\rho_1\Psi_1^0(\zeta) + \rho_2\Psi_2^0(\zeta) = R\frac{\tau_{xy}^\infty}{2}\frac{1}{\zeta}$$

and finally (considering the corresponding functions' domains of definition)

$$\begin{aligned}\Psi_1^0(\zeta_1) &= \frac{R}{2(\rho_1-\rho_2)}\left[\rho_2\sigma_y^\infty + \tau_{xy}^\infty\right]\frac{1}{\zeta_1}\\ \Psi_2^0(\zeta_2) &= \frac{-R}{2(\rho_1-\rho_2)}\left[\rho_1\sigma_y^\infty + \tau_{xy}^\infty\right]\frac{1}{\zeta_2}\end{aligned} \tag{28}$$

The finding of functions $\Psi_1^0(\zeta_1)$ and $\Psi_2^0(\zeta_2)$ solves the problem of determining the stress on the disk, without needing to investigate system (27) further, the solution of which would be necessary for the determination of the constants α_0, β_0, $\overline{\alpha_0}$ and $\overline{\beta_0}$.

Indeed if we use relations (23) which if we differentiate with respect to z_1 and z_2 respectively, we substitute in relations (15), we find:

$$\begin{aligned}\sigma_y &= \sigma_y^\infty + 2\Re\left\{\frac{-1}{\rho_1-\rho_2}\left(\rho_2\sigma_y^\infty + \tau_{xy}^\infty\right)\frac{1}{\zeta_1^2-1} + \frac{1}{\rho_1-\rho_2}\left(\rho_1\sigma_y^\infty + \tau_{xy}^\infty\right)\frac{1}{\zeta_2^2-1}\right\}\\ \sigma_x &= \sigma_x^\infty + 2\Re\left\{\frac{-\rho_1^2}{\rho_1-\rho_2}\left(\rho_2\sigma_y^\infty + \tau_{xy}^\infty\right)\frac{1}{\zeta_1^2-1} + \frac{\rho_2^2}{\rho_1-\rho_2}\left(\rho_1\sigma_y^\infty + \tau_{xy}^\infty\right)\frac{1}{\zeta_2^2-1}\right\}\\ \tau_{xy} &= \tau_{xy}^\infty + 2\Re\left\{\frac{\rho_1}{\rho_1-\rho_2}\left(\rho_2\sigma_y^\infty + \tau_{xy}^\infty\right)\frac{1}{\zeta_1^2-1} + \frac{-\rho_2}{\rho_1-\rho_2}\left(\rho_1\sigma_y^\infty + \tau_{xy}^\infty\right)\frac{1}{\zeta_2^2-1}\right\}\end{aligned} \tag{29}$$

Equations (29) give the solution to the problem.

2. Concluding, referring to the linear system (27), it wouldn't be pointless to add the following:

It is obvious that the equations are insufficient for the determination of the constants α_0 and β_0.

It is true that the structure of the functions of stress requires only the sums $\alpha_0+\beta_0+\overline{\alpha_0}+\overline{\beta_0}$, $\rho_1\alpha_0+\rho_2\beta_0+\overline{\rho_1\alpha_0}+\overline{\rho_2\beta_0}$, $\rho_1^2\alpha_0+\rho_2^2\beta_0+\overline{\rho_1^2\alpha_0}+\overline{\rho_2^2\beta_0}$, which are determined (27), but this is not true for the calculation of displacements (16), where the separate determination of the constants α_0 and β_0 is required.

In this case, we use the condition of vanishing of torque at infinity [7] [12] [17], which is expressed by the relation:

$$\lim e_\infty = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0$$

and because of (16)

$$\lim_{z\to\infty} 2\Re\left\{\lambda_1\Phi_1''(z_1) + \lambda_2\Phi_2''(z_2) - \rho_1\kappa_1\Phi_1''(z_1) - \rho_2\kappa_2\Phi_2''(z_2)\right\} = 0$$

because $\lim_{z_1 \to \infty} \Phi_1''(z_1) = \alpha_0$ and $\lim_{z_2 \to \infty} \Phi_2''(z_2) = \beta_0$, the above equation is written:

$$(\lambda_1 - \rho_1\kappa_1)\alpha_0 + (\lambda_2 - \rho_2\kappa_2)\beta_0 + (\overline{\lambda_1} - \overline{\rho_1\kappa_1})\overline{\alpha_0} + (\overline{\lambda_2} - \overline{\rho_2\kappa_2})\overline{\beta_0} = 0 \qquad (30)$$

Equation (30) is the missing fourth equation of the system (27).

We prove below that the determinant of the coefficients of the unknowns is (for $\rho_1 \neq \rho_2$) always non-zero.

If we examine the coefficient of α_0 in (30), it can be written because of (16),:

$$(\rho_1\kappa_1 - \lambda_1) = \alpha_{11}\rho_1^3 - \alpha_{13}\rho_1^2 + \alpha_{23} - \frac{\alpha_{22}}{\rho_1}$$

Going back to (11) we find the following relation:

$$-\frac{\alpha_{22}}{\rho_1} = \alpha_{11}\rho_1^3 - 2\alpha_{13}\rho_1^2 + (2\alpha_{12} + \alpha_{33})\rho_1 - 2\alpha_{23}$$

which we substitute in its previous. We therefore get:

$$(\rho_1\kappa_1 - \lambda_1) = 2\alpha_{11}\rho_1^3 - 3\alpha_{13}\rho_1^2 + (2\alpha_{12} + \alpha_{33})\rho_1 - \alpha_{23}$$

If we substitute this equation (as well as in those which result by exchanging ρ_1 with ρ_2) into (30), using also the first three equations of system (27), we transform (30) as follows:

$$2\alpha_{11}\rho_1^3\alpha_0 + 2\alpha_{11}\rho_2^3\beta_0 + 2\alpha_{11}\overline{\rho_1^3\alpha_0} + 2\alpha_{11}\overline{\rho_2^3\beta_0} = 3\alpha_{13}\sigma_x^\infty + (2\alpha_{12} + \alpha_{33})\tau_{xy}^\infty + \alpha_{23}\sigma_y^\infty$$

or after simplifying

$$\rho_1^3\alpha_0 + \rho_2^3\beta_0 + \overline{\rho_1^3\alpha_0} + \overline{\rho_2^3\beta_0} = \frac{3\alpha_{13}}{2\alpha_{11}}\sigma_x^\infty + \frac{2\alpha_{12} + \alpha_{33}}{2\alpha_{11}}\tau_{xy}^\infty + \frac{\alpha_{23}}{2\alpha_{11}}\sigma_y^\infty \qquad (31)$$

The determinant of the coefficients of the unknowns of system (27) and of (31) can be written as follows:

$$D = \begin{vmatrix} 1 & 1 & 1 & 1 \\ \rho_1 & \rho_2 & \overline{\rho_1} & \overline{\rho_2} \\ \rho_1^2 & \rho_2^2 & \overline{\rho_1^2} & \overline{\rho_2^2} \\ \rho_1^3 & \rho_2^3 & \overline{\rho_1^3} & \overline{\rho_2^3} \end{vmatrix}$$

it is therefore non-zero, being of type Vandermonde, for $\rho_1 \neq \rho_2$.

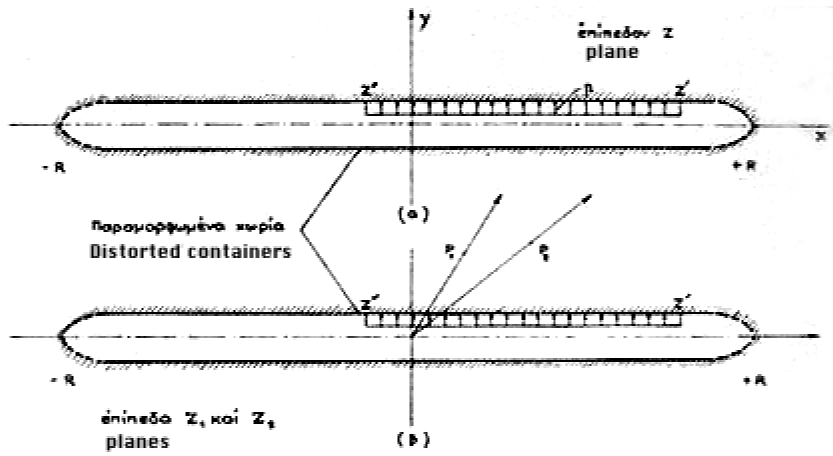

Figure 9: Charge on plane z and on planes z_1 and z_2

B.2 Continuous, uniformly distributed normal charge over section of the boundary

1. The image of the charge on the plane z, as well as that on the planes z_1 and z_2 are shown on Figs. (9a) and (9b) respectively.

The conformal mapping of Fig. 9b) using the transformations $z_i = \frac{R}{2}\left(\zeta_i + \frac{1}{\zeta_i}\right)$, $(i = 1, 2)$ gives the image of Fig. 10.

Additionally we have:

$$Y_n = -p,\ X_n = 0,\ ds = -dz$$

According to the above and after we set $\alpha_0 = \beta_0 = 0$ (vanishing stress at infinity)[20], the boundary conditions (19) transferred onto the unit circle take the following form:

$$\begin{aligned}
&\alpha_1 \ln(\sigma) + f_1^0(\sigma) + \beta_1 \ln(\sigma) + f_2^0(\sigma) - \overline{\alpha_1} \ln(\sigma) + \overline{f_1^0(\sigma)} - \overline{\beta_1} \ln(\sigma) + \overline{f_2^0(\sigma)} \\
&= -p \int_0^s dz \\
&\rho_1 \alpha_1 \ln(\sigma) + \rho_1 f_1^0(\sigma) + \rho_2 \beta_1 \ln(\sigma) + \rho_2 f_2^0(\sigma) \\
&+ \overline{\rho_1 \alpha_1} \ln(\sigma) + \overline{\rho_1 f_1^0(\sigma)} - \overline{\rho_2 \beta_1} \ln(\sigma) + \overline{\rho_2 f_2^0(\sigma)} = 0
\end{aligned}$$

or after simplifying:

$$\begin{aligned}
f_1^0(\sigma) + f_2^0(\sigma) + \overline{f_1^0(\sigma)} + \overline{f_2^0(\sigma)} &= -p \int_0^s dz - (\alpha_1 + \alpha_2 - \overline{\alpha_1} - \overline{\alpha_2}) \ln(\sigma) \\
\rho_1 f_1^0(\sigma) + \rho_2 f_2^0(\sigma) + \overline{\rho_1 f_1^0(\sigma)} + \overline{\rho_2 f_2^0(\sigma)} &= -\left(\rho_1 \alpha_1 + \rho_2 \beta_1 - \overline{\rho_1 \alpha_1} - \overline{\rho_2 \beta_1}\right) \ln(\sigma)
\end{aligned} \tag{32}$$

[20]See investigation of the function forms of Complex potential in A.2.4

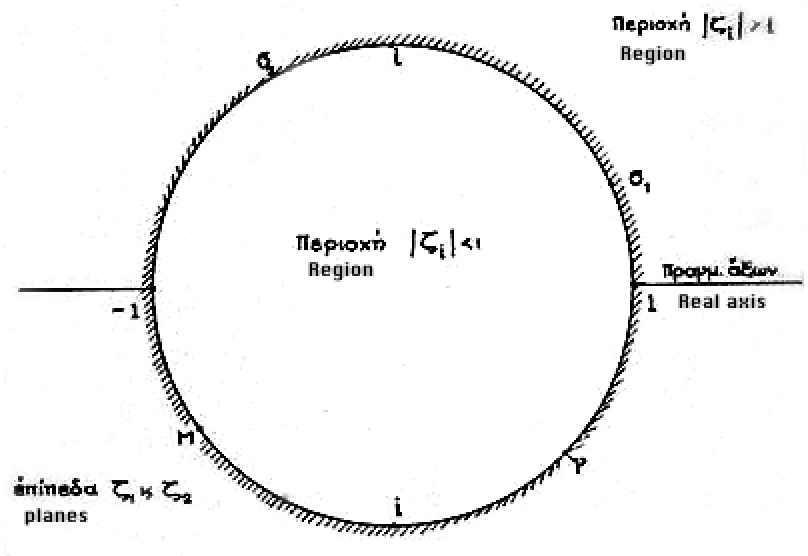

Figure 10: Fig. 9b) under the transformations $z_i = \frac{R}{2}\left(\zeta_i + \frac{1}{\zeta_i}\right)$, $(i = 1, 2)$

2. Before integrating equations (32) it is required on one hand the investigation of the integral $p\int_0^s dz$ and a search for necessary conditions for the determination of constants α_1 and β_1 on the other.

Because stress p (as an external charge) acts only on section $z'z''$ we assume that on this section of the contour $p\int dz$ takes the form $-pz$, whereas on section $z''Mz'$ (Fig. 9) it is constant and equal to $-pz''$. [17]

As far as the characterization of the constants α_1 and α_2 we think as follows:

Integrals $\int Y_n ds$ and $\int X_n ds$ for every traversal of the contour acquire increases $-X_0$ and Y_0 respectively, where X_0 and Y_0 are the components with respect to the axes x and y of the total applied charge.

They are therefore multi-valued functions.

Because the function $\ln(\sigma)$ is also multi-valued (being the inverse of a periodic function with period $2\pi i$) the following equations must hold, in order to ensure the single-valuedness of f_1^0 and f_2^0:

$$\begin{aligned} X_0 &= -\left(\rho_1\alpha_1 + \rho_2\beta_1 - \overline{\rho_1\alpha_1} - \overline{\rho_2\beta_1}\right)2\pi i \\ Y_0 &= \left(\alpha_1 + \beta_1 - \overline{\alpha_1} - \overline{\beta_1}\right)2\pi i \end{aligned} \tag{33a}$$

Thinking similarly for the deformation functions (16), we must have, because u and v are single-valued functions, the following relations:

$$\begin{aligned} \kappa_1\alpha_1 + \kappa_2\beta_1 - \overline{\kappa_1\alpha_1} - \overline{\kappa_2\beta_1} &= 0 \\ \lambda_1\alpha_1 + \lambda_2\beta_1 - \overline{\lambda_1\alpha_1} - \overline{\lambda_2\beta_1} &= 0 \end{aligned} \tag{33b}$$

The first of (33b) can be written:

$$\begin{aligned} 0 = &\alpha_1\left(\alpha_{11}\rho_1^2-\alpha_{13}\rho_1+\alpha_{12}\right)+\beta_1\left(\alpha_{11}\rho_2^2-\alpha_{13}\rho_2+\alpha_{12}\right)\\ &-\overline{\alpha_1}\left(\alpha_{11}\overline{\rho_1^2}-\alpha_{13}\overline{\rho_1}+\alpha_{12}\right)-\overline{\beta_1}\left(\alpha_{11}\overline{\rho_2^2}-\alpha_{13}\overline{\rho_2}+\alpha_{12}\right)\end{aligned}$$

or:

$$\begin{aligned} &\alpha_{11}\left(\rho_1^2\alpha_1+\rho_2^2\beta_1-\overline{\rho_1^2\alpha_1}-\overline{\rho_2^2\beta_1}\right)-\alpha_{13}\left(\rho_1\alpha_1+\rho_2\beta_1-\overline{\rho_1\alpha_1}-\overline{\rho_2\beta_1}\right)\\ &+\alpha_{12}\left(\alpha_1+\beta_1-\overline{\alpha_1}-\overline{\beta_1}\right)=0\end{aligned}$$

and because of (33a)

$$\rho_1^2\alpha_1+\rho_2^2\beta_1-\overline{\rho_1^2\alpha_1}-\overline{\rho_2^2\beta_1}=-\frac{\alpha_{13}X_0+\alpha_{12}Y_0}{\alpha_{11}2\pi i} \tag{33c}$$

The second of (33b) can be written:

$$\begin{aligned} &\alpha_1\left(\alpha_{12}\rho_1-\alpha_{23}+\frac{\alpha_{22}}{\rho_1}\right)+\beta_1\left(\alpha_{12}\rho_2-\alpha_{23}+\frac{\alpha_{22}}{\rho_2}\right)\\ &+\overline{\alpha_1}\left(\alpha_{12}\overline{\rho_1}-\alpha_{23}+\frac{\alpha_{22}}{\overline{\rho_1}}\right)+\overline{\beta_1}\left(\alpha_{12}\overline{\rho_2}-\alpha_{23}+\frac{\alpha_{22}}{\overline{\rho_2}}\right)=0\end{aligned}$$

and using similar to the previous case transformations, it takes the following form.

$$\frac{1}{\rho_1}\alpha_1+\frac{1}{\rho_2}\beta_1-\frac{1}{\overline{\rho_1}}\overline{\alpha_1}-\frac{1}{\overline{\rho_2}}\overline{\beta_1}=\frac{\alpha_{23}Y_0+\alpha_{12}X_0}{a_{22}2\pi i}$$

Concluding, we write the entire linear system which can be solved for the constants α_1 and β_1

$$\begin{aligned} \alpha_1+\beta_1-\overline{\alpha_1}-\overline{\beta_1}&=\frac{Y_0}{2\pi i}\\ \rho_1\alpha_1+\rho_2\beta_1-\overline{\rho_1\alpha_1}-\overline{\rho_2\beta_1}&=-\frac{X_0}{2\pi i}\\ \rho_1^2\alpha_1+\rho_2^2\beta_1-\overline{\rho_1^2\alpha_1}-\overline{\rho_2^2\beta_1}&=-\frac{\alpha_{13}X_0+\alpha_{12}Y_0}{\alpha_{11}2\pi i}\\ \frac{1}{\rho_1}\alpha_1+\frac{1}{\rho_2}\beta_1-\frac{1}{\overline{\rho_1}}\overline{\alpha_1}-\frac{1}{\overline{\rho_2}}\overline{\beta_1}&=\frac{\alpha_{12}X_0+\alpha_{23}Y_0}{a_{22}2\pi i}\end{aligned} \tag{34}$$

System (34) always has a solution because the determinant of the coefficients of the unknowns is non-zero. Indeed we have:

$$\begin{vmatrix} 1 & 1 & 1 & 1\\ \rho_1 & \rho_2 & \overline{\rho_1} & \overline{\rho_2}\\ \rho_1^2 & \rho_2^2 & \overline{\rho_1^2} & \overline{\rho_2^2}\\ \frac{1}{\rho_1} & \frac{1}{\rho_2} & \frac{1}{\overline{\rho_1}} & \frac{1}{\overline{\rho_2}}\end{vmatrix}=\frac{-1}{\rho_1\rho_2\overline{\rho_1\rho_2}}\begin{vmatrix} 1 & 1 & 1 & 1\\ \rho_1 & \rho_2 & \overline{\rho_1} & \overline{\rho_2}\\ \rho_1^2 & \rho_2^2 & \overline{\rho_1^2} & \overline{\rho_2^2}\\ \rho_1^3 & \rho_2^3 & \overline{\rho_1^3} & \overline{\rho_2^3}\end{vmatrix}$$

And this as of type Vandermonde for $\rho_1 \neq \rho_2$ never vanishes.

3. Coming back to the system of limit values of the functions $f_1(\zeta_1)$ and $f_2(\zeta_2)$ (32), we can proceed with its integration (according to Cauchy), as follows:

Multiplying both sides of the equations by $1/(2\pi i)d\sigma/(\sigma-\zeta)$ and integrating on the unit circle γ we get[21]:

$$\begin{aligned}\frac{1}{2\pi i}\int_\gamma \frac{f_1^0(\sigma)}{\sigma-\zeta}d\sigma + \frac{1}{2\pi i}\int_\gamma \frac{f_2^0(\sigma)}{\sigma-\zeta}d\sigma + \frac{1}{2\pi i}\int_\gamma \frac{\overline{f_1^0(\sigma)}}{\sigma-\zeta}d\sigma + \frac{1}{2\pi i}\int_\gamma \frac{\overline{f_2^0(\sigma)}}{\sigma-\zeta}d\sigma \\ = \frac{1}{2\pi i}\int_\gamma \frac{-\int_0^s Y_n ds}{\sigma-\zeta}d\sigma - \frac{Y_0}{(2\pi i)^2}\int_\gamma \frac{\ln(s)}{\sigma-\zeta}d\sigma \\ \frac{\rho_1}{2\pi i}\int_\gamma \frac{f_1^0(\sigma)}{\sigma-\zeta}d\sigma + \frac{\rho_2}{2\pi i}\int_\gamma \frac{f_2^0(\sigma)}{\sigma-\zeta}d\sigma + \frac{\overline{\rho_1}}{2\pi i}\int_\gamma \frac{\overline{f_1^0(\sigma)}}{\sigma-\zeta}d\sigma + \frac{\overline{\rho_2}}{2\pi i}\int_\gamma \frac{\overline{f_2^0(\sigma)}}{\sigma-\zeta}d\sigma = 0\end{aligned}$$

We also take into account that[22]

$$\begin{aligned}\frac{1}{2\pi i}\int_\gamma \frac{\overline{f_1^0(\sigma)}}{\sigma-\zeta}d\sigma = 0,\ \frac{1}{2\pi i}\int_\gamma \frac{\overline{f_2^0(\sigma)}}{\sigma-\zeta}d\sigma = 0 \\ \frac{1}{2\pi i}\int_\gamma \frac{f_1^0(\sigma)}{\sigma-\zeta}d\sigma = -f_1^0(\zeta),\ \frac{1}{2\pi i}\int_\gamma \frac{f_2^0(\sigma)}{\sigma-\zeta}d\sigma = -f_2^0(\zeta)\end{aligned}$$

The above system takes the form

$$\begin{aligned}f_1^0(\zeta) + f_2^0(\zeta) &= \frac{1}{2\pi i}\int_\gamma \frac{\int Y_n ds}{\sigma-\zeta}d\sigma + \frac{Y_0}{(2\pi i)^2}\int_\gamma \frac{\ln(\sigma)}{\sigma-\zeta}d\sigma \\ \rho_1 f_1^0(\zeta) + \rho_2 f_2^0(\zeta) &= 0\end{aligned} \tag{35}$$

The first integral of the right hand side of the first equation of system (35), in view of the remark in B.2.2, can be written:

$$\frac{1}{2\pi i}\int_\gamma \frac{\int Y_n ds}{\sigma-\zeta}d\sigma = \frac{1}{2\pi i}\int_{\sigma_1}^{\sigma_2} \frac{p\cdot z}{\sigma-\zeta}d\sigma + \frac{1}{2\pi i}\int_{\sigma_2}^{\sigma_1} \frac{pz''}{\sigma-\zeta}d\sigma$$

Calculating separately each integral we have:

$$\begin{aligned}\int_{\sigma_1}^{\sigma_2} \frac{pz}{\sigma-\zeta}d\sigma &= \frac{pR}{2}\int_{\sigma_1}^{\sigma_2} \frac{\left(\sigma+\frac{1}{\sigma}\right)}{\sigma-\zeta}d\sigma \\ &= \frac{pR}{2}\int_{\sigma_1}^{\sigma_2} \frac{(\sigma-\zeta)+\zeta+\frac{1}{\sigma}}{\sigma-\zeta}d\sigma \\ &= \frac{pR}{2}\left[(\sigma_2-\sigma_1)+\zeta\ln\left(\frac{\sigma_2-\zeta}{\sigma_1-\zeta}\right)+\frac{1}{\zeta}\left(\frac{\sigma_2-\zeta}{\sigma_1-\zeta}-\ln\left(\frac{\sigma_2}{\sigma_1}\right)\right)\right] \\ &= \frac{pR}{2}\left[(\sigma_2-\sigma_1)+\left(\zeta+\frac{1}{\zeta}\right)\ln\left(\frac{\sigma_2-\zeta}{\sigma_1-\zeta}\right)-\ln\left(\frac{\sigma_2}{\sigma_1}\right)\right]\end{aligned}$$

[21] For the writing of the system we made use of the equations $\alpha_1+\beta_1-\overline{\alpha_1}-\overline{\beta_1}=\frac{Y_0}{2\pi i}$ and $X_0=0$.

[22] See Appendix 2.2.b) and 1.8.b).i.

$$p\int_{\sigma_2}^{\sigma_1}\frac{z''}{\sigma-\zeta}d\sigma=-pz''\ln\left(\frac{\sigma_2-\zeta}{\sigma_1-\zeta}\right)\text{ consequently}$$

$$\frac{1}{2\pi i}\int_\gamma\frac{\int Y_n ds}{\sigma-\zeta}d\sigma=\frac{pR}{4\pi i}\left[(\sigma_2-\sigma_1)+(z-z'')\ln\left(\frac{\sigma_2-\zeta}{\sigma_1-\zeta}\right)-\frac{1}{\zeta}\ln\left(\frac{\sigma_2}{\sigma_1}\right)\right]\tag{36}$$

The second integral can be found as follows. If we represent it by $\Omega(\zeta)$, we will have:

$$\Omega(\zeta)=\frac{1}{2\pi i}\int_\gamma\frac{\ln(\sigma)}{\sigma-\zeta}d\sigma\text{ and differentiating with respect to }\zeta$$

$$\Omega'(\zeta)=-\frac{1}{2\pi i}\int_\gamma\ln(\sigma)d\left(\frac{1}{\sigma-\zeta}\right)=-\frac{1}{2\pi i}\left[\frac{\ln(\sigma)}{\sigma-\zeta}\right]_\gamma+\frac{1}{2\pi i}\int_\gamma\frac{d\sigma}{\sigma(\sigma-\zeta)}$$

but it is

$$\frac{1}{2\pi i}\int_\gamma\frac{d\sigma}{\sigma(\sigma-\zeta)}=-\frac{1}{\zeta}\text{ and }\frac{1}{2\pi i}\left[\frac{\ln(\sigma)}{\sigma-\zeta}\right]_{\sigma_1=e^{i\theta}}^{\sigma_2=e^{i(\theta+2\pi)}}=\frac{1}{\sigma_1-\zeta}$$

as such we conclude that [17]

$$\Omega'(\zeta)=-\frac{1}{\sigma_1-\zeta}-\frac{1}{\zeta}$$

therefore,

$$\Omega(\zeta)=\ln(\sigma_1-\zeta)-\ln(\zeta)=\ln\left(\frac{\sigma_1-\zeta}{\zeta}\right)+C\tag{37}$$

By substituting equations (36) and (37) in system (35), the latter transforms into:

$$\begin{aligned}f_1^0(\zeta)+f_2^0(\zeta)&=\frac{p}{2\pi i}\left[\frac{R}{2}(\sigma_2-\sigma_1)+(z-z'')\ln\left(\frac{\sigma_2-\zeta}{\sigma_1-\zeta}\right)-\frac{R}{2\zeta}\ln\left(\frac{\sigma_2}{\sigma_1}\right)\right]\\&\quad+\frac{Y_0}{2\pi i}\ln\left(\frac{\sigma_1-\zeta}{\zeta}\right)\\\rho_1 f_1^0(\zeta)+\rho_2 f_2^0(\zeta)&=0\end{aligned}\tag{38}$$

By solving (38) with respect to the unknown functions $f_1^0(\zeta)$ and $f_1^0(\zeta)$ (after we consider their domains of definition), we find[23].

$$f_1^0(\zeta_1)=\frac{-\rho_2}{\rho_1-\rho_2}\left[\frac{p}{2\pi i}\left\{-\frac{R}{2\zeta_1}\ln\left(\frac{\sigma_2}{\sigma_1}\right)+(z_1-z'')\ln\left(\frac{\sigma_2-\zeta_1}{\sigma_1-\zeta_1}\right)\right\}+\frac{Y_0}{2\pi i}\ln\left(\frac{\sigma_1-\zeta_1}{\zeta_1}\right)\right]$$

$$f_2^0(\zeta_2)=\frac{\rho_1}{\rho_1-\rho_2}\left[\frac{p}{2\pi i}\left\{-\frac{R}{2\zeta_2}\ln\left(\frac{\sigma_2}{\sigma_1}\right)+(z_2-z'')\ln\left(\frac{\sigma_2-\zeta_2}{\sigma_1-\zeta_2}\right)\right\}+\frac{Y_0}{2\pi i}\ln\left(\frac{\sigma_1-\zeta_2}{\zeta_2}\right)\right]$$

[23] In the final expressions of Complex Potential constants are ignored as not influencing the tensile situation.

therefore:

$$
\begin{aligned}
\Phi_1'(z_1) = f_1(\zeta_1) = \frac{-\rho_2}{\rho_1-\rho_2}\times & \\
\times\left[\frac{p}{2\pi i}\left\{-\frac{R}{2\zeta_1}\ln\left(\frac{\sigma_2}{\sigma_1}\right)+(z_1-z'')\ln\left(\frac{\sigma_2-\zeta_1}{\sigma_1-\zeta_1}\right)\right\}+\frac{Y_0}{2\pi i}\ln\left(\frac{\sigma_1-\zeta_1}{\zeta_1}\right)\right] & \\
+\alpha_1\ln(\zeta_1) & \\
\Phi_2'(z_2) = f_2(\zeta_2) = \frac{\rho_1}{\rho_1-\rho_2}\times & \\
\times\left[\frac{p}{2\pi i}\left\{-\frac{R}{2\zeta_2}\ln\left(\frac{\sigma_2}{\sigma_1}\right)+(z_2-z'')\ln\left(\frac{\sigma_2-\zeta_2}{\sigma_1-\zeta_2}\right)\right\}+\frac{Y_0}{2\pi i}\ln\left(\frac{\sigma_1-\zeta_2}{\zeta_2}\right)\right] & \\
+\beta_1\ln(\zeta_2) &
\end{aligned}
\tag{39}
$$

The above found functions $\Phi_1(z_1)$ and $\Phi_2(z_2)$ solve completely the initially posed problem.

The calculation of stresses can now be performed easily by differentiating with respect to z_1 and z_2 respectively the functions $f_1(\zeta_1)$ and $f_2(\zeta_2)$ as follows:

$$
\begin{aligned}
\Phi_1''(z_1) &= \frac{-\rho_2}{\rho_1-\rho_2}\left[\frac{p}{2\pi i}A_1-\frac{Y_0\sigma_1\zeta_1}{2\pi i\frac{R}{2}(\sigma_1-\zeta_1)(\zeta_1^2-1)}\right]+\frac{\alpha_1\zeta_1}{\frac{R}{2}(\zeta_1^2-1)}\\
\Phi_2''(z_1) &= \frac{\rho_1}{\rho_1-\rho_2}\left[\frac{p}{2\pi i}A_2-\frac{Y_0\sigma_1\zeta_2}{2\pi i\frac{R}{2}(\sigma_1-\zeta_2)(\zeta_2^2-1)}\right]+\frac{\beta_1\zeta_2}{\frac{R}{2}(\zeta_2^2-1)}
\end{aligned}
\tag{40}
$$

with[24]

$$
\begin{aligned}
A_1 &= \frac{1}{\zeta_1^2-1}\ln\left(\frac{\sigma_2}{\sigma_1}\right)+\ln\left(\frac{\sigma_2-\zeta_1}{\sigma_1-\zeta_1}\right)+\frac{(z_1-z'')(\sigma_2-\sigma_1)\zeta_1^2}{(\sigma_1-\zeta_1)(\sigma_2-\zeta_1)\frac{R}{2}(\zeta_1^2-1)}\\
A_2 &= \frac{1}{\zeta_2^2-1}\ln\left(\frac{\sigma_2}{\sigma_1}\right)+\ln\left(\frac{\sigma_2-\zeta_2}{\sigma_1-\zeta_2}\right)+\frac{(z_1-z'')(\sigma_2-\sigma_1)\zeta_2^2}{(\sigma_1-\zeta_2)(\sigma_2-\zeta_2)\frac{R}{2}(\zeta_2^2-1)}
\end{aligned}
$$

From equations (40), based on relations (15) and (16), we find the stresses and the displacements.

For a total charge on the boundary relations (39) are simplified as follows:

We have then

$$
\ln\left(\frac{\sigma_2}{\sigma_1}\right)=2\pi i,\ \ln\left(\frac{\sigma_2-\zeta_i}{\sigma_1-\zeta_i}\right)=0,\ Y_0=0
$$

so[25],

[24] Translator's Note: The terms A_1 and A_2 are given separately for easier reading.

[25] Equations (41) are found in a different form also in [7] because of the various constants of elasticity and parameters used.

$$\begin{aligned}\Phi_1'(z_1) &= \frac{\rho_2}{\rho_1-\rho_2}\cdot\frac{pR}{2\zeta_1}\\ \Phi_2'(z_2) &= \frac{-\rho_1}{\rho_1-\rho_2}\cdot\frac{pR}{2\zeta_2}\end{aligned} \tag{41}$$

4. In each case using appropriate limiting access we can, by using equations (39), solve the problem of the crack with concentrated charge (moving to the limit with respect to the charge) or the problems of the half-space with concentrated or distributed charges (moving to the limit with respect to the boundary).

a) Concentrated normal charge against the boundary of the crack at the point Z_0. In this case is: $p(z'-z'')=Y_0$ and $\sigma_2\to\sigma_1=\sigma_0$ or

$$p\frac{R}{2}\left[(\sigma_1-\sigma_2)-\frac{\sigma_1-\sigma_2}{\sigma_1\sigma_2}\right]=p\frac{R}{2}(\sigma_1-\sigma_2)\frac{\sigma_0^2-1}{\sigma_0^2}=Y_0 \tag{42}$$

If we set:

$$p(\sigma_1-\sigma_2)=A \text{ and } p\ln\left(\frac{\sigma_2}{\sigma_1}\right)=B$$

we will have:

$$\lim_{\sigma_2\to\sigma_1}\frac{B}{-A}=\frac{\ln(\sigma_2)-\ln(\sigma_1)}{-(\sigma_2-\sigma_1)}=-\frac{1}{\sigma_0} \text{ or } B=\frac{A}{\sigma_0}$$

and also

$$A\cdot\frac{R}{2}\cdot\frac{\sigma_0^2-1}{\sigma_0^2}=Y_0$$

Hence:

$$-\frac{p}{2\pi i}\cdot\frac{R}{2}\cdot\frac{1}{\zeta}\ln\left(\frac{\sigma_2}{\sigma_1}\right)=-\frac{B}{2\pi i}\cdot\frac{R}{2}\cdot\frac{1}{\zeta}=-\frac{A}{2\pi i}\cdot\frac{R}{2\sigma_0}\cdot\frac{1}{\zeta}=-\frac{Y_0}{2\pi i}\cdot\frac{\sigma_0^2}{\sigma_0^2-1}\cdot\frac{1}{\zeta}$$

or

$$\lim_{\sigma_2\to\sigma_1}-\frac{p}{2\pi i}\cdot\frac{R}{2}\cdot\frac{1}{\zeta}\ln\left(\frac{\sigma_2}{\sigma_1}\right)=-\frac{Y_0}{2\pi i}\cdot\frac{\sigma_0}{\sigma_0^2-1}\cdot\frac{1}{\zeta} \tag{43}$$

Similarly we find, that if (42) holds,

$$\lim_{\sigma_2\to\sigma_1}(z-z_0)\frac{p}{2\pi i}\ln\left(\frac{\sigma_2-\zeta}{\sigma_1-\zeta}\right)=(z-z_0)\frac{1}{2\pi i}\cdot\frac{Y_0}{\sigma_0-\zeta}\cdot\frac{\sigma_0^2}{\frac{R}{2}(\sigma_0^2-1)} \tag{44}$$

With the substitution of (43) and (44) in (39), we find finally:

$$\begin{aligned}
f_1(\zeta_1) &= \frac{-\rho_2 Y_0}{(\rho_1-\rho_2)2\pi i}\left[\frac{-\sigma_0}{(\sigma_0^2-1)\zeta_1} + \frac{z_1 - z_0}{\frac{R}{2}(\sigma_0-\zeta_1)} + \ln\left(\frac{\sigma_0-\zeta_1}{\zeta_1}\right)\right] + \alpha_1 \ln(\zeta_1) \\
f_2(\zeta_2) &= \frac{\rho_1 Y_0}{(\rho_1-\rho_2)2\pi i}\left[\frac{-\sigma_0}{(\sigma_0^2-1)\zeta_2} + \frac{z_2 - z_0}{\frac{R}{2}(\sigma_0-\zeta_2)} + \ln\left(\frac{\sigma_0-\zeta_2}{\zeta_2}\right)\right] + \beta_1 \ln(\zeta_2)
\end{aligned} \tag{45}$$

If we set $z_i = \frac{R}{2}\left(\zeta_i + \frac{1}{\zeta_i}\right)$, for $i = 1, 2$ in the equations (45) and perform the calculations, we find (ignoring constants):

$$\begin{aligned}
f_1(\zeta_1) &= \frac{-\rho_2}{\rho_1-\rho_2}\cdot\frac{Y_0}{2\pi i}\ln\left(\frac{\sigma_0-\zeta_1}{\zeta_1}\right) + \alpha_1 \ln(\zeta_1) \\
f_2(\zeta_2) &= \frac{\rho_1}{\rho_1-\rho_2}\cdot\frac{Y_0}{2\pi i}\ln\left(\frac{\sigma_0-\zeta_2}{\zeta_2}\right) + \beta_1 \ln(\zeta_2)
\end{aligned} \tag{46}$$

b) Concentrated charge (normal) at the point $z_0 = 0$ of the infinite disk.
In this case we are looking for

$$\lim_{R\to 0} f_i(\zeta_i), \text{ for } i = 1, 2$$

However, because for $R \to 0$, the "finiteness" of z assumes $|\zeta_i| \to \infty$, we conclude that:

$$\lim \ln\left(\frac{\sigma_0-\zeta_i}{\zeta_i}\right) = \pi i$$

Therefore, as long as we ignore the constants for the functions of Complex Potential, we have initially:

$$\Phi_1'(z_1) = \alpha_1 \ln(\zeta_1)$$

and

$$\Phi_1''(z_1) = \alpha_1 \frac{\zeta_1}{\frac{R}{2}(\zeta_1^2 - 1)} = \alpha_1 \frac{1}{\frac{R}{2}(\zeta_1 - \zeta_1^{-1})}$$

But for: $|\zeta_i| \to \infty$, $\zeta_i - \frac{1}{\zeta_i} = O(\zeta_i) = \zeta_i + \frac{1}{\zeta_i}$, $(i = 1, 2)$.
Hence,

$$\begin{aligned}
\Phi_1'' &= \frac{\alpha_1}{z_1} \\
\Phi_2'' &= \frac{\beta_1}{z_1}
\end{aligned} \tag{47}$$

Therefore the functions of Complex Potential, can be written, by integrating (47) as:

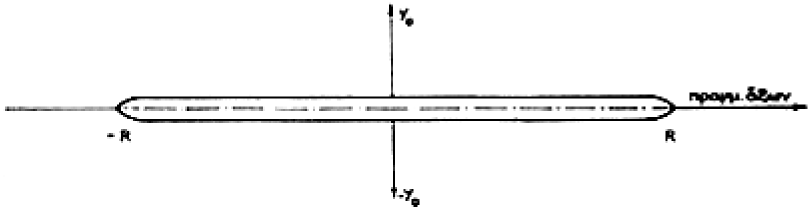

Figure 11: Two successive charges

$$\begin{aligned} \Phi_1'(z_1) &= \alpha_1 \ln(z_1) \\ \Phi_2'(z_2) &= \beta_1 \ln(z_1) \end{aligned} \tag{48}$$

where α_1, β_1 are determined as usual from the solution of the system (34).

c) Concentrated normal charge at a point $z_0 = 0$ of the half-space.

We consider the situation (as two successive charges) as in (Fig. 11),
and by applying (46) and because $\sigma_0 = i$, $\sigma_0' = -i$, we find:

$$\begin{aligned} \Phi_1'(z_1) &= \frac{-\rho_2}{\rho_1 - \rho_2} \cdot \frac{Y_0}{2\pi i} \ln\left(\frac{\zeta_1 - i}{\zeta_1 + i}\right) \\ \Phi_2'(z_2) &= \frac{\rho_1}{\rho_1 - \rho_2} \cdot \frac{Y_0}{2\pi i} \ln\left(\frac{\zeta_2 - i}{\zeta_2 + i}\right) \end{aligned} \tag{49}$$

Differentiating equations (49) with respect to z_1 and z_2 respectively, we find after performing the calculations:

$$\begin{aligned} \Phi_1''(z_1) &= \frac{-\rho_2}{\rho_1 - \rho_2} \cdot \frac{Y_0}{2\pi z_1} \cdot \frac{2\zeta_1}{\zeta_1^2 - 1} \\ \Phi_2''(z_2) &= \frac{\rho_1}{\rho_1 - \rho_2} \cdot \frac{Y_0}{2\pi z_2} \cdot \frac{2\zeta_2}{\zeta_2^2 - 1} \end{aligned} \tag{50}$$

In order to have z_i finite while $R \to \infty$ we conclude that

$$\lim_{R\to\infty} \zeta_i = \pm i$$

Therefore,

$$\begin{aligned} \lim_{R\to\infty} \Phi_1''(z_1) &= \frac{-\rho_2 Y_0}{2\pi i(\rho_1 - \rho_2)} \cdot \frac{1}{z_1} \\ \lim_{R\to\infty} \Phi_2''(z_2) &= \frac{\rho_1 Y_0}{2\pi i(\rho_1 - \rho_2)} \cdot \frac{1}{z_2} \end{aligned} \qquad \text{for } \Im(z) > 0 \tag{51a}$$

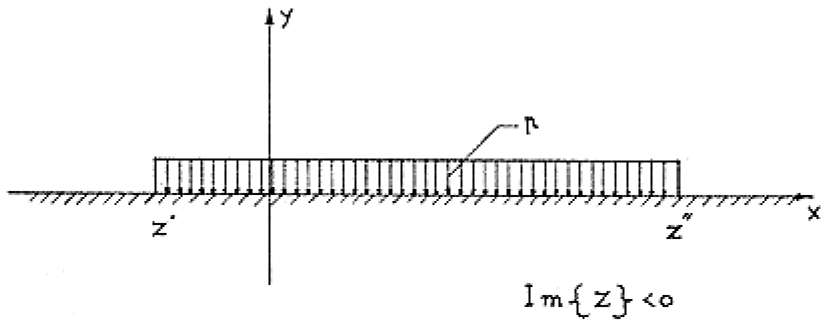

Figure 12: Charge between the points z' and z''

$$\begin{aligned}\lim_{R\to\infty}\Phi_1''(z_1) &= \frac{-\rho_2 Y_0}{2\pi i(\rho_1-\rho_2)}\cdot\frac{1}{z_1}\\ \lim_{R\to\infty}\Phi_2''(z_2) &= \frac{\rho_1 Y_0}{2\pi i(\rho_1-\rho_2)}\cdot\frac{1}{z_2}\end{aligned}\qquad \text{for } \Im(z)<0 \tag{51b}$$

d) Continuous, uniformly distributed normal charge p on the vicinity of the half-space $y=0$ and between the points z', z'' (Fig. 12).

Adding to the first of equations (39) the constant $\frac{\rho_2(z'-z'')}{\rho_1-\rho_2}\cdot\frac{p}{2\pi i}\ln(R)$ and setting $Y_0=p(z'-z'')$ we get

$$f_1(\zeta_1)=\frac{-\rho_2}{\rho_1-\rho_2}A_1+(\kappa+\alpha_1)\ln(\zeta_1) \tag{52}$$

where[26]

$$\begin{aligned}A_1 &= -\frac{R}{2}\cdot\frac{1}{\zeta_1}\ln\left(\frac{\sigma_2}{\sigma_1}\right)+(z_1-z'')\ln\left(\frac{\sigma_2-\zeta_1}{\sigma_1-\zeta_1}\right)+(z'-z'')\ln(R)(\sigma_1-\zeta_1)\\ \kappa &= \frac{\rho_2}{\rho_1-\rho_2}\cdot\frac{p}{2\pi i}(z'-z'')\end{aligned}$$

Because $R\to\infty$, we can set $\alpha_1+\kappa=0$ considering the charge on (Fig. 12) to be balanced with an equal and opposite force applied at infinity and thus not affecting the tensile situation on the considered domain.

The first term of A_1 of (52), differentiated with respect to z_1 gives:

$$A'(z_1)=\frac{R}{2}\cdot\frac{1}{\zeta_1^2}\cdot\frac{\zeta_1^2}{\frac{R}{2}(\zeta_1^2-1)}\ln\left(\frac{\sigma_2}{\sigma_1}\right)=\frac{1}{\zeta_1^2-1}\ln\left(\frac{\sigma_2}{\sigma_1}\right)$$

and because for $R\to\infty$, $\zeta_i\to -i$ and $\sigma_2\to\sigma_1$, we conclude: $\lim_{R\to\infty}A'_{z_1}=0$. Therefore A equals a constant, which can be ignored.

[26]Translator's Note: See footnote 24 on page 34.

The second term of A_1 of (52) can be written:

$$B = (z_1 - z'') \ln\left(\frac{\sigma_2 - \zeta_1}{\sigma_1 - \zeta}\right)$$

and for $R \to \infty$ (we accept the variable ζ in the region of $-i$) we will have:

$$\begin{aligned}
\sigma_2 &= x_2 - \sqrt{x_2^2 - 1} \sim x_2 - i\left(1 - \frac{x_2^2}{2}\right),\ x_2 = \frac{z''}{R} \to 0 \\
\sigma_1 &= x_1 - \sqrt{x_1^2 - 1} \sim x_1 - i\left(1 - \frac{x_1^2}{2}\right),\ x_1 = \frac{z'}{R} \to 0 \\
\zeta_1 &= x - \sqrt{x^2 - 1} \sim x - i\left(1 - \frac{x^2}{2}\right),\ x = \frac{z_1}{R} \to 0
\end{aligned}$$

which finally gives:

$$\begin{aligned}
\lim_{R\to\infty} (z_1 - z'') \ln\left(\frac{x_2 - x}{x_1 - x}\right) \cdot \frac{1 - i\frac{x+x_2}{2}}{1 - i\frac{x+x_1}{2}} &= (z_1 - z'') \ln\left(\frac{x_2 - x}{x_1 - x}\right) \\
&= (z_1 - z'') \ln\left(\frac{z'' - z_1}{z' - z_1}\right)
\end{aligned}$$

Similarly we find:

$$\lim_{R\to\infty} (z' - z'') \ln(R)(\sigma_1 - \zeta_1) = (z' - z'') \ln(z' - z_1)$$

Summarizing the above results and taking into account that in the distribution of stress the functions of Complex Potential are invariant under addition or subtraction of constants, we have:

$$\begin{aligned}
\Phi_1'(z_1) &= \frac{-\rho_2}{\rho_1 - \rho_2}\frac{p}{2\pi i}\left\{(z_1 - z'')\ln(z_1 - z'') - (z_1 - z')\ln(z_1 - z')\right\} \\
\Phi_2'(z_2) &= \frac{\rho_1}{\rho_1 - \rho_2}\frac{p}{2\pi i}\left\{(z_2 - z'')\ln(z_2 - z'') - (z_2 - z')\ln(z_2 - z')\right\}
\end{aligned} \tag{53}$$

for $\Im(z) < 0$.

Equations (53) solve the posed problem, much more generally than [23], showing for one more time the importance of the general solution (39).

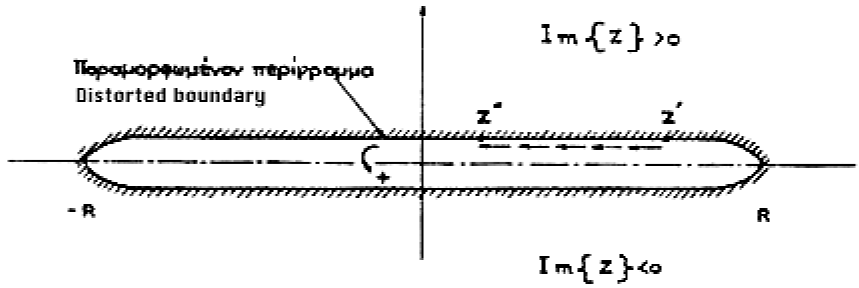

Figure 13: Distorted boundary

B.3 Continuous, uniformly distributed tangential charge over section of the boundary

1. In this case, we work exactly as in the problem of normal charge.
We have: $X_n = +q$, $Y_n = 0$, and $ds = -dz$
Functions (22) already take the following form:

$$\Psi_1'(z_1) = Q_1(\zeta_1) = \alpha_2 \ln(\zeta_1) + Q_1^0(\zeta_1)$$
$$\Psi_2'(z_2) = Q_1(\zeta_2) = \beta_2 \ln(\zeta_2) + Q_2^0(\zeta_2)$$

while the boundary conditions on the unit circle γ can be written:

$$Q_1^0(\sigma) + Q_2^0(\sigma) + \overline{Q_1^0(\sigma)} + \overline{Q_2^0(\sigma)} = -(\alpha_2 + \beta_2 - \overline{\alpha_2} - \overline{\beta_2}) \ln(\sigma)$$
$$\rho_1 Q_1^0(\sigma) + \rho_2 Q_2^0(\sigma) + \overline{\rho_1 Q_1^0(\sigma)} + \overline{\rho_2 Q_2^0(\sigma)} = \int X_n ds = -(\rho_1\alpha_2 + \rho_2\beta_2 - \overline{\rho_1\alpha_2} - \overline{\rho_2\beta_2}) \ln(\sigma)$$

By integrating according to Cauchy on the unit circle, we get the corresponding to (35) equations:

$$\begin{aligned} &Q_1^0(\zeta) + Q_2^0(\zeta) = 0 \\ &\rho_1 Q_1^0(\zeta) + \rho_2 Q_2^0(\zeta) = -\frac{1}{2\pi i}\int_\gamma \frac{\int X_n ds}{\sigma - \zeta} d\sigma - \frac{X_0}{(2\pi i)^2}\int_\gamma \frac{\ln(\sigma)}{\sigma - \zeta} d\sigma \end{aligned} \tag{54}$$

where the constants α_2, β_2 are calculated by solving the corresponding to (34) general system.
For $Y_0 = 0$ the system takes the form:

$$
\begin{aligned}
\alpha_2 + \beta_2 - \overline{\alpha_2} - \overline{\beta_2} &= 0 \\
\rho_1\alpha_2 + \rho_2\beta_2 - \overline{\rho_1\alpha_2} - \overline{\rho_2\beta_2} &= -\frac{X_0}{2\pi i} \\
\rho_1^2\alpha_2 + \rho_2^2\beta_2 - \overline{\rho_1^2\alpha_2} - \overline{\rho_2^2\beta_2} &= -\frac{\alpha_{13}}{\alpha_{11}} \cdot \frac{X_0}{2\pi i} \\
\frac{1}{\rho_1}\alpha_2 + \frac{1}{\rho_2}\beta_2 - \frac{1}{\overline{\rho_1}}\overline{\alpha_2} - \frac{1}{\overline{\rho_2}}\overline{\beta_2} &= -\frac{\alpha_{12}}{\alpha_{22}} \cdot \frac{X_0}{2\pi i}
\end{aligned}
\tag{55}
$$

After the above and in full correspondence to the solution of the functional system (38), we find:

$$
\begin{aligned}
Q_1^0(\zeta) + Q_2^0(\zeta) &= 0 \\
\rho_1 Q_1^0(\zeta) + \rho_2 Q_2^0(\zeta) &= + \frac{qR}{4\pi i}(\sigma_2 - \sigma_1) + \frac{q}{2\pi i}(z - z'') \ln\left(\frac{\sigma_2 - \zeta}{\sigma_1 - \zeta}\right) \\
&\quad + \frac{qR}{2\pi i} \cdot \frac{1}{\zeta} \ln\left(\frac{\sigma_2}{\sigma_1}\right) + \frac{X_0}{2\pi i} \ln\left(\frac{\sigma_1 - \zeta}{\zeta}\right)
\end{aligned}
$$

and finally: (in correspondence to the general solution (39) and (40))

$$
\begin{aligned}
\Psi_1'(z_1) = Q_1(\zeta_1) &= \frac{1}{\rho_1 - \rho_2} \times \\
&\times \left[\frac{q}{2\pi i}\left\{-\frac{R}{2\zeta_1}\ln\left(\frac{\sigma_2}{\sigma_1}\right) + (z_1 - z'')\ln\left(\frac{\sigma_2 - \zeta_1}{\sigma_1 - \zeta_1}\right)\right\} + \frac{X_0}{2\pi i}\ln\left(\frac{\sigma_1 - \zeta_1}{\zeta_1}\right)\right] \\
&+ \alpha_2 \ln(\zeta_1) \\
\Psi_2'(z_2) = Q_2(\zeta_2) &= \frac{-1}{\rho_1 - \rho_2} \times \\
&\times \left[\frac{q}{2\pi i}\left\{-\frac{R}{2\zeta_2}\ln\left(\frac{\sigma_2}{\sigma_1}\right) + (z_2 - z'')\ln\left(\frac{\sigma_2 - \zeta_2}{\sigma_1 - \zeta_2}\right)\right\} + \frac{X_0}{2\pi i}\ln\left(\frac{\sigma_1 - \zeta_2}{\zeta_2}\right)\right] \\
&+ \beta_2 \ln(\zeta_2)
\end{aligned}
\tag{56}
$$

By differentiating equations (56) we get

$$
\begin{aligned}
\Psi_1''(z_1) &= \frac{1}{\rho_1 - \rho_2}\left[\frac{q}{2\pi i}A_1 - \frac{X_0\sigma_1\zeta_1}{2\pi i\frac{R}{2}(\sigma_1 - \zeta_1)(\zeta_1^2 - 1)}\right] + \frac{\alpha_2\zeta_1}{\frac{R}{2}(\zeta_1^2 - 1)} \\
\Psi_2''(z_1) &= \frac{-1}{\rho_1 - \rho_2}\left[\frac{q}{2\pi i}A_2 - \frac{X_0\sigma_1\zeta_2}{2\pi i\frac{R}{2}(\sigma_1 - \zeta_2)(\zeta_2^2 - 1)}\right] + \frac{\beta_2\zeta_2}{\frac{R}{2}(\zeta_2^2 - 1)}
\end{aligned}
\tag{57}
$$

with[27]

[27]Translator's Note: See footnote 24 on page 34.

$$A_1 = \frac{1}{\zeta_1^2 - 1} \ln\left(\frac{\sigma_2}{\sigma_1}\right) + \ln\left(\frac{\sigma_2 - \zeta_1}{\sigma_1 - \zeta_1}\right) + \frac{(z_1 - z'')(\sigma_2 - \sigma_1)\zeta_1^2}{(\sigma_1 - \zeta_1)(\sigma_2 - \zeta_1)\frac{R}{2}(\zeta_1^2 - 1)}$$
$$A_2 = \frac{1}{\zeta_2^2 - 1} \ln\left(\frac{\sigma_2}{\sigma_1}\right) + \ln\left(\frac{\sigma_2 - \zeta_2}{\sigma_1 - \zeta_2}\right) + \frac{(z_1 - z'')(\sigma_2 - \sigma_1)\zeta_2^2}{(\sigma_1 - \zeta_2)(\sigma_2 - \zeta_2)\frac{R}{2}(\zeta_2^2 - 1)}$$

for a total charge of the boundary we find:

$$\Psi_1''(z_1) = \frac{+q}{\rho_1 - \rho_2} \cdot \frac{1}{\zeta_1^2 - 1} \qquad \Psi_2''(z_2) = \frac{-q}{\rho_1 - \rho_2} \cdot \frac{1}{\zeta_2^2 - 1} \tag{58}$$

The relations (56) and (57) solve completely the problem for the case of a constant tangential distributed charge between the points z' and z''. We can now apply these, using the same methods of limiting approach as in section B, to solve the corresponding problems in B.2.4. a), b), c) and d).

2. a) Concentrated tangential charge on the boundary at the point z_0 (from (46)):

$$\begin{aligned} Q_1(\zeta_1) &= \frac{+1}{\rho_1 - \rho_2} \cdot \frac{X_0}{2\pi i} \ln\left(\frac{\sigma_0 - \zeta_1}{\zeta_1}\right) + \alpha_2 \ln(\zeta_1) \\ Q_2(\zeta_1) &= \frac{-1}{\rho_1 - \rho_2} \cdot \frac{X_0}{2\pi i} \ln\left(\frac{\sigma_0 - \zeta_2}{\zeta_2}\right) + \beta_2 \ln(\zeta_2) \end{aligned} \tag{59}$$

b) Concentrated charge (horizontal) at the point $z_0 = 0$ of the infinite disk (from (47)):

$$\begin{aligned} \Psi_1''(z_1) &= \frac{\alpha_2}{z_1}, \text{ or } \Psi_1'(z_1) = \alpha_2 \ln(z_1) \\ \Psi_2''(z_2) &= \frac{\beta_2}{z_2}, \text{ or } \Psi_2'(z_2) = \beta_2 \ln(z_2) \end{aligned} \tag{60}$$

c) Charge (horizontal) concentrated at one point $z_0 = 0$ of the boundary $y = 0$ of the half-space (from (51a) and (51b)):

$$\begin{aligned} \Psi_1''(z_1) &= \frac{+1}{\rho_1 - \rho_2} \cdot \frac{X_0}{2\pi i z_1} \\ \Psi_2''(z_2) &= \frac{-1}{\rho_1 - \rho_2} \cdot \frac{X_0}{2\pi i z_2} \end{aligned} \qquad \text{for } \Im(z) > 0 \tag{61a}$$

$$\begin{aligned} \Psi_1''(z_1) &= \frac{-1}{\rho_1 - \rho_2} \cdot \frac{X_0}{2\pi i z_1} \\ \Psi_2''(z_2) &= \frac{+1}{\rho_1 - \rho_2} \cdot \frac{X_0}{2\pi i z_2} \end{aligned} \qquad \text{for } \Im(z) < 0 \tag{61b}$$

d) Constant tangential charge normally distributed at a section of the boundary $y = 0$ and between the points z' and z'' (from equations (53)):

$$
\begin{aligned}
\Psi'_1(z_1) &= \frac{+1}{\rho_1 - \rho_2} \cdot \frac{q}{2\pi i} \left\{ (z_1 - z'') \ln(z_1 - z'') - (z_1 - z') \ln(z_1 - z') \right\} \text{ for } \Im(z) > 0 \\
\Psi'_2(z_2) &= \frac{-1}{\rho_1 - \rho_2} \cdot \frac{q}{2\pi i} \left\{ (z_2 - z'') \ln(z_2 - z'') - (z_2 - z') \ln(z_2 - z') \right\} \text{ for } \Im(z) < 0
\end{aligned}
\tag{62}
$$

C Explanation of the Found Expressions - Conclusions

The general formulas (28), (39) and (56) found in section B solve completely the initially posed problem.

Via relations (16) we can also find the displacements on any point on the disk by determining based on initial conditions the undetermined constants u_0 and v_0.

In what follows we examine the operational details of the found expressions by applying them to the stress relations on the boundary.

The transition from the complex variable to the real variable can be performed easily via the known transformations, omitting mostly the operations on the usual trigonometric functions.

C.1 Uncharged Boundary

1. Relations (29) give the stresses on the disk for uniformly distributed stresses at infinity, that is for given σ_x^∞, σ_y^∞ and τ_{xy}^∞.

It can be proved initially that existence of a crack oriented parallel to the applied stresses does not affect the tensile situation of the disk (as in the case of isotropy).

Indeed for $\sigma_x^\infty = \sigma_y^\infty = 0$, it is: $\sigma_x = \sigma_x^\infty$ and $\sigma_y = \tau_{xy} = 0$.

2. If we look for the stresses on the boundary we will have:

a) $\sigma_y = \sigma_y^\infty + 2\Re\left\{\sigma_y^\infty \cdot \frac{1}{\sigma^2-1}\right\}$ where $\sigma = e^{i\theta}$ the common value of the variables ζ_1 and ζ_2 ($\theta \neq \frac{\pi}{2} \pm \frac{\pi}{2}$) or

$$\sigma_y = \sigma_y^\infty + 2\sigma_y^\infty \cdot \Re\left\{\frac{\cos(\theta) - i\sin(\theta)}{\sin(\theta)} \cdot \frac{1}{2i}\right\} = 0$$

similarly

$$\tau_{xy} = \tau_{xy}^\infty + 2\Re\left\{\tau_{xy}\frac{1}{\sigma^2-1}\right\} = 0 \qquad (\sigma \neq \pm 1)$$

The above results were expected and were found simply for the verification of the correctness of expressions (29).

b) To find σ_x on the boundary we set $\zeta_1 = \zeta_2 = \sigma = e^{i\theta}$, therefore:

$$\sigma_x = \sigma_x^\infty + 2\Re\left[\left\{\frac{\rho_1\rho_2(\rho_2-\rho_1)}{\rho_1-\rho_2}\sigma_y^\infty\right\}\frac{1}{\sigma^2-1}\right]$$

or

$$\sigma_x = \sigma_x^\infty - \frac{1}{\sin(\theta)}\left\{\left(\rho_1\rho_2\sigma_y^\infty + (\rho_1+\rho_2)\tau_{xy}^\infty\right)\frac{e^{-i\theta}}{i}\right\}$$

if in the relation above we substitute the value of the complex parameters.

$$\rho_1 = \gamma_1 + i\delta_1 \;\; \text{where } \delta_1 > 0$$
$$\rho_2 = \gamma_2 + i\delta_2 \;\; \text{where } \delta_2 > 0$$

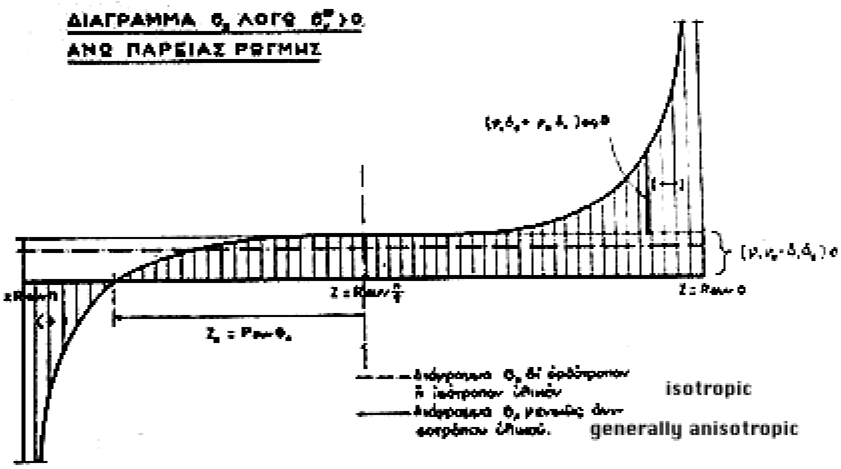

Figure 14: Diagram σ_x because of $\sigma_y^\infty > 0$, for upper section of the crack

we get after the calculations.

$$\begin{aligned}\sigma_x = \sigma_x^\infty &+ \left[(\gamma_1\gamma_2 - \delta_1\delta_2) - (\gamma_1\delta_2 - \gamma_2\delta_1)\cot(\theta)\right]\sigma_y^\infty \\ &+ \left[(\gamma_1 + \gamma_2) - (\delta_1 + \delta_2)\cot(\theta)\right]\tau_{xy}^\infty\end{aligned} \tag{63}$$

We can easily plot the graph of equation (63) for the two separate cases: the vanishing of τ_{xy} and of σ_y.

Taking into account the relations $|\gamma_1| < \delta_1$ and $|\gamma_2| < \delta_2$ (they hold since we are in the elliptic region) as well as $(\gamma_1\delta_2 + \gamma_2\delta_1) > 0$ and $\sigma_y > 0$ we draw on (Fig. 14) the function $\sigma_x(\theta)$ for the upper section of the crack.

The vanishing point of σ_x can be found from the relation:

$$\tan(\theta) = \frac{\gamma_1\gamma_2 - \delta_1\delta_2}{\gamma_1\delta_2 + \gamma_2\delta_1} \quad \text{or} \quad \theta_0 = \arctan\frac{\gamma_1\gamma_2 - \delta_1\delta_2}{\gamma_1\delta_2 + \gamma_2\delta_1}$$

therefore:

$$\theta_0 = \begin{cases}\arg(\rho_1) + \arg(\rho_2) & , \text{for } \arg(\rho_1\rho_2) < \pi, \\ \arg(\rho_1) + \arg(\rho_2) - \pi & , \text{for } \arg(\rho_1\rho_2) > \pi.\end{cases}$$

We find the diagram of σ_x for the lower section from (Fig. 14) by reflecting with respect to the axes x and y.

The diagram of σ_x because of τ_{xy}^∞ for the lower section is found by reflecting (Fig. 15) with respect to the x and y axes.

c) From equation (63) as well as from studying the diagrams (Fig. 14) and (Fig. 15) we conclude the following:

c1) In the case where $\tau_{xy}^\infty = 0$, σ_x becomes unbounded in absolute value close to the edges of the crack. This phenomenon appears only in the case of general anisotropy

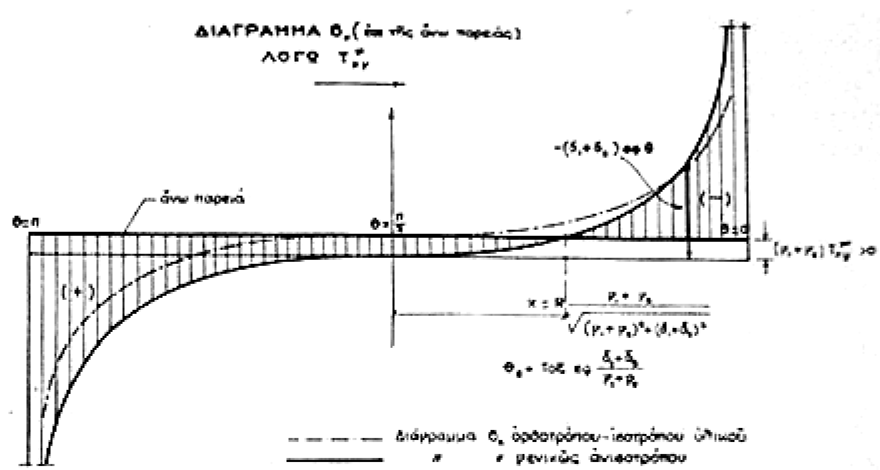

Figure 15: Diagram σ_x because $\tau_{xy}^{\infty} > 0$, for upper section of the crack

and happens because the material tends to slide towards the direction x, despite the fact that it is stressed only in the direction y.

c2) If we have orthotropic material or balance then σ_x is constant and compressive on the entire boundary and equal to $-\delta_1\delta_2\sigma_y^{\infty}$ in the first case, and equal to $-\sigma_y^{\infty}$ in the second case.

c3) In the case of general anisotropy the magnitudes of $|\gamma_1|$ and $|\gamma_2|$ are much smaller than δ_1 and δ_2, which means that the angle $|\pi - (\theta_1 + \theta_2)| = |\pi - \theta_0|$ is very small and the vanishing point of σ_x is found very close to the edges of the crack.

c4) If $\sigma_y^{\infty} = 0$ then σ_x in all three cases, general anisotropy, orthotropy and isotropy, becomes unbounded in magnitude the closer we are to the edges of the crack. The only difference between the first case and the other two is that in general anisotropy an additional constant σ_x^0 appears on the entire boundary which equals $(\gamma_1 + \gamma_2)\tau_{xy}^{\infty}$.

The vanishing point of σ_x in the above case is found at $R\frac{\gamma_1+\gamma_2}{\sqrt{(\gamma_1+\gamma_2)^2+(\delta_1+\delta_2)^2}}$ from the middle, which is quite small, but non-zero nevertheless for $\gamma_1 + \gamma_2 \neq 0$ or the same for $\alpha_{13} \neq 0$.

We can also add that the unboundedness of the stress at the end points happens because of the existence of an ideal cusp, a thing which cannot happen in reality on the one hand because of the nature of the material and on the other because of the immediate lamination which will happen in cases of such great stress on the aforementioned points.

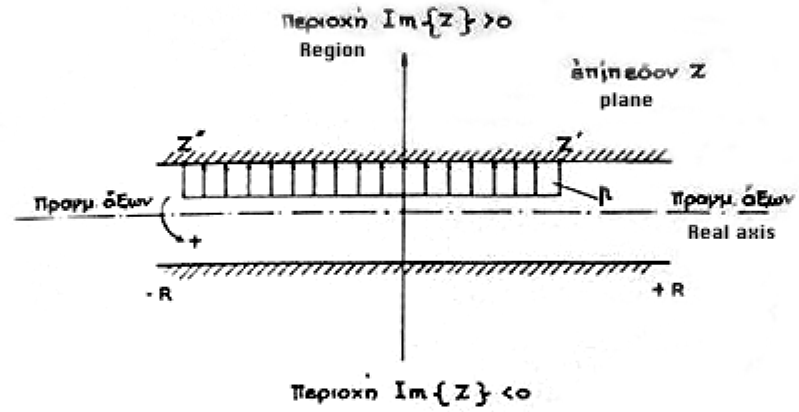

Figure 16: Normal charge on the disk

C.2 Normal Charge On the Boundary

1. Relations (40) give as is known the functions $\Phi_1''(z_1)$ and $\Phi_2''(z_2)$ based on which we can calculate the stresses on any point of the disk for a charge shown on (Fig. 16).

The mapping of the boundary of the crack on the circumference of the unit circle maps the points z' and z'' onto σ_1 and σ_2 (Fig. 17).

If we want to find σ_y on the boundary we will have:

$$\begin{aligned}\sigma_y = 2\Re\left\{\Phi_1''(\sigma) + \Phi_2''(\sigma)\right\} &= \Phi_1''(\sigma) + \Phi_2''(\sigma) + \overline{\Phi_1''(\sigma)} + \overline{\Phi_2''(\sigma)} \\ &= \Phi_1''(\sigma) + \Phi_2''(\sigma) + \overline{\Phi_1''}\left(\frac{1}{\sigma}\right) + \overline{\Phi_2''}\left(\frac{1}{\sigma}\right)\end{aligned}$$

given that for σ on the unit circle the relation $\sigma\overline{\sigma} = 1$ holds, we have:

$$\begin{aligned}\sigma_y = \frac{p}{2\pi i}&\left\{\frac{1}{\sigma^2-1}\ln\left(\frac{\sigma_2}{\sigma_1}\right) + +\ln\left(\frac{\sigma_2-\sigma}{\sigma_1-\sigma}\right) + \frac{(z-z'')(\sigma_2-\sigma_1)\sigma^2}{(\sigma_1-\sigma)(\sigma_2-\sigma)\frac{R}{2}(\sigma^2-1)}\right\} \\ &- \frac{Y_0\sigma_1\sigma}{2\pi i\frac{R}{2}(\sigma_1-\sigma)(\sigma^2-1)} + \frac{(\alpha_1+\beta_1)\sigma}{\frac{R}{2}(\sigma^2-1)} \\ &- \frac{p}{2\pi i}\left\{\frac{\sigma^2}{\sigma^2-1}\ln\left(\frac{\sigma_2}{\sigma_1}\right) + \ln\left(\frac{\overline{\sigma_2}-\overline{\sigma}}{\overline{\sigma_1}-\overline{\sigma}}\right) + \frac{(z-z'')(\sigma_2-\sigma_1)\sigma^2}{(\sigma_1-\sigma)(\sigma_2-\sigma)\frac{R}{2}(\sigma^2-1)}\right\} \\ &+ \frac{Y_0\sigma^2}{2\pi i\frac{R}{2}(\sigma_1-\sigma)(\sigma^2-1)} - \frac{(\overline{\alpha_1}+\overline{\beta_1})\sigma}{\frac{R}{2}(\sigma^2-1)}\end{aligned}$$

The above equation after simplifications and based on the relation $\alpha_1+\beta_1-\overline{\alpha_1}-\overline{\beta_1} = \frac{Y_0}{2\pi i}$ (see system (34)) results in the following:

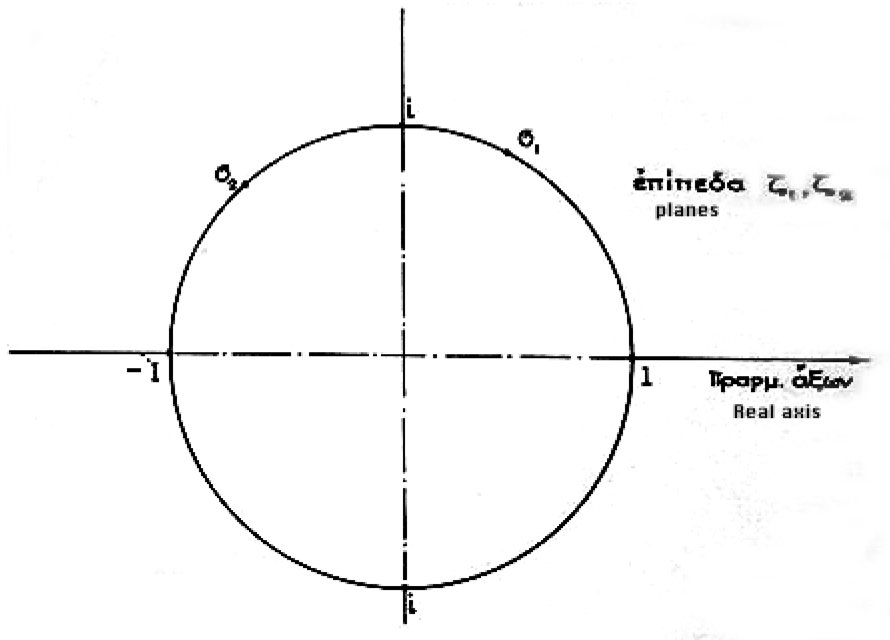

Figure 17: Correspondence between points z', z'' and σ_1, σ_2

$$\sigma_y = -\frac{p}{2\pi i}\ln\left(\frac{\sigma_2}{\sigma_1}\right) + \frac{p}{2\pi i}\left\{\ln\left(\frac{\sigma_2-\sigma}{\sigma_1-\sigma}\right) - \ln\left(\frac{\overline{\sigma_2}-\overline{\sigma}}{\overline{\sigma_1}-\overline{\sigma}}\right)\right\} \tag{64}$$

Setting,

$$\sigma_2 - \sigma = r_2 e^{i\phi_2} \qquad \overline{\sigma_2} - \overline{\sigma} = r_2 e^{-i\phi_2}$$
$$\sigma_1 - \sigma = r_1 e^{i\phi_1} \qquad \overline{\sigma_1} - \overline{\sigma} = r_1 e^{-i\phi_1}$$

We find

$$\begin{aligned}\ln\left(\frac{\sigma_2-\sigma}{\sigma_1-\sigma}\right) - \ln\left(\frac{\overline{\sigma_2}-\overline{\sigma}}{\overline{\sigma_1}-\overline{\sigma}}\right) &= \ln\left(\frac{r_2}{r_1}\right) + i(\phi_2-\phi_1) - \ln\left(\frac{r_2}{r_1}\right) + i(\phi_2-\phi_1)\\ &= 2i(\phi_2-\phi_1)\end{aligned}$$

and equation (64) takes the following final form:

$$\sigma_y = -\frac{p}{2\pi i}\left\{\ln\left(\frac{\sigma_2}{\sigma_1}\right) - 2i(\phi_2-\phi_1)\right\} \tag{65}$$

We examine below the following two cases:

a) The point σ is found outside the arc defined by $\sigma_1\sigma_2$ (outside the charged area), in which case we have the geometric image (Fig. 18).

One can easily deduce that in this case:

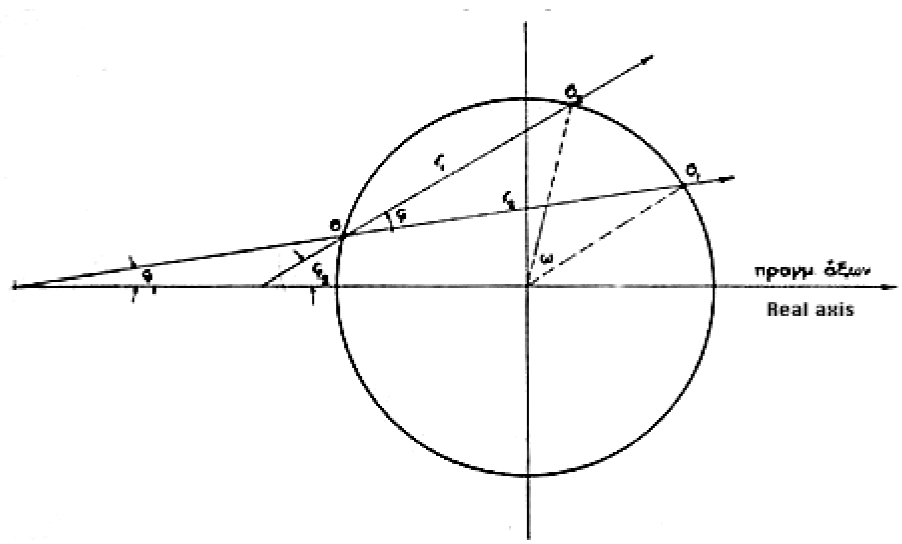

Figure 18: Unit circle with σ not in $\sigma_1\sigma_2$

$$\phi_2 - \phi_1 = \phi = \frac{\omega}{2} = \frac{1}{2i}\ln\left(\frac{\sigma_2}{\sigma_1}\right)$$

The above result substituted in (65) gives:

$$\sigma_y = -\frac{p}{2\pi i}\left[\ln\left(\frac{\sigma_2}{\sigma_1}\right) - \ln\left(\frac{\sigma_2}{\sigma_1}\right)\right] = 0$$

b) The point z is found inside the charge area hence σ lies between $\sigma_1\sigma_2$. In this case we will have the geometric image (Fig. 19), from which we conclude the following:

$$\omega' = \pi - \frac{\omega}{2} = (\phi_1 - \pi) + (\pi - \phi_2) = \phi_1 - \phi_2$$

hence

$$\phi_2 - \phi_1 = \frac{\omega}{2} - \pi = \frac{1}{2i}\ln\left(\frac{\sigma_2}{\sigma_1}\right) - \pi$$

The above result substituted in (5) gives

$$\sigma_y = -\frac{p}{2\pi i}\left[\ln\left(\frac{\sigma_2}{\sigma_1}\right) - \ln\left(\frac{\sigma_2}{\sigma_1}\right) + 2\pi i\right] = -p$$

The results of cases a) and b) were expected. The calculations were performed so that the operational details of the found functions could be shown. These functions being continuous on the entire planes ζ_1 and ζ_2 (except at the points ± 1) give limiting functions on the boundary which satisfy the initially posed conditions.

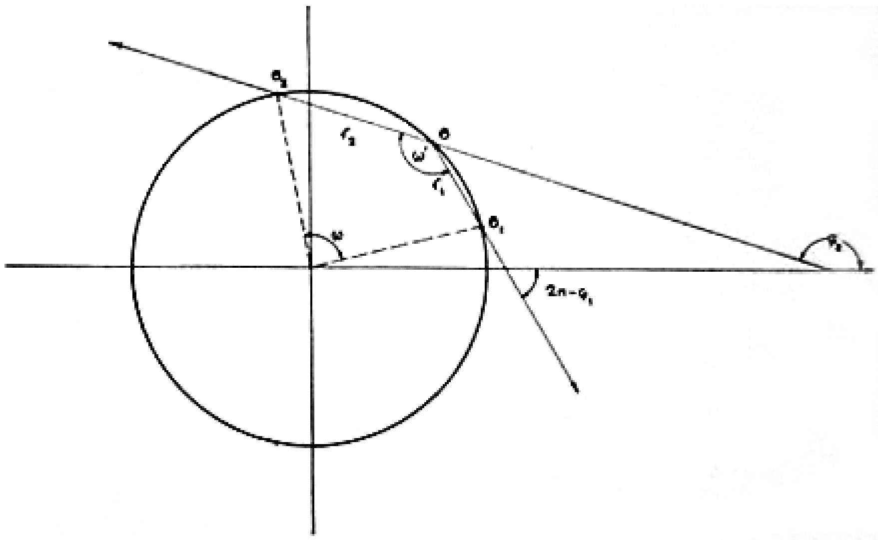

Figure 19: Unit circle with σ in $\sigma_1\sigma_2$

2. The calculation of τ_{xy} on the boundary is simpler, because it can be concluded immediately from the second of the limiting conditions (32) or by simple calculations, that $\tau_{xy} = 0$ on the entire boundary. (Exceptions are the usual points $\sigma = \pm 1$).

3. The calculation of σ_x on the boundary is a little involved because of the length of the used equations. Indeed, we have:

$$\sigma_x = 2\Re\left\{\rho_1^2 f_1'(\sigma) + \rho_2^2 f_2'(\sigma)\right\}$$

$$\begin{aligned}
\rho_1^2 f_1'(\sigma) + \rho_2^2 f_2'(\sigma) = {} & \frac{-\rho_1\rho_2 p}{2\pi i} \times \\
& \times \left[\frac{\ln\left(\frac{\sigma_2}{\sigma_1}\right)}{\sigma^2 - 1} + \ln\left(\frac{\sigma_2 - \sigma}{\sigma_1 - \sigma}\right) + \frac{(z - z'')(\sigma_2 - \sigma_1)\sigma^2}{(\sigma_1 - \sigma)(\sigma_2 - \sigma)\frac{R}{2}(\sigma^2 - 1)}\right] \\
& + \frac{Y_0 \sigma_1 \sigma \rho_1 \rho_2}{2\pi i \frac{R}{2}(\sigma_1 - \sigma)(\sigma^2 - 1)} + \frac{(\rho_1^2 \alpha_1 + \rho_2^2 \beta_1)\sigma}{\frac{R}{2}(\sigma^2 - 1)} \\
\overline{\rho_1^2 f_1'(\sigma)} + \overline{\rho_2^2 f_2'(\sigma)} = {} & \frac{-\rho_1\rho_2 p}{2\pi i} \times \\
& \times \left[\frac{\sigma \ln\left(\frac{\sigma_2}{\sigma_1}\right)}{\sigma^2 - 1} + \ln\left(\frac{\overline{\sigma_2} - \overline{\sigma}}{\overline{\sigma_1} - \overline{\sigma}}\right) + \frac{(z - z'')(\sigma_2 - \sigma_1)\sigma^2}{(\sigma_1 - \sigma)(\sigma_2 - \sigma)\frac{R}{2}(\sigma^2 - 1)}\right] \\
& - \frac{Y_0 \sigma^2 \overline{\rho_1 \rho_2}}{2\pi i \frac{R}{2}(\sigma_1 - \sigma)(\sigma^2 - 1)} - \frac{(\overline{\rho_1^2 \alpha_1} + \overline{\rho_2^2 \beta_1})\sigma}{\frac{R}{2}(\sigma^2 - 1)}
\end{aligned}$$

and by adding we get[28]

$$\sigma_x = \frac{p}{2\pi i}\left[\frac{-\rho_1\rho_2+\sigma^2\overline{\rho_1\rho_2}}{\sigma^2-1}\ln\left(\frac{\sigma_2}{\sigma_1}\right)-\rho_1\rho_2\ln\left(\frac{\sigma_2-\sigma}{\sigma_1-\sigma}\right)+\overline{\rho_1\rho_2}\ln\left(\overline{\frac{\sigma_2-\sigma}{\sigma_1-\sigma}}\right)\right]$$
$$-\frac{\alpha_{12}Y_0\sigma}{\alpha_{11}\pi iR(\sigma^2-1)}+\frac{Y_0\sigma\left(\rho_1\rho_2\sigma_1-\overline{\rho_1\rho_2}\sigma\right)}{\pi iR(\sigma_1-\sigma)(\sigma^2-1)} \tag{66}$$
$$-\frac{p\left(\rho_1\rho_2-\overline{\rho_1\rho_2}\right)(z-z'')(\sigma_2-\sigma_1)\sigma^2}{2\pi i(\sigma_1-\sigma)(\sigma_2-\sigma)\frac{R}{2}(\sigma^2-1)}$$

On the above expression (66) we perform the following transformations and substitutions:

$$\begin{aligned}
\rho_1 &= \gamma_1+i\delta_1 \quad \text{where } \delta_1>0\\
\rho_2 &= \gamma_2+i\delta_2 \quad \text{where } \delta_2>0\\
\sigma_1 &= e^{i\theta_1}\\
\sigma_2 &= e^{i\theta_2} \qquad \theta_2>\theta_1\\
\sigma &= e^{i\theta} \qquad \theta\neq\left(\frac{\pi}{2}\pm\frac{\pi}{2}\right)\\
Y_0 &= pR\left(\cos(\theta_1)-\cos(\theta_2)\right)
\end{aligned}$$

We also examine the expression

$$U=-\rho_1\rho_2\ln\left(\frac{\sigma_2-\sigma}{\sigma_1-\sigma}\right)+\overline{\rho_1\rho_2}\ln\left(\overline{\frac{\sigma_2-\sigma}{\sigma_1-\sigma}}\right)$$

We set

$$\begin{aligned}
\sigma_2-\sigma &= r_2e^{i\phi_2} \quad \text{hence } \overline{\sigma_2}-\overline{\sigma}=r_2e^{-i\phi_2}\\
\sigma_1-\sigma &= r_1e^{i\phi_1} \quad \text{hence } \overline{\sigma_1}-\overline{\sigma}=r_1e^{-i\phi_1}
\end{aligned}$$

from which we conclude:

$$U=(-\rho_1\rho_2+\overline{\rho_1\rho_2})\ln\left(\frac{r_2}{r_1}\right)-(-\rho_1\rho_2+\overline{\rho_1\rho_2})i(\phi_2-\phi_1)$$

or

$$U=-2i(\gamma_1\delta_2+\gamma_2\delta_1)\ln\left(\frac{r_2}{r_1}\right)-2i(\gamma_1\delta_1-\delta_1\delta_2)i(\phi_2-\phi_1)$$

With respect to the cases C.2.1 a) b) we have,

$$\begin{aligned}
(\theta-\theta_2)(\theta-\theta_1)>0 &\qquad \phi_2-\phi_1=\frac{\theta_2-\theta_1}{2}\\
(\theta-\theta_2)(\theta-\theta_1)<0 &\qquad \phi_2-\phi_1=\frac{\theta_2-\theta_1}{2}-\pi
\end{aligned}$$

[28] We also use the third equation from (34).

therefore

$$U_1 = U_2 + 2\pi i(\gamma_1\gamma_2 - \delta_1\delta_2) \tag{67}$$

Consequently, the found σ_x will have a jump discontinuity at the points z' and z'' of magnitude $(\gamma_1\gamma_2 - \delta_1\delta_2)p$.

According to the previous substitutions and transformations and after certain calculations expression (66) takes the following form.

$$\sigma_x = \frac{-p}{2\pi}\left\{(\gamma_1\delta_2 + \gamma_2\delta_1)A_1 + \frac{\cos(\theta_1) - \cos(\theta_2)}{\sin(\theta)}A_2\right\}$$

where[29]

$$\begin{aligned}A_1 &= [\theta_2 - \theta_1]\cot(\theta) + 2\ln\left(\frac{r_2}{r_1}\right)\\ &+ \frac{[\cos(\theta) - \cos(\theta_2)]\,[\sin(\theta - \theta_1) - \sin(\theta - \theta_2) + \sin(\theta_1 - \theta_2)]}{\sin(\theta)\,[1 - \cos(\theta - \theta_1)]\,[1 - \cos(\theta - \theta_2)]}\\ A_2 &= \frac{(\gamma_1\gamma_2 - \delta_1\delta_2)\,[1 - \cos(\theta - \theta_1)] - (\gamma_1\delta_2 + \gamma_2\delta_1)\sin(\theta - \theta_1)}{1 - \cos(\theta - \theta_1)} - \frac{\alpha_{12}}{\alpha_{11}}\end{aligned}$$

or after further calculations and simplifications the following final form

$$\begin{aligned}\sigma_x &= (\gamma_1\delta_2 + \gamma_2\delta_1)\left[\frac{(\theta_2 - \theta_1)\cos(\theta) + \sin(\theta_2) - \sin(\theta_1)}{\sin(\theta)} + 2\ln\left(\frac{r_2}{r_1}\right)\right]\\ &+ \left(\gamma_1\gamma_2 - \delta_1\delta_2 - \frac{\alpha_{12}}{\alpha_{11}}\right)\frac{\cos(\theta_1) - \cos(\theta_2)}{\sin(\theta)}\end{aligned} \tag{68}$$

Expression (68) gives the stress on the boundary as a function of the angle θ, which is connected as usual to the abscissa from the middle of the section based on the relation $x = R\cos(\theta)$.

We also observe that the expression (68) consists of two terms. The first stands only in the case of general anisotropy ($\gamma_1 \cdot \gamma_2 \neq 0$) and its magnitude becomes unbounded at the points which have abscissas equal to $R\cos(0)$, $R\cos(\theta_1)$, $R\cos(\theta_2)$ and $R\cos(\pi)$, because for θ tending to 0 or π, $\sin(\theta)$ vanishes, while for θ tending to θ_1 or θ_2 (given that $\ln\left(\frac{r_2}{r_1}\right) = \ln\left|\frac{\sin\left(\frac{\theta-\theta_2}{2}\right)}{\sin\left(\frac{\theta-\theta_1}{2}\right)}\right|$), $\left|\ln\left(\frac{r_2}{r_1}\right)\right|$ becomes unbounded (see Figs. 18, 19).

The second term gives σ_x tending to infinity for $\theta = 0$ and $\theta = \pi$ independently of the kind of anisotropy, a fact which happens because of the existence of a cusp on the boundary for these points.

After the above, we graph the diagram of the found σ_x on the boundary, considering the latter to be the superposition of two functions.

We also accept that the charge is symmetric with respect to y ($\sin(\theta_1) = \sin(\theta_2)$, $\cos(\theta_2) = -\cos(\theta_1)$), that $(\gamma_1\delta_2 + \gamma_2\delta_1) = A > 0$, as well as that $2\cos(\theta_1)\cdot(\gamma_1\gamma_2 - \delta_1\delta_2 - \alpha_{12}/\alpha_{11}) = B < 0$.

[29]Translator's Note: See footnote 24 on page 34.

Based on the above, we finally conclude:

$$\sigma_x = \frac{-p}{2\pi}\left[\frac{A(\theta_2-\theta_1)\cos(\theta)+B}{\sin(\theta)}\right] + \frac{-p}{\pi}A\ln\left|\frac{\sin\left(\frac{\theta-\theta_2}{2}\right)}{\sin\left(\frac{\theta-\theta_1}{2}\right)}\right|$$

The graphs of σ_x for the upper and lower sides are shown on Figs. 20 and 21, respectively.

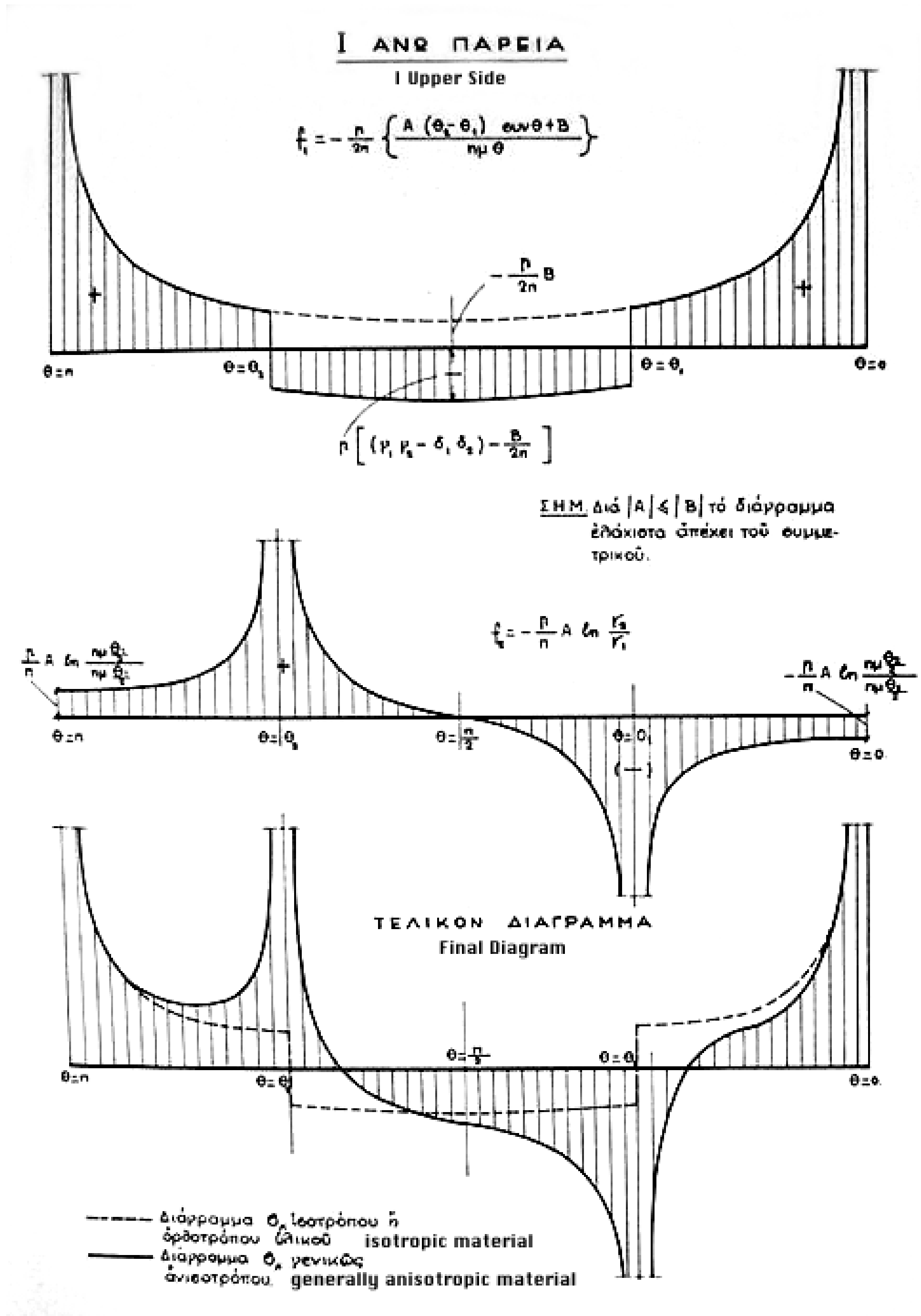

Figure 20: $\sigma_x(\theta)$ for upper disk side

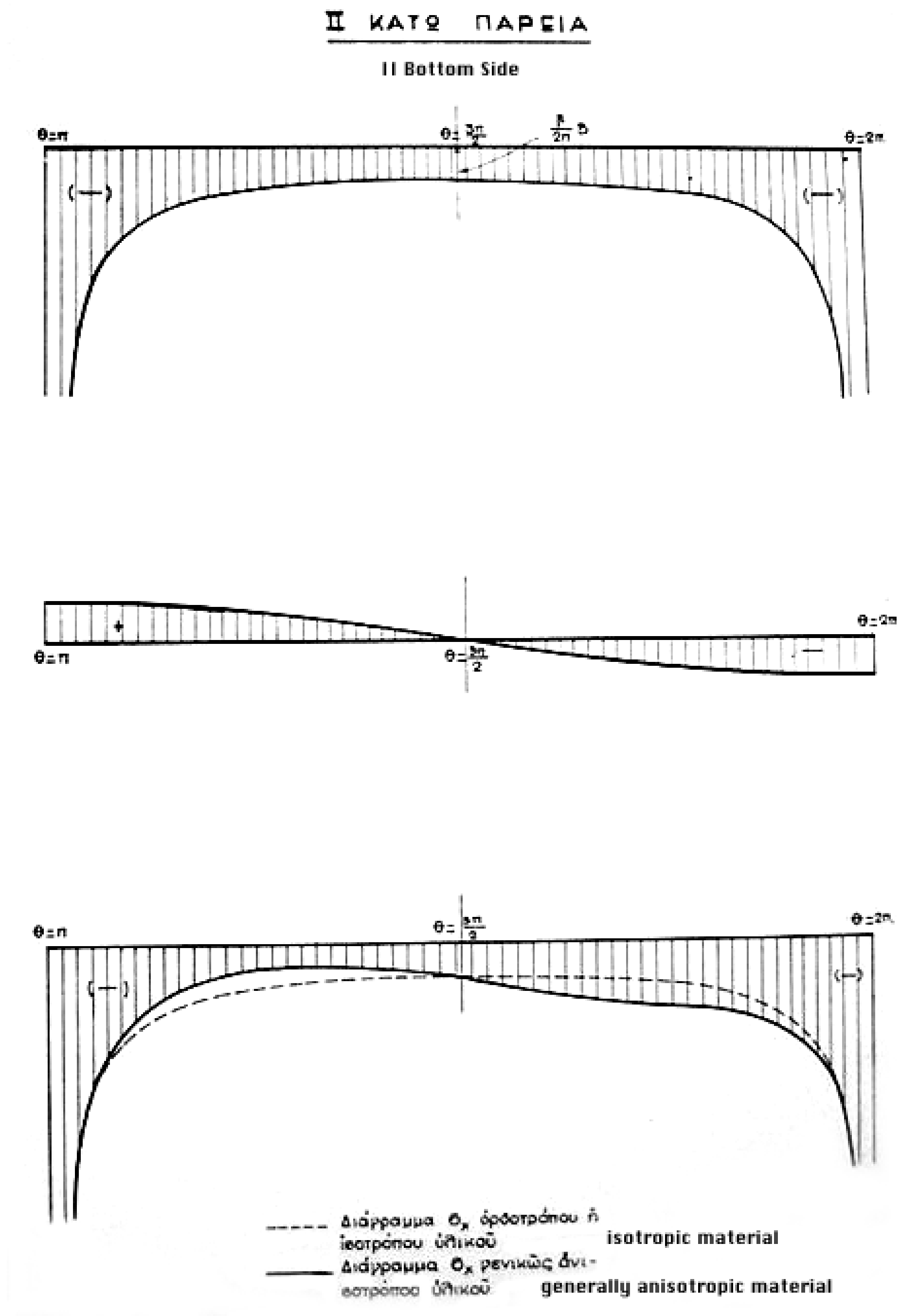

Figure 21: $\sigma_x(\theta)$ for lower disk side

C.3 Tangential Charge On the Boundary

1. The finding of the stress functions in this case is achieved equally simply and completely analogously.

The vanishing of τ_{xy} on the whole boundary is deduced immediately given that the expression for σ_y coincides with the first of the limiting conditions (54).

τ_{xy} (using calculations similar to those for the finding of σ_y for normal charge) is also found to vanish on the whole boundary except on the charged section on which it is equal to q. What remains for the research on the boundary is the calculation of σ_x which is done as follows.

We have: $\sigma_x = \rho_1^2\Psi_1''(z_1) + \rho_2^2\Psi_2''(z_2) + \overline{\rho_1^2\Psi_1''(z_1)} + \overline{\rho_2^2\Psi_2''(z_2)}$, where $\Psi_1''(z_1)$ and $\Psi_2''(z_2)$ are given by relations (57).

Setting $\zeta_1 = \zeta_2 = \sigma$ and $\overline{\zeta_1} = \overline{\zeta_2} = \sigma^{-1}$ we find the corresponding to (66) expression:

$$\begin{aligned}
\sigma_x = &\frac{q}{2\pi i}\left[\frac{-(\rho_1+\rho_2)+\sigma^2(\overline{\rho_1+\rho_2})}{\sigma^2-1}\ln\left(\frac{\sigma_2}{\sigma_1}\right)\right] \\
&+\frac{q}{2\pi i}\left[-(\rho_1-\rho_2)\ln\left(\frac{\sigma_2-\sigma}{\sigma_1-\sigma}\right)+(\overline{\rho_1}+\overline{\rho_2})\ln\left(\frac{\overline{\sigma_2}-\overline{\sigma}}{\overline{\sigma_1}-\overline{\sigma}}\right)\right] \\
&-\frac{\alpha_{13}X_0\sigma}{\alpha_{11}\pi i R(\sigma^2-1)} \\
&-\frac{q}{2\pi i}\frac{\left[(\rho_1+\rho_2)-(\overline{\rho_1}+\overline{\rho_2})\right](z-z'')(\sigma_2-\sigma_1)\sigma^2}{(\sigma_1-\sigma)(\sigma_2-\sigma)\frac{R}{2}(\sigma^2-1)} \\
&-\frac{X_0\sigma\left[(\rho_1+\rho_2)\sigma_1-(\overline{\rho_1}+\overline{\rho_2})\sigma\right]}{\pi i R(\sigma_1-\sigma)(\sigma^2-1)}
\end{aligned} \tag{69}$$

Knowing that we have[30]

$$\begin{aligned}
\rho_1 &= \gamma_1 + i\delta_1 \quad \text{where } \delta_1 > 0 \\
\rho_2 &= \gamma_2 + i\delta_2 \quad \text{where } \delta_2 > 0 \\
\sigma_1 &= e^{i\theta_1} \\
\sigma_2 &= e^{i\theta_2} \qquad \theta_2 > \theta_1 \\
\sigma &= e^{i\theta} \qquad \theta \neq \left(\frac{\pi}{2} \pm \frac{\pi}{2}\right) \\
\gamma_1 + \gamma_2 &= \frac{\alpha_{13}}{\alpha_{11}}
\end{aligned}$$

and also: (working in exactly similar way as in C.2.2)

$$\begin{aligned}
&-(\rho_1+\rho_2)\ln\left(\frac{\sigma_2-\sigma}{\sigma_1-\sigma}\right)+(\overline{\rho_1+\rho_2})\ln\left(\overline{\frac{\sigma_2-\sigma}{\sigma_1-\sigma}}\right) \\
&= -2i\left[(\delta_1+\delta_2)\ln\left(\frac{r_2}{r_1}\right)+(\gamma_1+\gamma_2)(\phi_2-\phi_1)\right]
\end{aligned} \tag{70}$$

[30] From the characteristic equation we deduce $\sum \rho_i = 2\frac{\alpha_{13}}{\alpha_{11}}$, hence $\gamma_1 + \gamma_2 = \frac{\alpha_{13}}{\alpha_{11}}$.

where

$$(\theta-\theta_2)(\theta-\theta_1)>0 \qquad \phi_2-\phi_1=\frac{\theta_2-\theta_1}{2}$$
$$(\theta-\theta_2)(\theta-\theta_1)<0 \qquad \phi_2-\phi_1=\frac{\theta_2-\theta_1}{2}-\pi$$

we get (69) in the following form[31].

$$\sigma_x=\frac{+q}{2\pi i}\left\{(\delta_1+\delta_2)A_1+\frac{\cos(\theta_2)-\cos(\theta_1)}{\sin(\theta)}A_2\right\}$$

with[32]

$$\begin{aligned} A_1 &= (\theta_2-\theta_1)\cot(\theta)+2\ln\left(\frac{r_2}{r_1}\right)\\ &+\frac{[\cos(\theta)-\cos(\theta_2)]\,[\sin(\theta-\theta_1)-\sin(\theta-\theta_2)+\sin(\theta_1-\theta_2)]}{\sin(\theta)\,[1-\cos(\theta-\theta_1)]\,[1-\cos(\theta-\theta_2)]}\\ A_2 &= \frac{\frac{\alpha_{13}}{\alpha_{11}}\,[1-\cos(\theta-\theta_1)]-(\delta_1+\delta_2)\sin(\theta-\theta_1)}{1-\cos(\theta-\theta_1)}+\frac{\alpha_{13}}{\alpha_{11}} \end{aligned}$$

or

$$\begin{aligned} \sigma_x &= \frac{q}{2\pi}\left[(\delta_1+\delta_2)\left(\frac{(\theta_2-\theta_1)\cos(\theta)+\sin(\theta_2)-\sin(\theta_1)}{\sin(\theta)}+2\ln\left(\frac{r_2}{r_1}\right)\right)\right]\\ &+2\frac{q}{2\pi}\frac{\alpha_{13}\,[\cos(\theta_1)-\cos(\theta_2)]}{\alpha_{11}\sin(\theta)} \end{aligned} \tag{71}$$

and if we accept for the sake of simplification a symmetric charge ($\theta_1+\theta_2=\pi$), then expression (71) is written as follows:

$$\sigma_x=\frac{+q}{2\pi}\left[(\delta_1+\delta_2)\left[(\theta_2-\theta_1)\cot(\theta)+2\ln\left(\frac{r_2}{r_1}\right)\right]+4\frac{\alpha_{13}\cos(\theta_1)}{\alpha_{11}\sin(\theta)}\right]$$

or

$$\sigma_x^I=\frac{q}{2\pi}\left[\frac{(\delta_1+\delta_2)(\theta_2-\theta_1)\cos(\theta)+4\frac{\alpha_{13}}{\alpha_{11}}\cos(\theta_1)}{\sin(\theta)}+2(\delta_1+\delta_2)\ln\left(\frac{r_2}{r_1}\right)\right] \tag{72a}$$

for $(\theta-\theta_2)(\theta-\theta_1)>0$, and

$$\sigma_x^{II}=\sigma_x^I-\frac{\alpha_{13}}{\alpha_{11}}\cdot q \tag{72b}$$

for $(\theta-\theta_2)(\theta-\theta_1)<0$

[31] At the points θ_1 and θ_2, σ_x will have a jump discontinuity because of (70) (as will σ_x for normal charge) of magnitude equal to $(\gamma_1+\gamma_2)q$.

[32] Translator's Note: See footnote 24 on page 34.

From the investigation of expressions (72a) and (72b) we immediately conclude that σ_x becomes unbounded at the points corresponding to 0, θ_1, θ_2 and π regardless of the elastic behavior of the material.

Worthy to be noted is that the first function inside the bracket is discontinuous at the points θ_1 and θ_2 (will have a jump discontinuity of magnitude $(\gamma_1 + \gamma_2)q$) only for the generally anisotropic material, being continuous for the orthotropic or isotropic material.

The result of course loses its significance in the end, because the stress becomes unbounded at the aforementioned points because of the second term of the function, nevertheless it is mentioned as a sample of the behavior of the anisotropic material.

For the graphing of the functions we accept:

$$\frac{\alpha_{13}}{\alpha_{11}} << \delta_1 + \delta_2 \qquad \theta_1 + \theta_2 = \pi \qquad 0 < \theta_1 < \theta_2 < \pi \qquad \alpha_{13} > 0$$

We also observe that the first function always vanishes at some point even for $\theta_2 - \theta_1 \to 0$

Indeed we have: $\theta_2 - \theta_1 = \pi - 2\theta_1 = 2\left(\frac{\pi}{2} - \theta_1\right)$, hence the function can be written:

$$2\left(\delta_1 + \delta_2\right)\left(\frac{\pi}{2} - \theta_1\right)\cos(\theta) + 4\frac{\alpha_{13}}{\alpha_{11}}\cos(\theta_1)$$

and because $\frac{\pi}{2} - \theta_1 > \cos(\theta_1)$ as well as $\delta_1 + \delta_2 > 2(\gamma_1 + \gamma_2) = 2\frac{\alpha_{13}}{\alpha_{11}}$ the equation $\cos(\theta_0) = \frac{2(\gamma_1+\gamma_2)\cos(\theta_1)}{(\delta_1+\delta_2)\left(\theta_1 - \frac{\pi}{2}\right)}$ always has a solution for $\frac{\pi}{2} < \theta_0 < \pi$.

The plotted graphs 20, 21, 22 and 23 for all charge cases and for the values of the variable z on the boundary of the crack or the same for the values of ζ on the unit circle, provide only a meager image of the significance of the general solutions (28), (39) and (56).

The fact that the solutions for all the problems of the anisotropic half space (which, as far as the author is concerned, have not been expressed elsewhere) are deduced as special cases of the solutions (39) and (56) as well as the fact that from these can be deduced conclusions which concern the diet of the anisotropic plane disk, shows on one hand the power of the used methods and on the other the significance of the found solutions.

Concluding we mention that the solution in the area of the Mathematical Theory of Anisotropic Elasticity for all the charge problem cases in this work in closed form is a very rare situation, not only for the area used in the current work, but also for most of the simplest of problems in the Plane - Isotropic Elasticity.

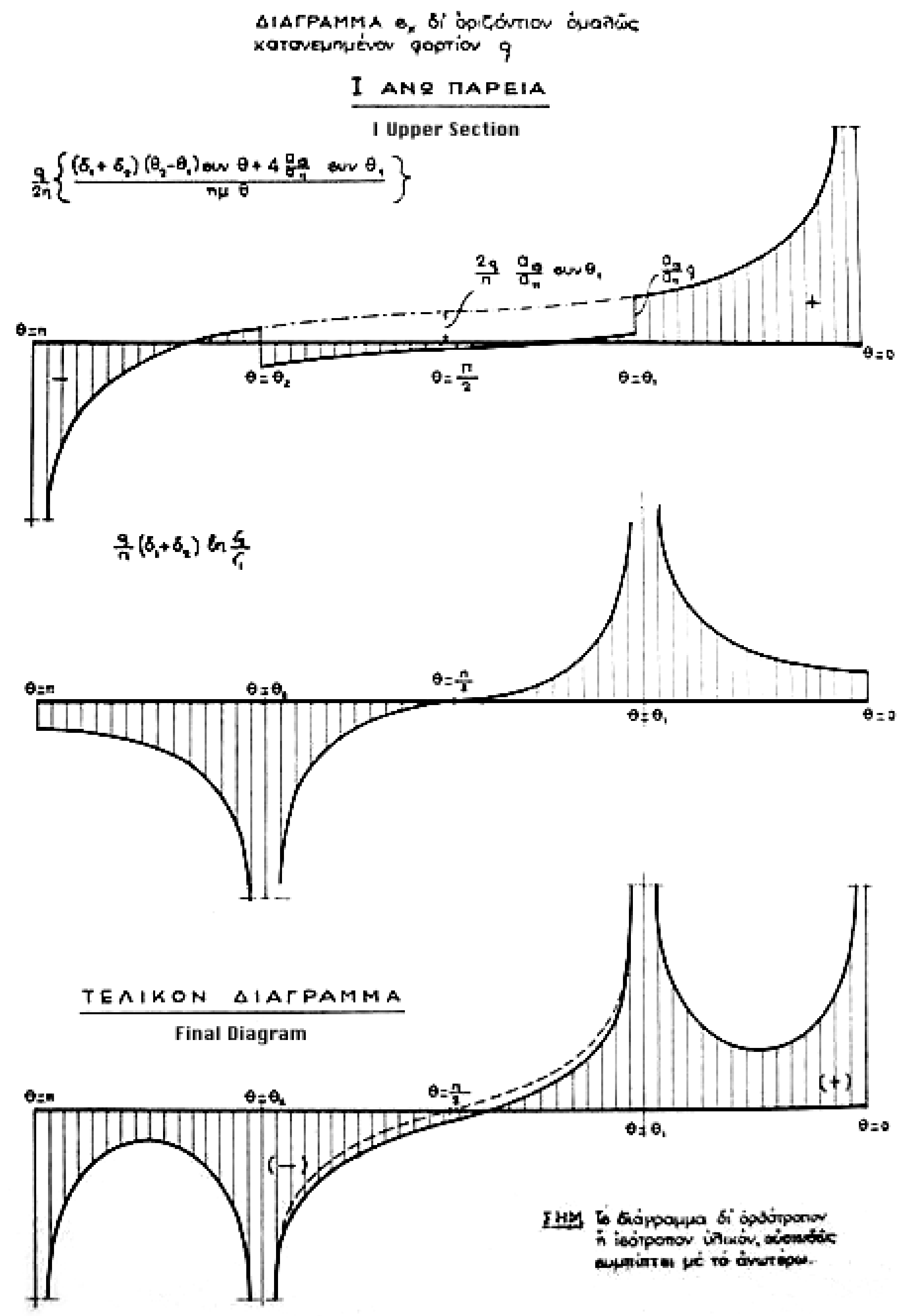

Figure 22: $\sigma_x(\theta)$ for upper disk side

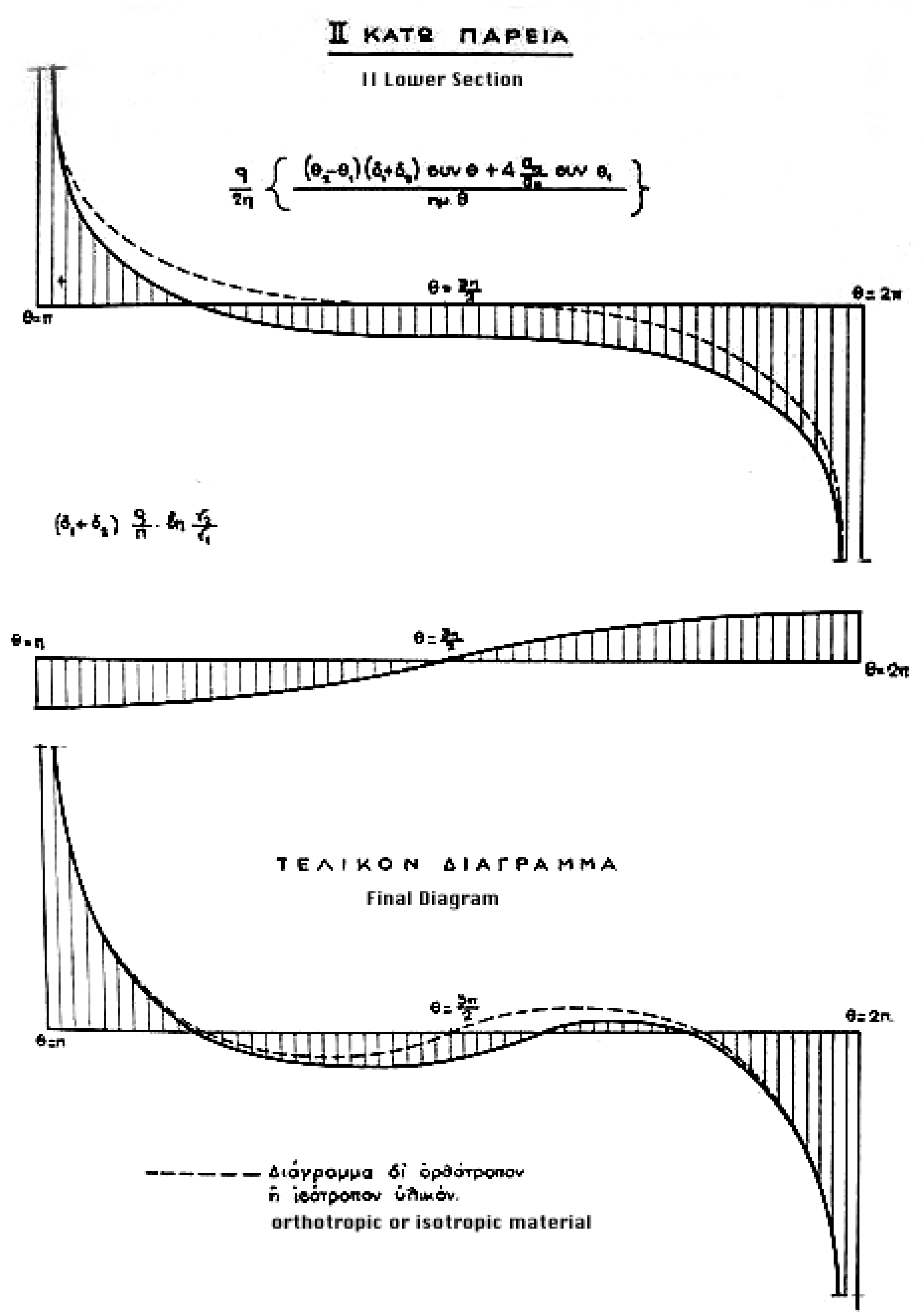

Figure 23: $\sigma_x(\theta)$ for lower disk side

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