Stability Condition of Continuous Takagi-Sugeno Fuzzy System based on Fuzzy Lyapunov Function

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Abstract— This paper deals with the stability of continuous Takagi-Sugeno (T-S) fuzzy models. Using non-quadratic Lyapunov function, new sufficient stability criteria is established in terms of Linear Matrix Inequality. Finally, numeric simulations are given to validate the developed approach.

Keywords— T-S fuzzy system, Linear Matrix Inequalities LMIs, Fuzzy Lyapunov Function.

I. INTRODUCTION

Fuzzy control systems have experienced a big growth of industrial applications in the recent decades, because of their reliability and effectiveness. Many researches are investigated on the Takagi-Sugeno models [1] last decades. Two classes of Lyapunov functions are used to analysis these systems: quadratic Lyapunov functions and non-quadratic Lyapunov ones which are less conservative than first class. Many researches are investigated with non-quadratic Lyapunov functions [5]–[8].

In this paper, a new stability conditions for Takagi-Sugeno sufficient fuzzy models based on the LMI approach and the use of Lyapunov non-quadratic function are presented. The presented method is less conservative than existing results.

The organization of the paper is as follows. In section 2, we present the problem formulation and we give some preliminaries which are needed to derive results. Section 3 will be concerned to stability analysis for Takagi-Sugeno fuzzy systems. Illustrative examples are given in section 4 for a comparison of previous results to demonstrate the advantage of proposed method. Finally section 5 makes conclusion.

II. SYSTEM DESCRIPTION AND PRELIMINARIES

Consider a Takagi-Sugeno fuzzy continuous model for a nonlinear system as follows:

\[ IF \ z_i(t) \ is \ M_j, \ and \ ...and \ z_p(t) \ is \ M_q, \ THEN \ \dot{x}(t) = A_i x(t) + B_i u(t) \quad i = 1, ..., r \]

where \( M_i \) (i = 1, 2, ..., r, j = 1, 2, ..., p) is the fuzzy set and \( r \) is the number of model rules.

\[ x(t) \in \mathbb{R}^n \] is the state vector, \( u(t) \in \mathbb{R}^m \) is the input vector, \( A_i \in \mathbb{R}^{n \times n}, \ B_i \in \mathbb{R}^{n \times m}, \) and \( z_i(t), ..., z_p(t) \) are known premise variables.

The final outputs of the fuzzy systems are:

\[ \dot{x}(t) = \sum_{i=1}^{r} h_i(z(t)) \{ A_i x(t) + B_i u(t) \} \quad (2) \]

where

\[ z(t) = [z_1(t), z_2(t), ..., z_p(t)] \]

\[ h_i(z(t)) = w_i(z(t)) \sum_{i=1}^{r} w_i(z(t)) \]

\[ w_i(z(t)) = \prod_{j=1}^{r} M_{ij}(z_j(t)) \]

for all \( t \). The term \( M_{ij}(z_j(t)) \) is the grade of membership of \( z_j(t) \) in \( M_{ij} \).

Since

\[ \sum_{i=1}^{r} w_i(z(t)) = 0 \]

\[ w_i(z(t)) \geq 0, \quad i = 1, 2, ..., r \]

We have

\[ \sum_{i=1}^{r} h_i(z(t)) = 1 \quad \text{For all } t \]

\[ h_i(z(t)) \geq 0, \quad i = 1, 2, ..., r \]

The time derivative of premise membership functions is given by:

\[ \frac{\partial h_i}{\partial z}(t) = \frac{\partial^2 z(t)}{\partial x(t)} \frac{dx(t)}{dt} = \sum_{i=1}^{p} \frac{\partial^2 z_j}{\partial x_j(t)} \frac{dx(t)}{dt} \]

We have the following property:

\[ \sum_{i=1}^{r} \dot{h}_i(z(t)) = 0 \quad (4) \]

The open-loop system is given by the equation (5)

\[ \dot{x}(t) = \sum_{i=1}^{r} h_i(z(t)) A_i x(t) \quad (5) \]
The fuzzy controller design is to determine the local feedback gains $F_i$ in the consequent parts.

**Assumption 1**

The time derivative of the premises membership function is upper bounded such that $|\dot{h}_k| \leq \phi_k$, for $k = 1, \ldots, r$, where, $\phi_k, k = 1, \ldots, r$ are given positive constants.

**Lemma 1 [3]**

The Takagi-Sugeno fuzzy system (5) is asymptotically stable if there exists $P = P^T > 0$ satisfying:

$$A_i^T P + P A_i < 0, \quad i \in \{1, \ldots, r\}$$  \hspace{1cm} (6)

**Lemma 2 [4]**

Assume that $|\dot{h}_k| \leq \phi_k$, $k = 1, \ldots, r$. The Takagi-Sugeno fuzzy system (5) is stable if the following LMIs are satisfied:

$$P_i = P_i^T > 0, \quad i = 1, \ldots, r$$ \hspace{1cm} (7)

$$P_i - P_j \geq 0, \quad i = 1, \ldots, r$$ \hspace{1cm} (9)

where $i,j \in \{1, \ldots, r\}$, $P_i = \sum_{k=1}^{r} \phi_k P_i$ and $\phi_k$ are scalars.

**Lemma 3 [4]**

Assume that $|\dot{h}_k| \leq \phi_k$, $k = 1, \ldots, r$. The Takagi-Sugeno fuzzy system (5) is stable if the following LMIs are satisfied:

$$P_i - P_j \geq 0, \quad i = 1, \ldots, r$$ \hspace{1cm} (8)

$$P_i = P_i^T > 0, \quad i = 1, \ldots, r$$ \hspace{1cm} (10)

where $i, j \in \{1, \ldots, r\}$, $\bar{P}_i = \sum_{k=1}^{r} \phi_k (P_i - P_j)$ and $\phi_k$ are scalars.

**Lemma 4 [8]**

Assume that $|\dot{h}_k| \leq \phi_k$, $k = 1, \ldots, r$. The Takagi-Sugeno fuzzy system (5) is stable if the following LMIs are satisfied:

$$P_i = P_i^T > 0, \quad i = 1, \ldots, r$$ \hspace{1cm} (11)

$$P_i - P_j \geq 0, \quad i = 1, \ldots, r$$ \hspace{1cm} (12)

where $i, j \in \{1, \ldots, r\}$, $\bar{P}_i = \sum_{k=1}^{r} \phi_k (P_i - P_j)$ and $\phi_k$ are scalars, and $X = X^T$.

**III. MAIN RESULT**

Consider the open-loop system (5). The aim of the next section is to find conditions for the stability of the unforced T-S fuzzy system by using the Lyapunov theory.

**Theorem 1**

Under assumption 1 and for $0 < \varepsilon < 1$, the T-S fuzzy system (5) is stable if there exist positive definite symmetric matrices $P_i, k = 1, 2, \ldots, r$, matrix $R = R^T$ such that the following LMIs hold:

$$P_i + \varepsilon R > 0, \quad k = 1, 2, \ldots, r$$  \hspace{1cm} (15)

$$P_i + \frac{1}{2} \left( A_i^T (P_i + \varepsilon R) + (P_i + \varepsilon R) A_i \right) \leq 0, \quad i \leq j$$ \hspace{1cm} (16)

where $P_i = \sum_{k=1}^{r} \phi_k (P_i + R)$

**Proof**

Let consider the Lyapunov function in the following form:

$$V(x(t)) = \sum_{i=1}^{r} \dot{h}_i(z(t)) \cdot \dot{V}_i(x(t))$$  \hspace{1cm} (17)

where

$$V_i(x(t)) = x^T(t)(P_i + \varepsilon R)x(t), \quad k = 1, 2, \ldots, r$$

and $P_i = P_i^T > 0, R = R^T$, $0 < \varepsilon < 1$, and $(P_i + \varepsilon R) \geq 0$.

This candidate Lyapunov function satisfies

1) $V(x(t)) = C^T$

2) $V(0) = 0$ and $V(x(t)) \geq 0$ for $x(t) \neq 0$.

3) $\|x(t)\| \to \infty \Rightarrow V(x(t)) \to \infty$.

The time derivative of $V(x(t))$ with respect to $t$ along the trajectory of the system (4) is given by:

$$\dot{V}(x(t)) = \sum_{i=1}^{r} \dot{h}_i(z(t)) \dot{V}_i(x(t)) + \sum_{i=1}^{r} \dot{h}_i(z(t)) \dot{V}_i(x(t))$$ \hspace{1cm} (18)

The equation (18) can be rewritten as,

$$\dot{V}(x(t)) = Y_1(x,z) + Y_2(x,z)$$ \hspace{1cm} (19)

where

$$Y_1(x,z) = x^T(t) \left( \sum_{i=1}^{r} \dot{h}_i(z(t)) \cdot (P_i + \varepsilon R) \right) x(t)$$ \hspace{1cm} (20)

$$Y_2(x,z) = \frac{1}{2} \sum_{i=1}^{r} \sum_{j=1}^{r} \dot{h}_i(z(t)) \dot{h}_j(z(t)) x^T(t) \left( A_i^T (P_i + \varepsilon R) + (P_i + \varepsilon R) A_i \right) \times$$ \hspace{1cm} (21)

$$\dot{h}_i(z(t)) \dot{h}_j(z(t)) x^T(t) \left( A_i^T (P_i + \varepsilon R) + (P_i + \varepsilon R) A_i \right) x(t)$$

Then, based on assumption 1, an upper bound of $Y_1(x,z)$ obtained as:

$$Y_1(x,z) \leq \sum_{i=1}^{r} \dot{h}_i(z(t)) (1 - \varepsilon) R = \bar{R} \leq 0$$ \hspace{1cm} (22)

Based on (4), it follows that $\sum_{i=1}^{r} \dot{h}_i(z(t))(1 - \varepsilon) R = \bar{R} = 0$ where $R$ is any symmetric matrix of proper dimension.
Adding $\bar{R}$ to (20), then
$$Y_t(x,z) \leq \sum_{i=1}^{r} \phi_i \cdot x(t)^T (P_i + \varepsilon R) x(t)$$
(23)

Then,
$$V(x(t)) \leq \sum_{i=1}^{r} \phi_i \cdot x(t)^T (P_i + \varepsilon R) x(t)$$
$$+ \frac{1}{2} \sum_{i=1}^{r} \sum_{j=1}^{r} h_i h_j x(t) \cdot \left[ A_i^T (P_i + \varepsilon R) + (P_i + \varepsilon R) A_j \right] x(t)$$
$$+ A_i^T (P_i + \varepsilon R) + (P_i + \varepsilon R) A_i \right] x(t)$$
$$= x(t)^T \left( \sum_{i=1}^{r} \phi_i \cdot (P_i + \varepsilon R) \right) x(t)$$
$$+ x(t)^T \frac{1}{2} \sum_{i=1}^{r} \sum_{j=1}^{r} h_i h_j \cdot \left[ A_i^T (P_i + \varepsilon R) + (P_i + \varepsilon R) A_i \right] x(t)$$
$$+ A_i^T (P_i + \varepsilon R) + (P_i + \varepsilon R) A_i \right] x(t)$$

If (16) holds, then $V(x(t)) < 0$ and (5) is stable. ■

IV. NUMERICAL EXAMPLES

In order to show the improvements of proposed approaches over some existing results, in this section, we present two numerical examples. The first one concerns the feasible area for a T-S fuzzy system. Indeed, we compare the fuzzy Lyapunov approach (Theorem 1) with the Lemma 1 in [3], Lemma 2 and Lemma 3 in [4], and Lemma 4 in [8]. However, the second example presents the state variables evolution for a two rules T-S fuzzy system.

Example 1. Consider the following T-S fuzzy system:
$$\dot{x}(t) = \sum_{i=1}^{r} h_i (z(t)) A_i x(t)$$
(24)

with: $r = 4$
$$A_1 = \begin{bmatrix} -5 & -4 \\ -1 & a \end{bmatrix}, \quad A_2 = \begin{bmatrix} -4 & -4 \\ \frac{1}{5} (3b - 2) & \frac{1}{5} (3b - 4) \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -3 & -4 \\ \frac{1}{5} (2b - 3) & \frac{1}{5} (2a - 6) \end{bmatrix}, \quad A_4 = \begin{bmatrix} -2 & -4 \\ b & -2 \end{bmatrix},$$

where $a \in [-10, 0], \quad b \in [0, 500]\]

and the premise functions are given by:
$$h_1 = \alpha_1 (x_1) \alpha_2 (x_2), \quad h_2 = \alpha_1 (x_1) \beta_1 (x_2)$$
$$h_3 = \beta_1 (x_1) \alpha_2 (x_2), \quad h_4 = \beta_1 (x_1) \beta_2 (x_2)$$

such that,
$$\alpha_i (x_i) = \begin{cases} \left(1 - \sin(x_i)\right)/2, & \text{for } |x_i| \leq \frac{\pi}{2}, \\
0, & \text{for } x_i > \frac{\pi}{2}, \\
1, & \text{for } x_i < -\frac{\pi}{2}, \\
\end{cases}$$

$$\beta_i (x_i) = 1 - \alpha_i (x_i).$$
The stability of this system is checked using Lemma 1 in [3], Lemma 2 and Lemma 3 in [4], Lemma 4 in [8] and proposed Theorem 1, respectively, for several values of pair \((a,b)\), \(a \in [-10,0]\), \(b \in [0,500]\) and \(h_i(z(t)) < 0.85\).

It is easy to see that if \(\varepsilon = 1\) and \(R = 0\), then Lemma 2 is recovered, if \(\varepsilon = 1\) and \(R = -P\), so is Lemma 3, and if \(\varepsilon = 1\) so is Lemma 4. Therefore, Lemma 2, 3 and 4 are particular cases of Theorem 1. Fig.1 shows the stability margin when we use a Common Quadratic Lyapunov Function. This method is more conservative.

Fig.2. and Fig.3 show that the Fuzzy Lyapunov Function promotes a larger stability margin than the Common Quadratic Lyapunov Function.

Fig. 4 presents a less conservative condition given by the Lemma 4 [8]. This result is obtained by introducing of a single variable \(R\).

Fig. 5 shows the proposed condition stability presented in Theorem 1. Note that the proposed condition involved all previous ones with a larger stability region. With the introduction of a positive constant \(\varepsilon < 1\), which multiply the variable \(R\), a new less conservative condition is obtained.

**Example 2.** Consider the following T-S fuzzy system:

\[
\dot{x}(t) = \sum_{i=1}^{r} h_i(z(t))A_i x(t)
\]

with: \(r = 2\)

the premise functions are given by:

\[
h_i(x_i(t)) = \frac{1 + \sin x_i(t)}{2} ; \quad h_i(x_i(t)) = \frac{1 - \sin x_i(t)}{2} ;
\]

\[
A_1 = \begin{bmatrix} -5 & -4 \\ -1 & -2 \end{bmatrix} ; \quad A_2 = \begin{bmatrix} -2 & -4 \\ 20 & -2 \end{bmatrix} ;
\]

Figure 6 shows the evolution of the state variables. As can be seen, the conservatism reduction leads to very interesting results regarding fast convergence of this Takagi-Sugeno fuzzy system.

This paper provided a new condition for the stability of T-S fuzzy systems in terms of a combination of the LMI approach and the use of non-quadratic Lyapunov function as Fuzzy Lyapunov function.

The stability condition proposed in this note is less conservative than some of those in the literature, which has been illustrated via example.

**REFERENCES**


