The existence of directed BIBDs

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Abstract

For any positive integers $k \geq 3$ and $\lambda$, let $c_d(k, \lambda)$ denote the smallest integer such that the necessary conditions $2\lambda(v-1) \equiv 0 \pmod{k-1}$ and $\lambda(v-1) \equiv 0 \pmod{(\frac{k}{2})}$ for the existence of a DB$(k, \lambda; v)$ are also sufficient for every $v \geq c_d(k, \lambda)$. In this article we provide an estimate for $c_d(k, \lambda)$ when $k \equiv 0 \pmod{4}$ and any $\lambda$. Combined with the results in (Discrete Math. 222 (2000) 27–40), we completely give an estimate of $c_d(k, \lambda)$ for any integers $k \geq 3$ and $\lambda$.

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1. Introduction

Let $v$, $k$ and $\lambda$ be positive integers. A transitive ordered $k$-tuple $(a_1, a_2, \ldots, a_k)$ is defined to be the set $\{(a_i, a_j) : 1 \leq i < j \leq k\}$ consisting of $\left(\begin{array}{c} k \\ 2 \end{array}\right)$ ordered pairs. A directed balanced incomplete block design (DBIBD) with parameters $v$, $k$ and $\lambda$, denoted by DB$(k, \lambda; v)$, is a pair $(X, \mathcal{A})$ where $X$ is a $v$-set and $\mathcal{A}$ is a collection of transitivity ordered $k$-tuples of $X$ (called blocks) such that every ordered pair of $X$ appears in exactly $\lambda$ blocks of $\mathcal{A}$. For the definition of BIBDs, PBDs, GDDs and TDs, the reader is referred to [2,3]. Applications of DBIBDs and other directed designs to computer

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science are discussed in [11]. Let \( DB(k, \lambda) \) (or \( B(k, \lambda) \)) be the set of all \( v \) such that there exists a \( DB(k, \lambda; v) \) (or \( B(k, \lambda; v) \)), and write \( DB(k, 1) \) (or \( B(k, 1) \)) briefly as \( DB(k) \) (or \( B(k) \)). Recall that \( B(K) \) is the set of all \( v \) such that there is a \( (v, K, 1) \)-PBD. A set \( K \) is said to be PBD-closed if \( B(K) = K \). It is easy to see that \( DB(k; 1) \) (or \( B(k; 1) \)) briefly as \( DB(k) \) (or \( B(k) \)).

If we ignore the order of the blocks, a \( DB(k, \lambda; v) \) becomes a \( B(k; 2 \lambda; v) \). It is well known that the necessary conditions for the existence of a \( DB(k, \lambda; v) \) are

\[
2 \lambda (v - 1) \equiv 0 \pmod{k - 1},
\]

\[
\lambda v (v - 1) \equiv 0 \pmod{k/2}.
\]

(1)

It has been shown in [10,12,13,1] that when \( k = 3, 4, 5 \) and \( 6 \) the necessary conditions (1) are also sufficient, with the exceptions of \( (v, k, \lambda) = (15, 5, 1), (21, 6, 1) \). It is worth noticing that the existence of a \( B(k, \lambda; v) \) implies the existence of a \( DB(k, \lambda; v) \). The DBIBD is obtained by writing each block of the BIBD twice—once in some order and the another in the reverse order.

Let \( c_d(k, \lambda) \) (or \( c(k, \lambda) \)) denote the smallest integer such that the necessary conditions for the existence of \( DB(k, \lambda; v) \) (or \( B(k, \lambda; v) \)) are also sufficient for every \( v \geq c_d(k, \lambda) \) (or \( c(k, \lambda) \)). In [9] the authors provided an estimate of \( c_d(k, \lambda) \) for \( k \not\equiv 0 \pmod{4} \) and any \( \lambda \) as follows.

**Theorem 1.1** (Chang and Lo Faro [9, Theorem]). Let \( k \geq 3 \) and \( \lambda \) be positive integers and \( k \not\equiv 0 \pmod{4} \). Then

\[
c_d(k, \lambda) \leq \begin{cases} 
\exp\{\exp\{k^6k^2\}\} & \text{if } k \equiv 1 \pmod{4}, \\
\exp\{k^3k^6\} & \text{if } k \equiv 2, 3 \pmod{4}.
\end{cases}
\]

In this article we give an estimate of \( c_d(k, \lambda) \) for \( k \equiv 0 \pmod{4} \) and any \( \lambda \), and establish the following theorem:

**Theorem 1.2.** Let \( k \geq 4 \) and \( \lambda \) be positive integers and \( k \equiv 0 \pmod{4} \). Then \( c_d(k, \lambda) \leq \exp\{k^3k^6\} \).

To prove Theorem 1.2 we always assume that \( k \equiv 0 \pmod{4} \) and \( k \geq 8 \) throughout this paper.

2. Useful lemmas

In this section we quote several lemmas which will be used later.

**Lemma 2.1** (Chang [3, Lemma 6.4]). Suppose that there exist a \( TD(l, n) \), a \( TD(l, n - 1) \) and a \( TD(n + 1, m) \). Let \( t \) be an integer such that \( 0 \leq t \leq m \). Then there exists an \( \{l, n\}\)-GDD having group type \( (ml - m)^{n-1} (mn - m)^{1} (tl - t)^{1} \).
Lemma 2.2 (Chang [5]). Let $D(x)$ denote the smallest number $n$ such that $TD(x, n)$ exists if $n \geq D(x)$. If $x \geq 3$ is a positive integer, then $D(x) < (x-1)2^{4(x-1)}$ and hence $D(x) < e^{4(x-1)}$.

Lemma 2.3 (Chang [4, Lemma 3.1]). Suppose that $q$ is a prime power, $u$ is a positive integer such that $q \geq u + 2$, and $d = \binom{u}{2}$. Suppose that there exists a $B(k, q; u)$. Then there is a $(k, \lambda)$-GDD with group type $(q^d)^2$.

Lemma 2.4 (Chang and Lo Faro [9]). If there exists a $DB(k, \lambda; u)$ and there exists a $TD(k, u - 1)$, then the existence of a $B(k, l; v)$ implies the existence of a $DB(k, \lambda; v(u - 1) + 1)$.

Lemma 2.5 (Chang and Lo Faro [9]). If there exists a $DB(k, \lambda; u)$ and there exists a $TD(k, u)$, then the existence of a $B(k, l; v)$ implies the existence of a $DB(k, \lambda; uv)$.

Lemma 2.6 (Chang [8]). Suppose that $v$ and $l$ are given, $v \geq l + 2$ and $\lambda_0(v, l) = (\frac{v-2}{2})^2 \times 4^{v-1}$. If $\lambda(v - 1) \equiv 0 \pmod{l - 1}$, $\lambda'(v - 1) \equiv 0 \pmod{l(l - 1)}$ and $\lambda \geq \lambda_0(v, l)$, then there is a $B(l, \lambda; v)$.

Lemma 2.7 (Chang [6]). Let $q, l$ be any positive integers and $\gcd(q, l) = 1$. If real number $x \geq \max\{e^{27}, e^{0.4/\sqrt{\ln q}}\}$, then there exists a prime power $Y$ between $x$ and $q$ such that $Y \equiv l \pmod{q}$.

For any positive integer $n$, let $I(n)$ denote the index of 2 in the prime factorization of $n$. In other words, $n = 2^{I(n)} \cdot n'$ where $n'$ is odd.

Theorem 2.8. Let $k \geq 8$ and $I(k) \geq 2$. For integer $j \in [0, 2^{(k-1)/2} - 1]$, there exists a prime power $Y(k, j)$ such that $Y(k, j) \equiv (k(k - 1)/2^{(k-1)/2} + 1 \pmod{k(k - 1))}$ and $(\frac{k}{2})^{k(k-1)/2} < Y(k, j) < k^{2(k-1)}$.

Proof. Let $q = k(k - 1)$ and $l = (k(k - 1)/2^{(k-1)/2}) + 1$ for $j \in [0, 2^{(k-1)/2} - 1]$. Then $\gcd(q, l) = 1$ as the two integers do not share any common divisor. Let $x = (\frac{k}{2})^{k(k-1)}$. If $k \geq 49$, then $x \geq \max\{e^{27}, e^{0.4/\sqrt{\ln q}}\}$. By Lemma 2.7 there exists a prime power $Y(k, j)$ such that $Y(k, j) \equiv (k(k - 1)/2^{(k-1)/2} + 1 \pmod{k(k - 1))}$ and $x < Y(k, j) < qx$. It is easy to check that

$$\left(\frac{k}{2}\right)^{k(k-1)/2} < Y(k, j) < \left(\frac{k}{2}\right)^{k(k-1)/2} \cdot k(k - 1) < k^{2(k-1)}.$$

If $8 \leq k \leq 48$ and $I(k) \geq 2$, for integer $j \in [0, 2^{(k-1)/2} - 1]$ let $p(k, j)$ be a prime power such that $p(k, j) \equiv (k(k - 1)/2^{(k-1)/2} + 1 \pmod{k(k - 1))}$ in the
following tables:

\[
\begin{array}{cccccccccccc}
(k,j) & (8,0) & (8,1) & (8,2) & (8,3) & (12,0) & (12,1) & (16,0) & (16,1) & (16,2) & (16,3) \\
p(k,j) & 113 & 71 & 29 & 43 & 397 & 67 & 241 & 31 & 61 & 331 \\
\end{array}
\]

\[
\begin{array}{cccccccccccc}
& (16,4) & (16,5) & (16,6) & (16,7) & (20,0) & (20,1) & (24,0) & (24,1) & (24,2) & (24,3) & (28,0) \\
& 121 & 151 & 181 & 211 & 761 & 191 & 1657 & 139 & 277 & 967 & 757 \\
\end{array}
\]

\[
\begin{array}{cccccccccccc}
& (28,1) & (32,0) & (32,1) & (32,2) & (32,3) & (32,4) & (32,5) & (32,6) & (32,7) & (32,8) & (32,9) \\
& 379 & 5953 & 5023 & 125 & 3163 & 4217 & 311 & 373 & 1427 & 1489 & 2543 \\
\end{array}
\]

\[
\begin{array}{cccccccccccc}
& (32,10) & (32,11) & (32,12) & (32,13) & (32,14) & (32,15) & (36,0) & (36,1) & (40,0) & (40,1) & (40,2) \\
& 1613 & 683 & 2729 & 2791 & 1861 & 3907 & 2521 & 631 & 3121 & 1951 & 2341 \\
\end{array}
\]

\[
\begin{array}{cccccccccccc}
& (40,3) & (44,0) & (44,1) & (48,0) & (48,1) & (48,2) & (48,3) & (48,4) & (48,5) & (48,6) & (48,7) \\
& 1171 & 9461 & 947 & 4513 & 283 & 5077 & 9871 & 1129 & 5923 & 1693 & 4231 \\
\end{array}
\]

It is easy to see that \(p(k,j)^2 \equiv I(k) - 1 \equiv (k(k-1)=2^i j+1) \equiv 1 \pmod{k(k-1)}\).

Let \(x\) be a real number such that \(p(k,j)^2 \leq x < (k(k-1))^2 \). Then

\[
p(k,j)^2 \equiv I(k) - 1 \equiv \left\lfloor\frac{x}{2(k-1)}\right\rfloor + 1 \\pmod{k(k-1)},
\]

where \(\left\lfloor x \right\rfloor\) denotes the smallest integer larger than \(x\). Direct verifications show that

\[
\left(\begin{array}{c}
\frac{k}{2}
\end{array}\right)^{k(k-1)} \leq p(k,j)^2 \leq \left(\begin{array}{c}
\frac{k}{2}
\end{array}\right)^{k(k-1)} < k^{2(k-1)}.
\]

This completes the proof.

**Lemma 2.9** (Chang [3, Lemma 5.5]). Let \(k \geq 8\). Then there exists a prime power \(q_0\) such that there is a \(B(k,1;q_0)\), \(q_0 \equiv 1 \pmod{k(k-1)}\) and \(e^{k^2 k^{2k}} < q_0 \leq k^{k^2-10}\).

**Lemma 2.10** (Chang [7]). Let \(k \geq 8\). If \(v \equiv 1, k \pmod{k(k-1)}\) and \(v > \exp\{k^{k^2-6}\}\), then \(v \in B(k)\).

**Lemma 2.11** (Chang [7]). Let \(k \geq 3\) and \(\lambda\) be positive integers. Then \(e(k,\lambda) < \exp\{k^{3^{k^3}}\}\) (that is, if \(\lambda(v-1) \equiv 0 \pmod{k-1}\) and \(\lambda(v-1) \equiv 0 \pmod{k(k-1)}\), then \(v \in B(k,\lambda)\) whenever \(v > \exp\{k^{3^{k^3}}\}\)).
Lemma 2.12 (Fundamental construction Wilson [14]). Suppose \((X, G, \mathcal{A})\) is a GDD, and let \(\omega: X \mapsto \mathbb{Z}^+ \cup \{0\}\) be any weighting function. For every \(x \in X\), let \(S(x)\) be a set of \(\omega(x)\) ‘copies’ of \(x\). For every \(A \in \mathcal{A}\), suppose that

\[
\left( \bigcup_{x \in A} S(x), \{S(x) : x \in A\}, \mathcal{B}(A) \right)
\]

is a GDD. Then

\[
\left( \bigcup_{x \in X} S(x), \left\{ \bigcup_{x \in G} S(x) : G \in \mathcal{G} \right\}, \bigcup_{A \in \mathcal{A}} \mathcal{B}(A) \right)
\]

is also a GDD.

3. A reduction of the necessary conditions (1)

In this section we will give a reduction to the necessary conditions (1) for the existence of a DB\((k; \lambda; v)\).

Lemma 3.1. If \(I(\lambda) \geq I(k) - 1\) and \(I(k) \geq 2\), then \(c_d(k, \lambda) \leq \exp\{k^{3k}\}\).

Proof. By \(I(\lambda) \geq I(k) - 1\) and \(I(k) \geq 2\), the necessary conditions (1) imply the congruences

\[
\lambda(v - 1) \equiv 0 \pmod{k - 1},
\]

\[
\lambda v(v - 1) \equiv 0 \pmod{k(k - 1)}.
\]

By Lemma 2.11, there exists a \(B(k, \lambda; v)\) and hence a DB\((k, \lambda; v)\) for \(v \geq \exp\{k^{3k}\}\). Hence, \(c_d(k, \lambda) \leq \exp\{k^{3k}\}\). \(\square\)

Lemma 3.2. If \(0 \leq I(\lambda) \leq I(k) - 2\), then \(u\) is a solution of the congruences

\[
\lambda(v - 1) \equiv 0 \pmod{k - 1},
\]

\[
\lambda v(v - 1) \equiv 0 \pmod{k(k - 1))}
\]

if and only if \(u + k(k - 1)/2^{I(\lambda) + 1}\) is a solution of the congruences

\[
\lambda(v - 1) \equiv 0 \pmod{k - 1},
\]

\[
\lambda v(v - 1) \equiv \binom{k}{2} \pmod{k(k - 1)).
\]

(3)
Theorem 3.3. Let \( u \) be a solution of the congruences (2), then \( u + k(k - 1)/2^{I(l) + 1} \) is a solution of (3). In fact, \( \lambda(u + k(k - 1)/2^{I(l) + 1} - 1) = \lambda(u - 1 + \lambda k(k - 1)/2^{I(l) + 1}) \equiv 0 \pmod{k - 1} \), and
\[
\lambda \left( u + \frac{k(k - 1)}{2^{I(l) + 1}} \right) \left( u + \frac{k(k - 1)}{2^{I(l) + 1}} - 1 \right)
\equiv \frac{k}{2} \pmod{k - 1}.
\]

Proof. If \( u \) is a solution of the congruences (2), then \( u + k(k - 1)/2^{I(l) + 1} \) is a solution of (3). In fact, \( \lambda(u + k(k - 1)/2^{I(l) + 1} - 1) = \lambda(u - 1 + \lambda k(k - 1)/2^{I(l) + 1}) \equiv 0 \pmod{k - 1} \), and
\[
\lambda \left( u + \frac{k(k - 1)}{2^{I(l) + 1}} \right) \left( u + \frac{k(k - 1)}{2^{I(l) + 1}} - 1 \right)
\equiv \frac{k}{2} \pmod{k - 1}.
\]

On the other hand, if \( u \) is a solution of the congruences (3), direct checks show that \( \lambda(u - (k(k - 1)/2^{I(l) + 1}) - 1) \equiv 0 \pmod{k - 1} \) and \( \lambda(u - k(k - 1)/2^{I(l) + 1})(u - (k(k - 1)/2^{I(l) + 1}) - 1) \equiv 0 \pmod{k - 1} \). Hence \( u - k(k - 1)/2^{I(l) + 1} \) is a solution of the congruences (2). \( \square \)

If \( u \) is a solution of the congruences (2), by Lemma 3.2, \( u + (k(k - 1)/2^{I(l)})j \) is also a solution of (2) for any non-negative integer \( j \). We can easily calculate all the possible residue classes \( v \) modulo \( k(k - 1)/2^{I(l)} \) satisfying the conditions (2). Suppose that \( u_1, u_2, \ldots, u_s \) are all the representatives of those residue classes modulo \( k(k - 1)/2^{I(l)} \) of the congruences (2). Next we give a reduction of the necessary conditions (1) as follows.

Theorem 3.3. Let \( u_1, u_2, \ldots, u_s \) be all the representatives of the residue classes modulo \( k(k - 1)/2^{I(l)} \) of the congruences (2). If \( 0 \leq I(\lambda) \leq I(k) - 2 \), then \( u_1, u_2, \ldots, u_s \) are all the representatives of the residue classes modulo \( k(k - 1)/2^{I(l) + 1} \) of the congruences (1).

Proof. It is easy to see that the set of solutions of (1) is the union of the solution sets of (2) and (3) as \( I(k) \geq 2 \). By Lemma 3.2, \( u_1 + k(k - 1)/2^{I(l) + 1}, u_2 + k(k - 1)/2^{I(l) + 1}, \ldots, u_s + k(k - 1)/2^{I(l) + 1} \) are all the representatives of the residue classes modulo \( k(k - 1)/2^{I(l)} \) of the congruences (3). Then \( u_1, u_2, \ldots, u_s, u_1 + k(k - 1)/2^{I(l) + 1}, u_2 + k(k - 1)/2^{I(l) + 1}, \ldots, u_s + k(k - 1)/2^{I(l) + 1} \) are all the representatives of the residue classes modulo \( k(k - 1)/2^{I(l)} \) of the congruences (1) and hence the conclusion follows. \( \square \)

Note that \( k \equiv 0 \pmod{4} \) is equivalent to \( I(k) \geq 2 \). By Lemma 3.1, we only need to prove Theorem 1.2 for the case \( 0 \leq I(\lambda) \leq I(k) - 2 \). In what follows we always assume that \( k \geq 8 \) and \( 0 \leq I(\lambda) \leq I(k) - 2 \).

4. The existence of DB\((k, \lambda; v) \) with \( v \equiv 1 \pmod{k(k - 1)/2^{I(l) + 1}} \)

Lemma 4.1 (Chang and Lo Faro [9]). If \( q \) is a prime power such that \( \lambda(q - 1) \equiv 0 \pmod{(\frac{k}{\lambda})} \) and \( q > (\frac{k}{\lambda})^{k(k - 1)} \), then there exists a DB\((k, \lambda; q) \).
Lemma 4.2. Let \( k \geq 8 \) and \( 0 \leq I(\lambda) \leq I(k) - 2 \). For any integer \( i \in [0, 2^{I(\lambda)+1} - 1] \), there exists a prime power \( p_i \) such that \( p_i \equiv 1 + (k(k-1)/2^{I(\lambda)+1})j \pmod{k(k-1)} \), \( p_i \in \text{DB}(k, \lambda) \) and \( (\frac{k}{2})(k-1) < p_i < k^2(k-1) \).

Proof. It immediately follows by Theorem 2.8 and Lemma 4.1. \( \square \)

In this section we always take \( q_0 \) as the prime power provided in Lemma 2.9. Then \( q_0 \in \text{B}(k) \subseteq \text{DB}(k), q_0 \equiv 1 \pmod{k(k-1)}, q_0 > D(k) \) and \( e^{k^2+2k\ln k} < q_0 \leq k^{k^2-10} \). Let

\( C = \text{exp}\{k^{k^2-6}\} \).

Lemma 4.3. If \( k \geq 8 \) and \( 0 \leq I(\lambda) \leq I(k) - 2 \), then there exist \( \tilde{p}_i \in \text{DB}(k, \lambda) \) such that \( \tilde{p}_i \equiv 1 + (k(k-1)/2^{I(\lambda)+1})j \pmod{k(k-1)} \). By Lemmas 2.5 and 4.2, \( \tilde{p}_i \in \text{DB}(k, \lambda) \).

\( \frac{q_0^2}{q_0} \) for \( i \in [0, 2^{I(\lambda)+1} - 1] \), then exist positive integer \( x \) such that

\[ \exp\{4q_0\} < q_0^x \tilde{p}_i \leq q_0 \exp\{4q_0\} \]

By Lemma 2.2, \( \tilde{p}_i > D(q_0) \). \( \square \)

Lemma 4.4. Let \( C_i = C + (\tilde{p}_i - 1)/(q_0^{I(\lambda)+1} - 1) \) for \( i \in [0, 2^{I(\lambda)+1} - 1] \) where \( x \) is the integer such that \( 2C < q_0^x \leq 2Cq_0 \). If \( k \geq 8 \) and \( 0 \leq I(\lambda) \leq I(k) - 2 \), then \( (C_i + a)(q_0 - 1) + \tilde{p}_i \in \text{DB}(k, \lambda) \) for \( i \in [0, 2^{I(\lambda)+1} - 1] \) and \( a \in [0, C] \). Moreover, \( 2C\exp\{4q_0\} < C_i < Ckq_0^x \exp\{4q_0\} \).

Proof. There is a TD(\( \tilde{p}_1, 1,q_0^\lambda \)) as \( q_0^\lambda \) is a prime power. There are a TD(q_0, \( \tilde{p}_1 \)) and a TD(q_0, \( \tilde{p}_1 \)) by Lemma 4.3. Let \( m = q_0^\lambda \) and \( t = C + a + 1 \) where \( a \in [0, C] \). Then \( C + 1 \leq t < m \). Apply Lemma 2.1 with \( l = q_0 \) and \( n = \tilde{p}_i \) to get a \( \{q_0, \tilde{p}_i\}\)-GDD with group type \( (mq_0 - m)^{\tilde{p}_i - 1}(m\tilde{p}_1 - m)^t(q_0 - t)^t \). Hence,

\[ (\tilde{p}_1 - 1)mq_0 + t(q_0 - 1) + 1 \in B(\{q_0, \tilde{p}_i, m(q_0 - 1) + 1, m(\tilde{p}_1 - 1) + 1, t(q_0 - 1) + 1\}) \]

By Lemmas 2.9 and 4.3, \( q_0, \tilde{p}_i \in \text{DB}(k, \lambda) \). Note that \( m(q_0 - 1) + 1 \equiv 1 \pmod{k(k-1)} \) and \( t(q_0 - 1) + 1 \equiv 1 \pmod{k(k-1)} \). By Lemma 2.10, \( m, m(q_0 - 1) + 1 \) and \( t(q_0 - 1) + 1 \in \text{B}(k) \subseteq \text{DB}(k, \lambda) \). By Lemma 4.3, \( \tilde{p}_i > D(q_0) > D(k) \).

By Lemma 4.4, \( m(\tilde{p}_1 - 1) + 1 \in \text{DB}(k, \lambda) \). Hence, \( (C_i + a)(q_0 - 1) + \tilde{p}_i = (\tilde{p}_1 - 1)mq_0 + t(q_0 - 1) + 1 \in \text{DB}(k, \lambda) \). Note that \( C_i = C + (\tilde{p}_1 - 1)(q_0^{I(\lambda)+1} - 1)/(q_0 - 1) + 1 \) for \( i \in [0, 2^{I(\lambda)+1} - 1] \).
where $x$ is the integer such that $2C < q_0^x \leq 2Cq_0$. Then, for $k \geq 8$ we have

$$2C(\hat{p}_i - 1) + 1 < C + (\hat{p}_i - 1) \frac{2Cq_0 - 1}{q_0 - 1} + 1 < Cq_0$$

Using $\exp\{4q_0\} < \hat{p}_i \leq q_0 \exp\{4q_0\}$ by Lemma 4.3 we obtain $2C \exp\{4q_0\} < Cq_0 < Cq_0^2 \exp\{4q_0\}$ for $i \in [0, 2^{l(k)} + 1 - 1]$. □

**Lemma 4.5.** If $k \geq 8$, then $q_0 > D(k)$ and $C > D(kq_0 + 1)$.

**Proof.** By Lemmas 2.2 and 2.9, $C = [\exp\{k^3\}] > \exp\{4kq_0\} > D(kq_0 + 1)$. Similarly for $q_0 > D(k)$. □

For $i \in [0, 2^{l(k)} + 1 - 1]$ and $j \in [1, k]$, let

$$C_{ij} = C_{i0} + k(kq_0 - 1) \sum_{b=0}^{j-1} (q_0^{y+2b} - 1)/(q_0 - 1),$$

where $y$ is the integer such that $Ckq_0^y \exp\{4q_0\} < q_0^y \leq Ckq_0^y \exp\{4q_0\}$. It is readily checked that

$$C_{ij} = C_{i0} + k(kq_0 - 1)q_0^y(q_0^{2j} - 1)/(q_0 - 1)^2 - k(kq_0 - 1)/(q_0 - 1)j$$

$$< q_0^y + k(kq_0 - 1)q_0^{y+2j}/(q_0 - 1)^2 < k^3q_0^{y+2j+1}.$$

**Lemma 4.6.** Let $k \geq 8$ and $0 \leq I(\lambda) \leq I(k) - 2$. Then $(C_{ij} + a)(q_0 - 1) + k(kq_0 - 1)j + \hat{p}_i \in \text{DB}(k, \lambda)$ for $i \in [0, 2^{l(k)} + 1 - 1], j \in [0, k]$ and $a \in [0, C]$. Moreover, $C_{ij} < C^3$ for $i \in [0, 2^{l(k)} + 1 - 1]$ and $j \in [0, k]$.

**Proof.** By induction on $j$ for the first statement. When $j = 0$, the conclusion follows immediately by Lemma 4.4. Suppose that the conclusion is true for the case $j - 1$ ($j \geq 1$), i.e., $(C_{i,j-1} + a)(q_0 - 1) + k(kq_0 - 1)(j - 1) + \hat{p}_i \in \text{DB}(k, \lambda)$ for $i \in [0, 2^{l(k)} + 1 - 1]$ and $a \in [0, C]$. Next we consider the case $j$.

There are a TD$(k, kq_0 - 1)$, a TD$(k, q_0)$ and a TD$(kq_0 + 1, q_0^{y+2j+1})$ by Lemma 4.5. Let $m = q_0^{y+2j+1}$ and $t = ((C_{i,j-1} + a)(q_0 - 1) + k(kq_0 - 1)(j - 1) + \hat{p}_i)/(k - 1)$ where $a \in [0, C]$. If $j = 1$, then $t = ((C_{i0} + a)(q_0 - 1) + \hat{p}_i - 1)/(k - 1)$. By Lemmas 4.3 and 4.4, and noting that $C_{i0} > \max\{C, \hat{p}_i\}$, we have

$$0 < t < 2Cq_0^3(k - 1) < Cq_0^3 \exp\{4q_0\} < q_0^y = m.$$

If $j \geq 2$, it is easily checked that $0 < t < (k^3q_0^{y+2j} + C)q_0/(k - 1) < k^3q_0^{y+2j-3} < m$. Apply Lemma 2.1 with $l = k$ and $n = kq_0$ to get a $\{k, kq_0\}$-GDD with group type
Hence, 

\[ k(kq_0 - 1)m + (k - 1)t + 1 \in B(\{k, kq_0, m(k - 1) + 1, (kq_0 - 1)m + (k - 1) + 1\}). \]

By Lemma 2.5, \( kq_0 \in DB(k, \lambda) \). As \( m(k - 1) + 1 \equiv k \pmod{k - 1} \) and \( m(kq_0 - 1) + 1 \equiv k \pmod{k - 1} \), by Lemma 2.10, \( m(k - 1) + 1 \) and \( m(kq_0 - 1) + 1 \in B(k) \subseteq DB(k, \lambda) \).

By the induction, \( (k - 1)t + 1 = (C_{i,j-1} + a)(q_0 - 1) + k(kq_0 - 1)(j - 1) + \hat{p}_i \in DB(k, \lambda) \).

Then \( k(kq_0 - 1)m + (k - 1)t + 1 \in DB(k, \lambda) \). It is readily checked that

\[
(C_{ij} + a)(q_0 - 1) + k(kq_0 - 1)j + \hat{p}_i \\
= k(kq_0 - 1)q_0^{1+2(j-1)} + (C_{i,j-1} + a)(q_0 - 1) + k(kq_0 - 1)(j - 1) + \hat{p}_i \\
= k(kq_0 - 1)m + (k - 1)t + 1.
\]

Hence \( (C_{ij} + a)(q_0 - 1) + k(kq_0 - 1)j + \hat{p}_i \in DB(k, \lambda) \). Moreover, for \( i \in [0, 2^{I(\lambda)+1} - 1] \) and \( j \in [0, k] \), it is easy to see that

\[
C_{ij} < k^3 q_0^{1+2(j-1)} < Ck^4 q_0^{2k+2} \exp(4q_0) < C^2. \]

\[ \square \]

**Theorem 4.7.** Let \( k \geq 8 \) and \( 0 \leq I(\lambda) \leq I(k) - 2 \). If \( v \equiv 1 \pmod{k(k - 1)/2^{I(\lambda)+1}} \) and \( v > C^2 k^2 q_0^2 \), then \( v \in DB(k, \lambda) \).

**Proof.** For such \( v \), by Lemma 4.3 there is an integer \( i \in [0, 2^{I(\lambda)+1} - 1] \) such that \( v \equiv 1 + (k(k - 1)/2^{I(\lambda)+1})i \equiv \hat{p}_i \pmod{k(k - 1)} \). By \( k(k - 1)(q_0 - 1) \) and \( kq_0 - 1 \equiv k - 1 \pmod{q_0 - 1} \), we can find integers \( j \in [1, k] \) and \( h \in [0, q_0 - 1] \) satisfying

\[
v \equiv \hat{p}_i + (hk + j)k(k - 1) \equiv \hat{p}_i + (hk + j)k(kq_0 - 1) \pmod{q_0 - 1}.
\]

By Lemma 4.6 direct computation shows that

\[
v > C^2 k^2 q_0^2 > kq_0(kq_0 - 1)C_{ij} \\
> k(kq_0 - 1)(q_0 - 1)C_{ij} + kh + j + \hat{p}_i + (q_0 - 1)C_{ij}.
\]

Then \((v - \hat{p}_i - k(kq_0 - 1)(hk + j))/(q_0 - 1) - C_{ij} - k(kq_0 - 1)C_{ij}\) is a positive integer and hence can be written as \( a + k(kq_0 - 1)d \), where \( d \geq 0 \) and \( 0 < a \leq k(kq_0 - 1) < C \).

Take \( m = (q_0 - 1)(d + C_{ij}) + kh \) and

\[
t = \frac{(q_0 - 1)(a + C_{ij})}{k - 1} + \frac{k(kq_0 - 1)j}{k - 1} + \frac{\hat{p}_i - 1}{k - 1}.
\]

It is readily checked that \( 0 < t < m \) and

\[
v = k(kq_0 - 1)m + (k - 1)t + 1. \tag{4}
\]

By Lemma 4.5 there are a TD\((k, kq_0 - 1)\), a TD\((k, kq_0)\) and a TD\((kq_0 + 1, m)\). Apply Lemma 2.1 with \( l = k \) and \( n = kq_0 \) to get a \( \{k, kq_0\}\)-GDD with group type
$$(m(k-1))^{kq_0-1}(m(kq_0-1))^i((k-1)t)^j.$$ Hence,
$$k(kq_0-1)m+(k-1)t+1 \in B(\{k,kq_0,m(k-1)+1,(kq_0-1)m+1,t(k-1)+1\}).$$

By Lemma 2.5, $kq_0 \in DB(k,\lambda)$. Note that $m(k-1)+1 \equiv 1 \pmod{m(k-1)}$ and $m(kq_0-1)+1 \equiv 1 \pmod{m(k-1)}$. By Lemma 2.10, $m(k-1)+1$ and $m(kq_0-1)+1 \in B(k) \subseteq DB(k,\lambda)$. As $(k-1)t+1=(a+C_{ij})(q_0-1)+k(kq_0-1)j+\hat{p}_i$, by Lemma 4.6, $(k-1)t+1 \in DB(k,\lambda)$. Then $k(kq_0-1)m+(k-1)t+1 \in DB(k,\lambda)$. Hence, $v \in DB(k,\lambda)$ from (4). □

5. Small examples

We can easily calculate all the possible residue classes $v$ modulo $k(k-1)/2^{l(\lambda)}$ satisfying conditions (2). Suppose that $u_1, u_2, \ldots, u_s$ are small positive members of all those classes such that $k+1 < u_i \leq (k(k-1)/2^{l(\lambda)}) + k + 1$ for all $i=1,2,\ldots,s$. By Theorem 3.3, $u_1, u_2, \ldots, u_s$ are all representatives of the residue classes modulo $k(k-1)/2^{l(\lambda)+1}$ of congruences (1).

In what follows we always take $p_1$ as the one provided in Lemma 4.2. Then $p_1 \in DB(k)$. $p_1 \equiv (k(k-1)/2^{l(\lambda)+1})+1 \pmod{m(k-1)}$, $p_1 > D(k)$ and $(k/2)^{(k-1)} < p_1 \leq k2^{(k-1)}$.

**Lemma 5.1.** Let $k \geq 8$ and $0 \leq I(\lambda) \leq I(k)-2$. Then there exist $DB(k,\lambda;\hat{u}_{ij})$ such that
$$e^{k(p_1-1)} < \hat{u}_{ij} \leq p_1 \cdot e^{k(p_1-1)}$$

and $\hat{u}_{ij} \equiv u_i + (k(k-1)/2^{l(\lambda)})j \pmod{p_1-1}$ for $i \in [1,s]$ and $j \in [0,2^{l(\lambda)}k-1]$.

**Proof.** Let $q = p_1^{2k}$, $u_{ij} = u_i + (k(k-1)/2^{l(\lambda)})j$ and $d_{ij} = \binom{u_{ij}}{2}$ for $i \in [1,s]$ and $j \in [0,2^{l(\lambda)}k-1]$. Then $k+2 \leq u_{ij} \leq k^2(k-1)+k+1 < k^3$. Clearly
$$q = p_1^{2k} > \left(\frac{k}{2}\right)^{2k(k-1)} > \left(\frac{u_{ij} - 2}{k - 2}\right)^2 \times 4^{u_{ij} - k - 1}.$$ 

By Lemma 2.6, there exists a $B(k,\lambda q;u_{ij})$. It is easy to check that for any $i \in [1,s]$ and $j \in [0,2^{l(\lambda)}k-1]$
$$u_{ij}q^{d_{ij}} < k^3 \cdot p_1^{2k(\lambda/2)} < p_1^{k^2} < e^{k(p_1-1)}.$$ 

Then there exist positive integers $x_{ij}$ such that
$$e^{k(p_1-1)} < u_{ij}q^{d_{ij}}p_1^{x_{ij}} \leq p_1 \cdot e^{k(p_1-1)}.$$ 

Let $\hat{u}_{ij} = u_{ij}q^{d_{ij}}p_1^{x_{ij}}$ for $i \in [1,s]$ and $j \in [0,2^{l(\lambda)}k-1]$. Obviously, $\hat{u}_{ij} \equiv u_i + (k(k-1)/2^{l(\lambda)})j \pmod{p_1-1}$ and $e^{k(p_1-1)} < \hat{u}_{ij} \leq p_1 \cdot e^{k(p_1-1)}$. Since $q \geq u_{ij}+2$, by Lemma 2.3 there exists a $(k,\lambda)$-GDD with group type $(q^{d_{ij}})^{x_{ij}}$. Give weight $p_1^{x_{ij}}$ to every
Lemma 5.2. Suppose that \( k \geq 8 \) and \( 0 \leq l(\lambda) \leq I(k) - 2 \). Let \( \bar{u}_i = u_ip_1^{-2/(ui)} \) for \( i \in [1, s] \). Then \( \bar{u}_i \in \text{DB}(k, \lambda) \), \( \bar{u}_i \equiv u_i \pmod{p_1 - 1} \) and \( p_1^{k^2} < \bar{u}_i < (k^2 + 1)p_1^{k^2(k^2 + 1)} \) for any \( i \in [1, s] \).

Proof. Let \( q = p_1^2 \) and \( d_i = (\frac{u_i}{2}) \) for \( i \in [1, s] \). Then \( k + 2 \leq u_i \leq k^2 + 1 \). The conclusion follows by imitating the proof of Lemma 5.1. \( \square \)

Lemma 5.3. \( \hat{u}_{ij} > D(p_1) \) for any \( i \in [1, s] \) and \( j \in [0, 2^{l(\lambda)}k - 1] \).

Proof. It immediately follows by Lemmas 2.2 and 5.1. \( \square \)

Lemma 5.4. \( \bar{u}_i > D(k) \) for any \( i \in [1, s] \).

Proof. It immediately follows by Lemmas 2.2 and 5.2. \( \square \)

Lemma 5.5. \( p_1^{b/(\hat{u}_{ij} - 1)} + 1 \in \text{DB}(k, \lambda) \) for any integers \( b \geq 1 \), \( i \in [1, s] \) and \( j \in [0, 2^{l(\lambda)}k - 1] \).

Proof. When \( b = 1 \) by Lemma 5.3 there is a TD\( (p_1, \hat{u}_{ij} - 1) \) and hence there is a \( B(\{p_1, \hat{u}_{ij}\}, 1; p_1(\hat{u}_{ij} - 1) + 1) \), which implies \( p_1(\hat{u}_{ij} - 1) + 1 \in \text{DB}(k, \lambda) \) by Lemma 5.1.

Next to consider the general case \( b \geq 2 \). Similar discussions give a pairwise balanced design \( B(\{p_1, p_1^{b/(\hat{u}_{ij} - 1)}(\hat{u}_{ij} - 1) + 1\}, 1; p_1^{b/(\hat{u}_{ij} - 1)}(\hat{u}_{ij} - 1) + 1) \), which gives a \( p_1^{b/(\hat{u}_{ij} - 1)} + 1 \in \text{DB}(k, \lambda) \) by the induction. \( \square \)

6. The proof of Theorem 1.2

Let \( C_1 = \exp\{k^{x^2 - 6}\} \). Then it is easily checked that \( C_1 > k^{q_0^2}C_2 \) where \( C = \exp\{k^{x^2 - 6}\} \) and \( q_0 \) come from Lemma 2.9.

Lemma 6.1. Let \( m = p_1^x \) where \( x \) is the integer such that \( 2C_1 < m \leq 2p_1C_1 \). Then \( (\hat{u}_{ij} - 1)m + t(p_1 - 1) + 1 \in \text{DB}(k, \lambda) \) for \( i \in [1, s] \), \( j \in [0, 2^{l(\lambda)}k - 1] \) and \( C_1 \leq t \leq m \).

Proof. There is a TD\( (\hat{u}_{ij} + 1, m) \) as \( m \) is a prime power. There are a TD\( (p_1, \hat{u}_{ij}) \) and a TD\( (p_1, \hat{u}_{ij} - 1) \) by Lemma 5.3. Apply Lemma 2.1 with \( l = p_1 \) and \( n = \hat{u}_{ij} \) to get a \( \{p_1, \hat{u}_{ij}\} \)-GDD with group type \( (mp_1 - m)^{q_{ij} - 1}(m\hat{u}_{ij} - m)^l(p_1 - 1)^l \). Hence,
\[
(\hat{u}_{ij} - 1)m + t(p_1 - 1) + 1 \in B(\{p_1, \hat{u}_{ij}, m(p_1 - 1) + 1, m(\hat{u}_{ij} - 1) + 1, t(p_1 - 1) + 1\}).
\]
By Lemmas 4.2 and 5.1, \( p_1, \hat{u}_{ij} \in DB(k, \lambda) \). Note that \( m(p_1 - 1) + 1 \equiv 1 \pmod{k(k - 1)/2^{l(\lambda)+1}} \) and \( t(p_1 - 1) + 1 \equiv 1 \pmod{k(k - 1)/2^{l(\lambda)+1}} \). As \( m > C_1 > C^2k^2q_0^2 \) and \( t(p_1 - 1) + 1 > C_1 \), by Theorem 4.7, \( m(p_1 - 1) + 1 \) and \( t(p_1 - 1) + 1 \in DB(k, \lambda) \). By Lemma 5.5, \( m(\hat{u}_{ij} - 1) + 1 \in DB(k, \lambda) \). Thus, \( (\hat{u}_{ij} - 1)m p_1 + t(p_1 - 1) + 1 \in DB(k, \lambda) \). 

**Lemma 6.2.** If \( k \geq 8 \), then \( p_1 > D(k) \) and \( C_1 > D(k p_1 + 1) \).

**Proof.** By Lemmas 2.2 and 4.2, \( C_1 = \exp\{k^{2l^2-6}\} > \exp\{4k p_1\} > D(k p_1 + 1) \). Similarly for \( p_1 > D(k) \). 

For \( i \in [1, s] \) and \( j \in [0, 2^{l(\lambda)}k - 1] \), let

\[
D_{ij} = C_1 + (\hat{u}_{ij} - 1) \frac{p_1^{l+1} - 1}{p_1 - 1} + 1,
\]

where \( x \) is the integer such that \( 2C_1 < p_1^x \leq 2p_1C_1 \). With this notation Lemma 6.1 can be restated as follows.

**Lemma 6.3.** \( (D_{ij} + a)(p_1 - 1) + \hat{u}_{ij} \in DB(k, \lambda) \) for \( i \in [1, s] \), \( j \in [0, 2^{l(\lambda)}k - 1] \) and \( a \in [0, C_1] \).

**Theorem 6.4.** Let \( k \geq 8 \), \( 0 \leq l(\lambda) \leq l(k) - 2 \) and \( v > k^3 p_1^4 C_1 \). If \( (u_i - 1)/(k-1) \) is a positive integer and \( v \equiv u_i \pmod{k(k - 1)/2^{l(\lambda)+1}} \) for \( i \in [1, s] \), then \( v \in DB(k, \lambda) \).

**Proof.** Note that \( (p_1 - 1)2^{l(\lambda)+1}/(k(k - 1)) \) is odd as \( p_1 \equiv k(k - 1)/2^{l(\lambda)+1} + 1 \pmod{k(k - 1)} \). There are integers \( \tilde{f} \in [0, 2^{l(\lambda)}k - 1] \) and \( \tilde{h} \in [0, p_1 - 1] \) satisfying

\[
v \equiv u_i + \frac{k(k - 1)}{2^{l(\lambda)+1}} j + k^2(k - 1) \tilde{h} \pmod{p_1 - 1}.
\]

If \( \tilde{j} \) is even, and we let \( 2j = \tilde{j} \) and \( h = \tilde{h} \). If \( \tilde{j} \) is odd, then \( A = (p_1 - 1)2^{l(\lambda)+1}/(k(k - 1)) + \tilde{j} \) is even. Let \( 2j = A - [A/(2^{l(\lambda)+1}k)] \cdot 2^{l(\lambda)+1}k \) and \( h = \tilde{h} + [A/(2^{l(\lambda)+1}k)] \pmod{p_1 - 1} \). Then \( j \in [0, 2^{l(\lambda)}k - 1] \) and \( h \in [0, p_1 - 1] \). In both cases (5) becomes

\[
v \equiv u_i + \frac{k(k - 1)}{2^{l(\lambda)}} j + k^2(k - 1) h \pmod{p_1 - 1},
\]

where \( j \in [0, 2^{l(\lambda)}k - 1] \) and \( h \in [0, p_1 - 1] \). By Lemma 5.1 \( \hat{u}_{ij} \equiv u_i + (k(k - 1)/2^{l(\lambda)}) \tilde{j} \pmod{p_1 - 1} \). Since \( k p_1 - 1 \equiv k - 1 \pmod{p_1 - 1} \), the above formula can be rewritten as

\[
v \equiv \hat{u}_{ij} + h k^2(k p_1 - 1) \pmod{p_1 - 1}.
\]

Direct computation shows that

\[
v > k^3\ p_1^4\ e^{4(p_1 - 1)}\ C_1 > k p_1(k p_1 - 1) \left[ C_1 + (\hat{u}_{ij} - 1) \frac{p_1^{l+1} - 1}{p_1 - 1} + 1 \right] = k p_1(k p_1 - 1)D_{ij} > k(k p_1 - 1)\{(p_1 - 1)D_{ij} + k p_1\} + p_1D_{ij} > k(k p_1 - 1)\{(p_1 - 1)D_{ij} + h k\} + \hat{u}_{ij} + (p_1 - 1)D_{ij}.
\]
From this calculation we see that
\[
\frac{v - \hat{u}_{ij} - hk^2(kp_1 - 1)}{p_1 - 1} - D_{ij} - k(kp_1 - 1)D_{ij}
\]
is a positive integer and hence can be written as \(a + k(kp_1 - 1)d\), where \(d \geq 0\) and \(0 < a \leq k(kp_1 - 1) < C_i\). Take \(m = (p_1 - 1)(d + D_{ij}) + kh\) and
\[
t = \frac{(p_1 - 1)(a + D_{ij})}{k - 1} + \frac{\hat{u}_{ij} - 1}{k - 1},
\]
where \(t\) is an integer because \((u_i - 1)/(k - 1)\) is an integer! It is easy to check that
\[
v = k(kp_1 - 1)m + (k - 1)t + 1. \tag{6}
\]
By Lemma 6.2 there are a TD\((k, kp_1 - 1)\), a TD\((k, kp_1)\) and a TD\((kp_1 + 1, m)\). Apply Lemma 2.1 with \(l = k\) and \(n = kp_1\) to get a \(\{k, kp_1\}\)-GDD with group type \((m - 1)k^{kp_1 - 1}((k - 1)t)^1\). Hence,
\[
k(kp_1 - 1)m + (k - 1)t
\]
\(+ 1 \in B\{k, kp_1, m(k - 1) + 1, (kp_1 - 1)m + 1, t(k - 1) + 1\}\).
By Lemmas 2.5 and 4.2 \(kp_1 \in DB(k, \lambda)\). Note that \(m(k - 1) + 1 \equiv 1 \pmod{k - 1} / 2(k - 1)\) and \(m(kp_1 - 1) + 1 \equiv 1 \pmod{k - 1} / 2(k - 1)\). By Theorem 4.7, \((m - 1) + 1\) and \((m - 1) + 1 \in DB(k, \lambda)\). Since \((k - 1)t + 1 = (a + D_{ij})(p_1 - 1) + \hat{u}_{ij}\), by Lemma 6.3, \((k - 1)t + 1 \in DB(k, \lambda)\). Then \((k(kp_1 - 1)m + (k - 1)t + 1 \in DB(k, \lambda)\). Hence, from (6) \(v \in DB(k, \lambda)\). □

**Lemma 6.5.** Let \(k \geq 8\) and \(C^* = \exp\{4(k^2 + 1)p_1^{k^2(k^2 + 1)}\}\). Then \(C^* > D(\tilde{u}_i + 1)\) for \(i \in [1, s]\).

**Proof.** By Lemmas 2.2 and 5.2, \(C^* > \exp\{4\tilde{u}_i\} > D(\tilde{u}_i + 1)\). □

**Theorem 6.6.** Let \(k \geq 8, 0 \leq I(\lambda) \leq I(k) - 2\) and \(v > k^2(k - 1)(k^2 + 1)p_1^{k^2(k^2 + 1)}C^*\). If congruences (1) hold, then \(v \in DB(k, \lambda)\).

**Proof.** Recall that \(u_1, u_2, \ldots, u_s\) are all the representatives of the residue classes modulo \(k(k - 1)/2(k^2 + 1)\) of congruences (1). For such \(v\), there is an integer \(i \in [1, s]\) such that \(v \equiv u_i \equiv \tilde{u}_i \pmod{k(k - 1)/2(k^2 + 1)}\) by Lemma 4.2. So, we can find an integer 
\(h \in \{0, 2(k^2 + 1) - 1\}\) such that 
\(v \equiv \tilde{u}_i + (k(k - 1)/2(k^2 + 1))h \pmod{k - 1}\). By Lemma 5.2, 
\(v > k^2(k - 1)\tilde{u}_i C^* > k^2(k - 1)(\tilde{u}_i - 1)C^* + k(k - 1)C^* + \tilde{u}_i + (k(k - 1)/2(k^2 + 1))h\), we easily know that
\[
\frac{v - \tilde{u}_i - k(k - 1)h/2(k^2 + 1)}{k(k - 1)} - k(\tilde{u}_i - 1)C^* - C^*
\]
is a positive integer and hence can be written as \(a + k(\tilde{u}_i - 1)\) where \(d \geq 0\) and \(0 < a \leq k(\tilde{u}_i - 1)\). Take \(m = k(k - 1)(C^* + d) + 1\) and \(t = k(C^* + a) - \tilde{u}_i + (k/2(k^2 + 1)h + 1)\). It is easy to check that
\[
v = k(\tilde{u}_i - 1)m + (k - 1)t + 1. \tag{7}
\]
By Lemmas 5.4 and 6.5, there are a TD($k, \tilde{u}_i$), a TD($k, \tilde{u}_i - 1$) and a TD($\tilde{u}_i + 1$).

By Lemma 2.1 with $l = k$ and $n = \tilde{u}_i$ to get a \{k, $\tilde{u}_i$\}-GDD with group type (m($k - 1$))^$\tilde{u}_i - 1$($m(\tilde{u}_i - 1)$)($k - 1)t)$. Hence,

\[ k(\tilde{u}_i - 1)m + (k - 1)t \]

\[ + 1 \in \text{DB}(\{k, \tilde{u}_i, m(k - 1) + 1, (\tilde{u}_i - 1)m + 1, t(k - 1) + 1\}). \]

By Lemma 5.2, $\tilde{u}_i \in \text{DB}(k, \lambda)$. Since $m \equiv 1 \pmod{k(k - 1)}$, $m(k - 1) + 1 \equiv k \pmod{k(k - 1)}$ and $m > C^*$, by Lemma 2.10, $m(k - 1) + 1 \in \text{B}(k) \subseteq \text{DB}(k, \lambda)$. By Lemma 5.4 there exists a TD($k, \tilde{u}_i - 1$). By Lemma 2.4 $m(\tilde{u}_i - 1) + 1 \in \text{DB}(k, \lambda)$. Let $f$ be an integer such that $k + 1 < f \leq (k(k - 1)/2^j(\tilde{u}_i)) + k + 1$ and $f \equiv (k - 1)t + 1 \pmod{k(k - 1)/2^j(\tilde{u}_i) + 1}$. It is readily checked that $\lambda f(f - 1) \equiv 0 \pmod{\binom{t}{2}}$ and $2\lambda f(f - 1) \equiv 0 \pmod{m(k - 1)}$ and so $f$ is a solution of the congruences (1). By Lemma 3.2 $f$ or $f + k(k - 1)/2^j(\tilde{u}_i) + 1$ is a solution of congruences (2) and hence $f$ or $f + k(k - 1)/2^j(\tilde{u}_i) + 1$ is $u_j$ for some $j \in [1, s]$. Now $(u_j - 1)/(k - 1)$ is an integer because $(f - 1)/(k - 1)$ is an integer, and $(k - 1)t + 1 > C^* > k^3 P_1^k C_1$, by Theorem 6.4 $(k - 1)t + 1 \in \text{DB}(k, \lambda)$. Then $k(\tilde{u}_i - 1)m + (k - 1)t + 1 \in \text{DB}(k, \lambda)$. Hence, from (7) $v \in \text{DB}(k, \lambda)$. \qed

**The proof of Theorem 1.2.** When $I(k) \geq 2$ and $I(\lambda) \geq I(k) - 1$, the conclusion follows by Lemma 3.1. When $0 \leq I(\lambda) \leq I(k) - 2$, it is not difficult to check

\[ \exp\{k^{3e^8}\} > k^2(k - 1)(k^2 + 1)P_1^{k^2(\lambda^2 + 1)}C^*. \]

By Theorem 6.6, the conclusion follows immediately.

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**References**