GEOMETRIC PROPERTIES OF SOME BANACH SPACES ON HYPERGROUPS

KOSSI R. ETSE[1], ANATÉ K. LAKMON[2], YAOGAN MENSAH[3,*]

Abstract. In this paper, we study some geometric properties of the Fourier space of a hypergroup and other related Banach spaces. We are mainly concerned by the Radon-Nikodym property, the Dunford-Pettis property and the Schur property. Among other results, we proved that if H is a commutative hypergroup, then the Fourier space of $A(H)$ has the Dunford-Pettis property; if H is a compact hypergroup then $A(H)$ has the Schur property and consequently the Dunford-Pettis property. We also showed that the Figà-Talamanca–Herz space $A_p(H)$ does not have the Schur property if H is not compact.

1. Introduction

The Fourier algebra $A(G)$ and the Fourier-Stieltjes algebra $B(G)$ of a locally compact group G is the center of much attention from researchers in harmonic analysis since Eymard published his article [8] which deals with these objects. Many authors have examined conditions under which certain Banach spaces related to the Fourier algebra of the locally compact group G possess specific geometric properties such as the Radon-Nikodym property (RNP), the Dunford-Pettis property (DDP) and the Schur property (SP). For instance, in [15], Lau and Ülger extensively examined the aforementioned properties for the algebras $A(G)$ and $B(G)$ with C^*-algebras techniques. In the same sense, we may cite the work by Taylor [22] who studied the geometry of the Fourier algebras on locally compact groups with atomic representations. Subsequently, Miao [16] investigated these properties for the Figà -Talamanca-Herz (FTH) algebras $A_p(G)$ the category of which generalizes the category of Fourier algebras. It is noteworthy to mention the significant works of E. Granirer in the study of geometric properties of Banach spaces related to the Fourier algebra of various classes of groups [11, 12, 13]. In [9], Finet investigated the RNP for a subspace of the Lebesgue space $L^1(K)$, where K is a compact hypergroup.

Hypergroups are topological spaces which, without being groups, exhibit some of the characteristic structures of groups. These structures include the potential for defining convolution on the space of all finite regular Borel measures on these toological spaces, similar to the group case. It was Dunkl [6], Jewett [14] and Spector [19] who, independently of each other, initiated the study of hypergroups through their papers in the 1970s. Hypergroups generalize locally compact groups in various aspects. Consequently, any result from group harmonic analysis could have it analogous version (possibly under additional conditions)

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in the realm of hypergroups, particularly in the context of commutative hypergroups and compact hypergroups. Details regarding the theory of hypergroups, standard examples and most of their fundamental properties can be found in the book [2] by Bloom and Heyer.

Muruganandam studied several hypergroups the Fourier spaces of which are Banach algebras under pointwise multiplication. For commutative hypergroups, he successfully constituted a list of sufficient conditions for the Fourier space to be a Banach algebra in [17]. Additionally, in the article [18], he introduced a new class of hypergroups called spherical hypergroups, along with a subclass named ultraspherical hypergroups which includes double coset hypergroups. He established that the Fourier spaces of ultraspherical hypergroups are Banach algebras via pointwise multiplication. Vrem [23] also conducted harmonic analysis on compact hypergroups and characterized their Fourier spaces. In [1], the author introduce the Figá-Talamanca–Herz space $A_p(H)$ for a locally compact measured hypergroup H. The Chapter 5 of Degenfeld-Schonburg's thesis [3] on multipliers for hypergroups is dedicated to these algebras in the case of commutative hypergroups.
In [7], the authors investigate the RNP for Fourier spaces and other spaces related to commutative hypergroups and compact hypergroups. The present article aims to explore the geometric properties (mainly the DPP and the SP) for the Fourier spaces of hypergroups and other spaces related to these Fourier spaces.
The rest of the paper is organized as follows. In Section 2, we provide some definitions, facts, and notations which are essential in the article. Section 3 is devoted to an overview of the RNP, the DPP and the SP. In Section 4, we investigate the SP and the DPP for the Fourier spaces of hypergroups. In Section 5, we obtain, among other results, a necessary condition on a hypergroup in order that its Figà-Talamanca–Herz space has the SP.

2. Hypergroups : Definitions and basic facts

Let H be a locally compact Hausdorff space. Denote by $C(H)$, $C_b(H)$, $C_c(H)$ and $C_0(H)$ the space of complex-valued continuous functions on H, the space of the space of complex-valued continuous functions with compact support and the space of complex-valued continuous functions which vanish at infinity respectively. Denote by $M(H)$ the set of bounded Radon measures on H. The topology on $M(H)$ is given by the weak topology $\sigma(M(H), C_b(H))$. Let $M^1(H)$ denote the space of all probability measures on H equipped with the weak topology and let $M_+(H)$ denote the subspace of $M(H)$ consisting of positive measures.
Let $\mathfrak{K}(H)$ denote the space of all not empty compact subsets of H. For subsets U and V of H, set

$$\mathfrak{K}_U(V) = \{A \in \mathfrak{K}(H) : A \cap U \neq \emptyset, A \subset V\}.$$

The set $\mathfrak{K}(H)$ is given the topology generated by the sets $\mathfrak{K}_U(V)$ where U and V are open subsets of H. This topology is called the Michael topology on $\mathfrak{K}(H)$ [2, page 7].

We recall the following definition of hypergroup from [18] using Jewett's axioms [14].

Definition 2.1. *A nonempty locally compact Hausdorff space H is called a* hypergroup*, if the following conditions hold.*

H1: *There exists a binary operation* $*$ *called* convolution *on* $M(H)$ *under which* $M(H)$ *is an associative algebra. Moreover, for every* $x, y \in H$, *the product* $\delta_x * \delta_y$ *is a probability measure and the mapping* $(x, y) \to \delta_x * \delta_y$ *is continuous from* $H \times H$ *into* $M^1(H)$.

H2: *There exists an element (necessarily unique)* e *in* H *such that* $\delta_e * \delta_x = \delta_x * \delta_e = \delta_x$ *for all* $x \in H$.

H3: *There exists a (necessarily unique) homeomorphism* $x \to x^-$ *of* H *called* involution *satisfying the following :*

(1) $(x^-)^- = x$ *for all* $x \in H$.

(2) *If* μ^- *is defined by* $\int_H f(x) d\mu^-(x) = \int_H f(x^-) d\mu(x)$ *for all* $f \in C_c(H)$, *then* $(\delta_x * \delta_y)^- = \delta_{y^-} * \delta_{x^-}$ *for all* $x, y \in H$.

(3) e *belongs to* $supp(\delta_x * \delta_y)$ *if and only if* $y = x^-$.

H4: *For every* $x, y \in H$, $supp(\delta_x * \delta_y)$ *is compact. Moreover, the mapping* $(x, y) \to supp(\delta_x * \delta_y)$ *is continuous from* $H \times H$ *into* $\mathfrak{K}(H)$, *with respect to the Michael topology.*

For a continuous function f on H, $f(x * y)$ is defined by

$$f(x * y) = \langle f, \delta_x * \delta_y \rangle = \int_H f(z) d(\delta_x * \delta_y)(z).$$

If f is continuous on H and $x \in H$, the left translation $L_x f$ of f by x is defined by

$$L_x f(y) = f(x * y).$$

If f is a function on H, set

$$\check{f}(x) = f(x^-) \text{ and } \tilde{f}(x) = \overline{f(x^-)}.$$

Let H be a hypergroup with a left Haar measure. The Banach spaces $L^p(H)$, $1 \leq p \leq \infty$, are understood as usual.
For $f, g \in C_c(H)$, define the convolution $f * g$ by

$$(f * g)(x) = \int_H f(x * y) g(y^-) dy.$$

It is well known that, for $1 \leq p \leq \infty$, if $f \in L^1(H)$ and $g \in L^p(H)$ $f * g \in L^p(H)$ and $\|f * g\|_p \leq \|f\|_1 \|g\|_p$.

For a Hilbert space $\mathcal{H}$, denote by $B(\mathcal{H})$ the involutive Banach algebra of all bounded linear operators on $\mathcal{H}$.

Definition 2.2. [2, Definition 2.1.1] *We refer to* π *as a* representation *of the hypergroup* H *in some Hilbert space* $\mathcal{H}_\pi$ *if*

(1) π *is a* $*$*-homomorphism from the involutive Banach algebra* $M(H)$ *into* $B(\mathcal{H}_\pi)$,
(2) $\pi(\delta_e) = I$, *where* I *is the identity operator,*
(3) *For elements* $\xi, \eta \in \mathcal{H}_\pi$, *the mapping* $\mu \longmapsto \langle \pi(\mu)\xi, \eta \rangle_{\mathcal{H}_\pi}$ *is continuous on* $M_+(H)$ *with respect to the weak topology.*

The Hilbert space $\mathcal{H}_\pi$ is called the representation space of π. A representation π is said to be unitary if for all $\mu \in M(H)$, the operator $\pi(\mu)$ is a unitary operator on $\mathcal{H}_\pi$. The representation π is said to be *irreducible* if there is no closed proper subspace of $\mathcal{H}_\pi$ that is invariant by $\pi(\mu)$ for all $\mu \in M(H)$.

Let us say some words about commutative hypergroups [17, Section 4] and [3].
A hypergroup H is said to be *commutative* if the convolution is commutative. That is for each $x, y \in H, \delta_x * \delta_y = \delta_y * \delta_x$. In [20], the author proved that a left Haar measure exists on every commutative hypergroup.

Let H be a commutative hypergroup. The dual $\widehat{H}$ of H is the space of continuous complex-valued functions χ such that

$$\chi(x * y) = \chi(x)\chi(y) \text{ and } \chi(x^-) = \overline{\chi(x)} \text{ for all } x, y \in H.$$

The elements of $\widehat{H}$ are called (hermitian) characters of H.
For μ in $M(H)$ the *Fourier-Stieltjes transform* of μ is defined by

$$\mathcal{F}(\mu)(\chi) = \int_H \overline{\chi(x)} d\mu(x),\ \chi \in \widehat{H}$$

and for f in $L^1(H)$, the *Fourier transform* of f is defined by

$$\mathcal{F}(f)(\chi) = \int_H f(x)\overline{\chi(x)} dx,\ \chi \in \widehat{H}.$$

Let us denote by S the subset of $\widehat{H}$ defined by

$$S = \{\chi \in \widehat{H} : |\mathcal{F}(\mu)(\chi)| \leq \|\lambda(\mu)\|, \forall \mu \in M(H)\}.$$

There exists a unique non-negative measure ν on $\widehat{H}$ such that

$$\int_H |f(x)|^2 dx = \int_{\widehat{H}} |\mathcal{F}(f)(\chi)|^2 d\nu(\chi)$$

for all $f \in L^1(H) \cap L^2(H)$. The measure ν is called the *Plancherel-Levitan* measure; moreover, S is the support of ν. Let us notice that if H is a commutative hypergroup, then the Fourier space $A(H)$ is isometrically isomorphic to $L^1(S, \nu)$ [17, page 69] (see Section 4 for the definiton of $A(H)$).

3. Overview of some geometric properties of Banach spaces

In this section, we take an overview of the geometric properties we are going to investigate.

Definition 3.1 (Radon-Nikodym property). *A Banach space X has the Radon-Nikodym property (RNP) if every bounded subset D of X is dentable; that is, for each $\varepsilon > 0$, there exists some $x_\varepsilon \in D$ such that $x_\varepsilon \notin \overline{co}\big(D\backslash B_\varepsilon(x_\varepsilon)\big)$, where $B_\varepsilon(x_\varepsilon) = \{y \in X : \|y - x_\varepsilon\| < \varepsilon\}$ and $\overline{co}\big(D\backslash B_\varepsilon(x)\big)$ is the norm closed convex-hull of $D\backslash B_\varepsilon(x)$.*

For further insight into the Radon-Nikodym property, we refer to the interested readers the book [4] where various aspects of this property are discussed. We extract from this book the following properties that we may need.

- If X has the RNP, then every subspace of X has the RNP.
- The space X has the RNP if every separable closed linear subspace of X has the RNP.
- If X has the RNP, then any closed subspace of X has the RNP.
- Let Γ be a discrete set. Then $\ell^1 := \ell^1(\Gamma)$ has the RNP and every Banach space which is norm isomorphic to ℓ^1 has the RNP.

Definition 3.2 (Dunford-Pettis property)**.** *The Banach space X is said to have the Dunford-Pettis property (DPP) if, for any Banach space Y, every weakly compact linear operator $T : X \to Y$ is completely continuous, i.e. T sends weakly Cauchy sequences into norm convergent sequences.*

Definition 3.3 (Schur property)**.** *The Banach space X is said to have the Schur property (SP) if every weakly convergent sequence in X is norm convergent.*

Remark 3.4. Note that if a Banach space has the SP, then it has the DPP. Also, for every discrete set Γ, the space $\ell^1(\Gamma)$ has the SP.

4. Geometric properties of the Fourier space $A(H)$ of a hypergroup H

In the rest of the paper, H is assumed to be a hypergroup with a left Haar measure.
Let us denote by λ the left regular representation of H on $L^2(H)$ given by

$$\lambda(x)f(y) = f(x^- * y)$$

where $x, y \in H$ and $f \in L^2(H)$. This can be extended to $L^1(H)$ by setting

$$\lambda(f)(g) = f * g$$

for $f \in L^1(H)$ and $g \in L^2(H)$. Let $C^*(H)$ denote the C^*-algebras of H and by $C^*_\lambda(H)$ the norm closure of the space $\{\lambda(f) : f \in L^1(H)\}$ in the algebra $B(L^2(H))$ of bounded linear operators on $L^2(H)$.

Definition 4.1. [17, Definition 2.2] *Let H be a hypergroup. The Banach space dual of the C^*-algebra $C^*(H)$ is called the* Fourier-Stieltjes space *of H and it is denoted by $B(H)$.*

The Banach space dual of $C^*_\lambda(H)$ is denoted by $B_\lambda(H)$ and it is a closed subspace of $B(H)$.

Definition 4.2. [17, Section 2.3] *The closed subspace spanned by $\{f * \tilde{f} : f \in C_c(H)\}$ in $B_\lambda(H)$ is called the* Fourier space *of H and it is denoted by $A(H)$.*

Denote by $[\lambda(H)]''$ the bicommutant of $\lambda(H) = \{\lambda(x) : x \in H\}$. The space $[\lambda(H)]''$ is a von Neumann algebra called the von Neumann algebra of H and it is denoted by $VN(H)$ [17, Definition 2.17].
The space $VN(H)$ is called an *atomic* von Neumann algebra if the representation λ is atomic. That is, λ is the direct sum of irreducible representations [21, Section I.9 and Section III.6].
It is noteworthy to recall the following result about the RNP related to the theory of von Neumann algebra : Let M be a von Neumann algebra with predual M_*. Then, the following assertion are equivalent[22, Theorem 3.5].

(1) The space M is an atomic von Neumann algebra.
(2) The space M_* has the RNP.

Theorem 4.3. *If H is a commutative hypergroup, then $A(H)$ has the DPP.*

Proof. If H is a commutative hypergroup, then $A(H)$ is isometrically isomorphic to $L^1(S, d\nu)$ [17, Proposition 4.2]. Thus $A(H)$ has the DPP since any L^1-spaces has the DPP by [5]. □

Let M be a von Neumann algebra and M_* be its predual. Then, the following asssertions are equivalent [15, Theorem 3.4].

(1) M is the direct summand of finite dimensional C^*-algebras.
(2) The space M_* has the SP.

Theorem 4.4. *If H is a compact hypergroup, then $A(H)$ has the SP, hence the DPP.*

Proof. If H is a compact hypergroup, then the left regular representation λ can be written as a direct sum of continuous irreductible subrepresentations each of which is finite-dimensional ([23, Theorem 2.2]) and thus generate finite dimensional subalgebras of $B(L^2(H))$. This implies that λ is atomic. Then, the von Neumann algebra $VN(H) = [\lambda(H)]''$ is the direct summand of finite dimensional C^*-algebras. Therefore, $A(H)$ which is the predual of the von Neumann algebra $VN(H)$ has the SP. It also has the DPP since SP implies DPP. □

5. Geometric properties of the Figà-Talamanca–Herz spaces $A_p(H)$ of a hypergroup H

The results in this section are the analogue for hypergroups of some results on locally compact groups in [12, 15, 16].

Definition 5.1. *([1] or [3]) Let H be a hypergroup. For $1 < p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, define $A_p(H)$ to be the set*

$$\Big\{h \in C_0(H) : h = \sum_{n=1}^{\infty} f_n * \check{g}_n,\ f_n \in L^p(H),\ g_n \in L^q(H),\ \forall n \in \mathbb{N},\ \sum_{n=1}^{\infty} \|f_n\|_p \|g_n\|_q < \infty\Big\}.$$

with the norm

$$\|h\|_{A_p(H)} = inf \left\{ \sum_{n=1}^{\infty} \|f_n\|_p \|g_n\|_q : h = \sum_{n=1}^{\infty} f_n * \check{g}_n \right\},$$

where the infimum is taken over all possible representations of h. Equipped with this norm, $A_p(H)$ a Banach space called the Figà-Talamanca-Herz space.

Fact 5.2. *Hereafter are somes facts about $A_p(H)$.*

(1) *If $p = 2$, $A_2(H) = A(H)$ the Fourier space of H.*
(2) *If $u \in A_p(H)$, then there exists sequences (f_n) and (g_n) in $C_c(H)$ such that $u = \sum_{n=1}^{\infty} f_n * \check{g}_n$.* [1, Lemma 2.2].

Definition 5.3. *Let $1 < p < \infty$. The topology on $B(L^p(H))$ associated to the family of semi-norms*

$$T \mapsto \Big| \sum_{n=1}^{\infty} \int_H T(f_n)(x) g_n(x) dm(x) \Big|$$

with $\sum_{n=1}^{\infty} f_n * \check{g}_n \in A_p(H)$, is called the ultraweak topology.

This topology is Hausdorff.

Definition 5.4. *Let H be a hypergroup. For $1 < p < \infty$, the map $\lambda_p : M(H) \to B(L^p(H))$ defined by*

$$(\lambda_p(\mu))(f) = \mu * f \text{ for } \mu \in M(H) \text{ and } f \in L^p(H)$$

is a representation of $M(H)$ in the Banach space $B(L^p(H))$.
The closure of $\lambda_p(L^1(H))$ in $B(L^p(H))$ with respect to the ultraweak topology is denoted by $PM_p(H)$. The elements of $PM_p(H)$ are called p-pseudomeasures *on H.*

If $p = 2$, then λ_2 is the left regular representation of H and $PM_2(H)$ is the von Neumann algebra $VN(H)$.
The dual space of $A_p(H)$ can be isometrically identified with the Banach space $PM_q(H)$ by the following fact.

Fact 5.5. [1, Theorem 2.9] *Let $1 < p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. If $F \in A_p(H)^*$, then there exists a unique $F' \in PM_q(H)$ such that for all $f \in L^p(H)$ and $g \in L^q(H)$,*

$$F(f * \check{g}) = \int_H F'(g)(x) f(x) dx = \langle F'(g), f \rangle.$$

Moreover, the mapping

$$A_p(H)^* \to PM_q(H), F \mapsto F'$$

is a surjective isometry ; it carries the weak-topology of $A_p(H)^*$ to the ultraweak topology of $PM_q(H)$.*

Lemma 5.6. *Let H be a hypergroup. Consider a sequence $\{f_n\} \subset L^p(H)$. If $\{f_n\}$ converges weakly to 0 in $L^p(H)$, then $\forall g \in L^q(H)$, the sequence $\{f_n * \check{g}\}$ converges weakly to 0 in $A_p(H)$.*

Proof. Let $F \in A_p(H)^*$. By Fact 5.5, there exists a unique $F' \in PM_q(H)$ such that

$$F(f_n * \check{g}) = \int_H F'(g)(x) f_n(x) dx$$
$$= \langle F'(g), f_n \rangle$$

Since $\{f_n\}$ converges weakly 0 in $L^p(H)$ and $F'(g) \in L^q(H)$, we have that the sequence $\{\langle F'(g), f_n \rangle\}$ converges to 0. Therefore, the sequence $\{F(f_n * \check{g})\}$ converges to 0 for every $F \in A_p(H)^*$. It follows that $\{f_n * \check{g}\}$ converges weakly to 0 in $A_p(H)$. □

Lemma 5.7. *Let H be a hypergroup. For $x \in H$ and $f, g \in C_c(H)$, we have*

$$L_x(f * \check{g}) = (L_x f) * \check{g}.$$

Proof. Let $z \in H$, we have

$$\begin{aligned} L_x(f * \check{g})(z) &= (f * \check{g})(x * z) \\ &= \int_H f((x * z) * y) g(y) dy \\ &= \int_H f(x * (z * y)) g(y) dy \text{ (by associativity of } *) \\ &= \int_H (L_x f)(z * y) g(y) dy \\ &= ((L_x f) * \check{g})(z). \end{aligned}$$

□

The following theorem is the analogue in the framework of hypergroups of Lemme 3.1 in [16].

Theorem 5.8. *Let H be a hypergroup. If H is not compact, then there is a sequence $\{x_n\}$ in H such that $x_n \to \infty$; that is, for any compact subset K of H there is an $N \in \mathbb{N}$ with $x_n \notin K$ for all $n > N$. Furthermore, if $w \in A_p(H)$, then the sequence $\{L_{x_n} w\}$ converges weakly to 0 in $A_p(H)$.*

Proof. If the space H is not compact, then let us choose a σ-compact subset $H' = \cup_{n=1}^{\infty} K_n$ of H, whith K_n an open relatively compact set and $K_n \subset K_{n+1}$ for all n. For any compact subset K of H, there is some $N \in \mathbb{N}$ such that $K \cap H' \subseteq \cup_{n=1}^{N} K_n \cap K$. For each $n \geq 1$, if we choose $x_n \in K_{n+1} \backslash K_n$, then we obtain that $x_n \notin K$ for all $n > N$.
Now, let $w \in A_p(H)$. By Fact 5.2 (2), there exists sequences (f_i) and (g_i) in $C_c(H)$ such that $w = \sum_{i=1}^{\infty} f_i * \check{g}_i$. Thus, $L_{x_n} w = \sum_{i=1}^{\infty} L_{x_n}(f_i * \check{g}_i)$. By Lemma 5.7 , $L_{x_n}(f_i * \check{g}_i) = (L_{x_n} f_i) * \check{g}_i$.
Let us denote by V_i the compact support of f_i. We have $|f_i(y)| \leq \|f_i\|_\infty 1_{V_i}(y)$ for all $y \in H$, where 1_{V_i} is the characteristic function of V_i.
Let $h \in L^q(H)$. Then,

$$\begin{aligned} \Big|\langle h, L_{x_n} f_i \rangle\Big| &= \Big|\int_H h(y) L_{x_n} f_i(y) dy\Big| \\ &= \Big|\int_H h(y) f_i(x_n * y) dy\Big| \\ &\leq \int_H \Big|h(y)\Big| . \Big|f_i(x_n * y)\Big| dy \\ &\leq \int_H \Big|h(y)\Big| . \|f_i\|_\infty 1_{V_i}(x_n * y) dy \\ &\leq \int_{x_n^- * V_i} \Big|h(y)\Big| . \|f_i\|_\infty 1_{V_i}(y) dy \\ &\leq \|f_i\|_\infty \left(\int_{x_n^- * V_i} \Big|h(y)\Big|^q dy \right)^{1/q} . \left(\int_{x_n^- * V_i} \Big|1_{V_i}(y)\Big|^p dy \right)^{1/p} \\ &\text{(Hölder inequality)} \\ &\leq \|f_i\|_\infty m(V_i)^{1/p} \left(\int_{x_n^- * V_i} \Big|h(y)\Big|^q dy \right)^{1/q}, \end{aligned}$$

where m is the Haar measure of H. Moreover, $\left(\int_{x_n^- * V_i} \Big|h(y)\Big|^q dm(y) \right)^{1/q}$ goes to 0 when n tends to ∞ since for any compact subset K of H, $x_n^- * V_i \cap K = \emptyset$ for sufficiently large

n. Hence, the sequence $\{\langle h, L_{x_n} f_i\rangle\}$ converges to 0. Thus, $\{L_{x_n} f_i\}$ converges weakly to 0 in $L^p(H)$. This implies, by Lemma 5.6, that for every $F \in A_p(H)^*$, $F\left(L_{x_n}(f_i * \check{g}_i)\right) = F\left((L_{x_n} f_i) * \check{g}_i\right)$ converges to 0. It follows that $F\left(L_{x_n} w\right) = \sum_{i=1}^{\infty} F\left(L_{x_n}(f_i * \check{g}_i)\right)$ converges to 0 whenever n goes to ∞. □

For any $x \in H$ and $f \in PM_q(H)$, let us define ${}_x f \in PM_q(H)$ by $\langle {}_x f, u\rangle = \langle f, L_{x^-} u\rangle$, $u \in A_p(H)$.

Corollary 5.9. *Let H be a non-compact hypergroup and let $\{x_n\}$ be a sequence in H such that $x_n \to \infty$ as in Theorem 5.8. If $f \in PM_q(H)$, then the sequence $\{{}_{x_n} f\}$ tends to 0 in the weak-* topology of $PM_q(H)$.*

Proof. Let f be in $PM_q(H)$. Let $u \in A_p(H)$. We have $\langle {}_{x_n} f, u\rangle = \langle f, (L_{x_n^-})u\rangle$. It follows from Theorem 5.8 that the sequence $\{(L_{x_n^-})u\}$ converges weakly to 0 in $A_p(H)$. Thus, $\{\langle {}_{x_n} f, u\rangle\}$ converges to 0. □

Theorem 5.10. *Let H be a hypergroup. If H is not compact, then $A_p(H)$ does not have the SP.*

Proof. Let V be a relatively compact symmetric neighbourhood of e in H. If H is not compact, then let x_n be as in Theorem 5.8. Set $u_n = L_{x_n}(1_V * 1_V)$. By the choice of V, $1_V \in C_c(H)$ and $1_V = \check{1_V}$. Thus $1_V * 1_V \in A_p(H)$. Using Theorem 5.8, we get that the sequence $\{u_n\}$ converges weakly to 0 in $A_p(H)$. Moreover,

$$1_V * 1_V(e) = m\{e * V \cap V\} = m\{V\},$$

where m is the left Haar measure on H. Hence,

$$\|u_n\|_{A_p(H)} = \|1_V * 1_V\|_{A_p(H)} \geq \|1_V * 1_V\|_\infty = m\{V\} > 0.$$

It follows that $\{u_n\}$ is not norm convergent in $A(H)$. Thus, $A_p(H)$ does not have the SP. □

Theorem 5.11. *The space $B_\lambda(H)$ has the RNP and the DPP if only if H is compact.*

To prove this theorem, we may need the following well-known results.

(1) The dual A^* of a C^*-algebra A has the RNP if and only if A does not contain an isomorphic copy of ℓ^1 [10, Corollary VII.10].
(2) If X^* has the DPP, then so does X [5, Corollary 2].
(3) If X has the DPP and contains no copy of ℓ^1, then X^* has the SP [5, Theorem 3].

Proof. (of Theorem 5.11) We recall that $B_\lambda(H) = (C^*_\lambda(H))^*$. If $B_\lambda(H)$ has the RNP, then $C^*_\lambda(H)$ does not contain an isomorphic copy of ℓ^1. If $B_\lambda(H)$ has the DPP, then $C^*_\lambda(H)$ has also the DPP. Since $C^*_\lambda(H)$ does not contain an isomorphic copy of ℓ^1 and has the DPP, then its dual $(C^*_\lambda(H))^* = B_\lambda(H)$ has the SP. Therefore, $A(H) := A_2(H)$ which is a subspace of $B_\lambda(H)$ has the SP. Thus H is compact by Theorem 5.10.
In the converse, if H is compact, then by [17, Corollary 2.14], $A(H) = B_\lambda(H)$. Thus, $B_\lambda(H)$ has the DPP by Theorem 4.4 and the RNP by [7, Theorem 3.3]. □

Conclusion

We have established that if H is a commutative hypergroup, then the Fourier space of $A(H)$ has the Dunford-Pettis property ; if H is a compact hypergroup then $A(H)$ has the Schur property and consequently the Dunford-Pettis property. We also showed that the Figà-Talamanca–Herz space $A_p(H)$ does not have the Schur property if H is not compact. As future work, we can study the geometric properties for spherical hypergroups and ultra-spherical hypergroups (see the definitions of such hypergroups in [18]). Roughly speaking, they are hypergroups constructed from locally compact groups. The challenge could be to predict geometric properties of spherical hypergroups or ultra-spherical hypergroups from the properties of the underlying locally compact groups.

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[1] University of Lomé
Department of Mathematics
PO Box 1515, Lomé, Togo.
E-mail address: rolandek@yahoo.com

[2] University of Lomé
Department of Mathematics
PO Box 1515, Lomé, Togo.
E-mail address: davidlakmon@gmail.com

[3] University of Lomé
Department of Mathematics
PO Box 1515, Lomé, Togo
and University of Abomey-Calavi
International Chaire in Mathematical Physics and Applications
Benin.
E-mail address: mensahyaogan2@gmail.com, ymensah@univ-lome.tg