Unsteady flow and heat transfer of pseudo-plastic nanoliquid in a finite thin film on a stretching surface with variable thermal conductivity and viscous dissipation

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A R T I C L E   I N F O

Article history:
Received 10 November 2014
Received in revised form 8 January 2015
Accepted 18 January 2015
Available online 26 January 2015

Keywords:
Nanoliquid
Nanoparticle
Thin film
Stretching sheet
Heat transfer
Nonlinear equation

A B S T R A C T

This paper studies flow and heat transfer of pseudo-plastic nanoliquid in a finite thin film over an unsteady stretching surface with variable thermal conductivity and viscous dissipation effects. Four different types of nanoparticles, Cu, Al2O3, CuO and TiO2 are considered with sodium carboxymethyl cellulose (CMC)-water used as a base fluid. Unlike most classical works, a modified Fourier’s law of heat conduction for power-law fluids is adopted by assuming that the thermal conductivity is power-law-dependent on the velocity gradient. Similarity transformations are applied to reduce the governing partial differential equations into a system of nonlinear ordinary differential equations, which are solved numerically by a shooting method coupled with Runge–Kutta method and BVP4C. The effects of solid volume fraction, types of nanoparticles, power-law index, unsteadiness parameter, modified Prandtl number and Eckert number on film thickness, velocity and temperature fields are graphically illustrated and discussed.

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1. Introduction

The flow and heat transfer of a thin film over a stretching sheet have attracted much attention due to its applications arising from many fields of science and technology. Such applications include wire and fiber coating, metal and polymer extrusion, foodstuff processing, continuous casting, drawing of plastic sheets, exchangers, transpiration cooling, reactor fluidization, chemical processing equipment, etc. All coating processes demand a smooth glossy surface to meet requirements for appearance, low friction, transparency and strength. The analysis of flow and heat transfer within a thin film on a continuously stretching surface is important. In view of these applications, Wang [11] studied the flow of a Newtonian fluid within a finite thin film over an unsteady stretching sheet. It showed that the exact similarity solutions of the unsteady thin film Navier–Stokes equations may be obtained by restricting the motion to a specified family of time dependence. Later, Andersson et al. [2] studied the flow of an incompressible fluid in a thin liquid film obeying a power law model. Afterwards, Andersson and coworkers [3,4] explored the heat transfer characteristics of the hydrodynamical problem solved by Wang [11]. Chen [5–7] examined the flow and heat transfer of power law fluid in a thin liquid film on an unsteady surface and took the effects of viscous dissipation or Marangoni convection into account. Wang and Pop [8] considered the flow of a power law fluid film on an unsteady stretching sheet. Analytical solutions are obtained using the homotopy analysis method (HAM) and a critical value for unsteadiness parameter is derived. Abel and coworkers [9,10] studied MHD flow and heat transfer of a laminar liquid film over an unsteady stretching surface with external magnetic field and viscous dissipation effects. Aziz and coworkers [11–13] extended the problem of flow and heat transfer of finite thin film over an unsteady stretching sheet by considering a general surface temperature. A list of key references in the literature concerning this field can be found in Refs. [14–20].

All of the abovementioned investigations are restricted to pure fluids (Newtonian or non-Newtonian). As a novelty, nanofluid, proposed by Choi [21], is physically described as a heat transfer basic fluid containing a suspension of submicron solid particles (nanoparticles). It is expected that a mixture of the base fluid and nanoparticles will develop advanced heat transfer fluids with higher conductivities. Wang and Mujumdar [22] and Haddad et al. [23] offered an overview of the literature on the recent developments in nanofluids. The problems of flow and heat transfer within a finite thin film over an unsteady stretching sheet are extended by Bachok et al. [24], Xu et al. [25] and Narayana and Sibanda [26] to nanoliquid films. It should be noted that...
the base fluids of nanoliquid are restricted to Newtonian (pure) fluids in Refs. [24–26]. Pseudo-plastic non-Newtonian fluids are important due to its many industrial applications. Recently, considerable attention has been devoted to the problem of predicting the behavior of non-Newtonian fluids. Pop and coworkers [27,28] proposed a model that the thermal conductivity of power law fluids was power-law-dependent on the velocity gradient. Chamikha [29] analyzed a steady laminar boundary layer flow and heat transfer in a quiescent non-Newtonian fluid driven by a stretched porous surface. The flow and heat transfer of power law fluids are further studied by considering various thermal conductivity models such as Pop’s or Zheng’s by Lin et al. [30–33].

In this paper we investigate the flow and heat transfer of pseudo-plastic nanoliquid in a finite thin film over an unsteady stretching surface with variable thermal conductivity and viscous dissipation effects. We assume that the temperature field is similar to the velocity and the thermal conductivity of the fluid is power-law-dependent on the velocity gradient by modified Fourier’s law. Four types of nanoparticles, i.e., copper (Cu), aluminum oxide (Al2O3), copper oxide (CuO), and titanium oxide (TiO2) are considered. The CMC-water (carboxyl methyl cellulose) is used as the base fluid of nanoliquid [33]. The partial differential governing equations are reduced and solutions are obtained numerically by BVP4C and the shooting method coupled with the Runge–Kutta method. The effects of solid volume fraction, power-law index, unsteadiness parameter, modified Prandtl number and Eckert number are illustrated graphically via the velocity and temperature profiles.

The novel contributions of this paper are the following:

(i) Pseudo-plastic nanoliquid constitutive law modeling by incorporating import physical effect;
(ii) The consideration of finite thickness of the thin film with surface stretching;
(iii) The distinctive resulting properties of various nanoparticles.

2. Formulation for stretching surface problem

Consider the flow and heat transfer of pseudo-plastic nanoliquid in a finite film over a horizontal sheet issuing from a narrow slot. The fluid motion is caused by the stretching of the elastic sheet. A schematic of the physical model and coordinate system is shown in Fig. 1. The CMC-water-based nanoliquid contains different types of nanoparticles. Experimental studies show that the carboxymethyl cellulose (CMC) water solution with concentration of 0.0%–0.3% is used as a base fluid of nanoliquid. The viscos properties of the CMC-water are in Table 1 [33]. Thermophysical properties of the nanoliquid are given in Table 2 [33–37]. We assume that the nanoliquid is incompressible, the flow is laminar, the base fluid and the nanoparticles are in thermal equilibrium and that no slippage occurs between them. Furthermore, it is assumed that the stretching of elastic sheet has viscous dissipation, the interface of the nanoliquid is incompressible, the nanoliquid is in thermal equilibrium and that no slippage occurs between them. Furthermore, it is assumed that the stretching of elastic sheet has viscous dissipation, the interface of the nanoliquid is incompressible, the nanoliquid is in thermal equilibrium and that no slippage occurs between them. Furthermore, it is assumed that the stretching of elastic sheet has viscous dissipation, the interface of the nanoliquid is incompressible, the nanoliquid is in thermal equilibrium and that no slippage occurs between them.

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \\
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y} \left( \frac{\mu_n}{\rho C_p} \frac{\partial u}{\partial y} \right) + \frac{\mu_f}{\rho C_p} \frac{\partial v}{\partial y} \left( \frac{\partial u}{\partial y} \right)^2. 
\]

Note that the coefficient term \( \frac{\mu_n}{\rho C_p} \) in Eq. (2), \( \alpha_n \) in Eq. (3) model the pseudo-plastic property of the fluid. The boundary conditions are

\[
y = 0: u = u_w, v = 0, T = T_w, \\
y = h(x, t): \frac{\partial u}{\partial y} + \frac{\partial T}{\partial y} = 0, v = u \frac{\partial h}{\partial x} + \frac{\partial h}{\partial t}. 
\]

where \( u \) and \( v \) are the velocity components along the \( x \) and \( y \) directions, respectively, \( t \) is time, \( \tau_{xy} = \mu豆腐|\partial u/\partial y|^n-1\partial u/\partial y \) is shear stress, \( \mu \) is the effective density of nanoliquid, \( \mu_f \) is modified consistency coefficient, \( \rho \) is effective density of nanoliquid and \( n \) is power law index. The case \( n = 1 \) corresponds to a Newtonian fluid, \( 0 < n < 1 \) describes a pseudo-plastic fluid while \( n > 1 \) is for a dilatant fluid. \( T \) is temperature, \( \rho \) is specific heat at constant pressure, \( (\rho C_p)_f \) is effective heat capacity of the nanoliquid, and \( h(x, t) \) is thickness of the nanoliquid film. It is noted that the condition \( v = u \frac{\partial h}{\partial x} + \frac{\partial h}{\partial t} \) imposes a kinematic constraint of the fluid motion [1–7]. The effects of power law viscosity on temperature fields are taken into account by assuming that the temperature field is similar to the velocity field and the thermal conductivity is power-law-dependent on velocity as \( k = \alpha_{nf}(\rho C_p)_f|\partial u/\partial y|^n-1 \) (k is the effective thermal diffusivity of the nanoliquid, \( \alpha_{nf} \) is the modified thermal diffusivity of the nanoliquid) and \( \alpha_{nf}|\partial u/\partial y|^n-1 \) is the effective thermal diffusivity of nanoliquid. The term \( \mu_f/(\rho C_p)_f|\partial u/\partial y|^n-1(\partial u/\partial y)^2 \) accounts for the viscous dissipation effect. The fluid is caused by stretching the wall surface (the elastic surface) at \( y = 0 \) such that the continuous sheet moves in the \( x \)-direction with the velocity \( u_w \) as

\[
u_w = \beta x/(1-\alpha t),
\]

\[
|\begin{array}{c}
\text{Slot} \\
|\begin{array}{c}
\text{Free surface} \\
|\begin{array}{c}
\text{nanoliquid} \\
\text{u} \\
\text{T} \\
\text{h(x,t)}
\end{array}
|\begin{array}{c}
\text{Wall surface} \\
\text{T_w}
\end{array}
|\begin{array}{c}
\text{Stretching sheet}
\end{array}
\end{array}
\end{array}
|\begin{array}{c}
\text{Fig. 1. Schematic of the physical system.}
\end{array}
\]

<p>| Table 1 |
| Viscous properties of base fluid (CMC-water) [32,33]. |</p>
<table>
<thead>
<tr>
<th>Physical properties</th>
<th>CMC-water (0.0%)</th>
<th>CMC-water (0.1%)</th>
<th>CMC-water (0.2%)</th>
<th>CMC-water (0.3%)</th>
<th>CMC-water (0.4%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n )</td>
<td>1.00</td>
<td>0.91</td>
<td>0.81</td>
<td>0.76</td>
<td></td>
</tr>
<tr>
<td>( K ) (N·s/m²)</td>
<td>0.000855</td>
<td>0.006319</td>
<td>0.017540</td>
<td>0.0313603</td>
<td>0.0785254</td>
</tr>
</tbody>
</table>

<p>| Table 2 |
| Thermophysical properties of base fluid and nanoparticles [32–37]. |</p>
<table>
<thead>
<tr>
<th>Physical properties</th>
<th>CMC-water (0.0%-0.4%)</th>
<th>Cu</th>
<th>Al2O3</th>
<th>CuO</th>
<th>TiO2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_p ) (J/kg K)</td>
<td>4179</td>
<td>385</td>
<td>765</td>
<td>535.6</td>
<td>688.2</td>
</tr>
<tr>
<td>( \rho ) (kg m⁻³)</td>
<td>997.1</td>
<td>8933</td>
<td>3970</td>
<td>6500</td>
<td>4250</td>
</tr>
<tr>
<td>( k ) (W/m K)</td>
<td>0.613</td>
<td>400</td>
<td>40</td>
<td>20</td>
<td>8.9538</td>
</tr>
</tbody>
</table>
Eqs. (6) and (7) are valid for time $t$ as a constant. It should be noted that the expressions given by [32,33,39] are the base fluid parameters are given by\[34,35,39\]

\[
T 
\]

is the solid volume fraction of nanoliquid. The other parameters are given by [34,35,39–44]:

\[
\rho_{nf} = (1 - \phi) \rho_f + \phi \rho_s, 
\]

\[
\mu_{nf} = \mu_f/(1 - \phi)^{2.5}, 
\]

where $\phi$ is the solid volume fraction of nanoliquid. The other parameters are given by [34,35,39–44]:

\[
\rho_{nf} = (1 - \phi) \rho_f + \phi \rho_s, 
\]

\[
\alpha_{nf} = \frac{k_{nf}}{(\mu C_p)_f} = \frac{k_f}{(\mu C_p)_f}, 
\]

where $\rho_s$ is density of the solid, $(\mu C_p)_f$ heat capacity of the base fluid, $(\mu C_p)_f$ heat capacity of solid, $k_{nf}[\partial u/\partial y]^n$ effective thermal conductivity of nanoliquid, $k_{nf}$ modified thermal conductivity of the nanoliquid, $k_f$ thermal conductivity of base fluid, and $k_s$ thermal conductivity of solid.

\[
\text{Table 3} 
\]

Comparison of $\beta$ and $-f'(0)$ for various values of $S$ with $n = 1.0, \phi = 0.0 \%$.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>$-f'(0)$</td>
<td>$\beta$</td>
<td>$-f'(0)$</td>
<td>$\beta$</td>
</tr>
<tr>
<td>0.8</td>
<td>2.15199</td>
<td>1.24581</td>
<td>2.15199</td>
<td>1.24580</td>
</tr>
<tr>
<td>1.0</td>
<td>1.54362</td>
<td>1.27777</td>
<td>1.5438</td>
<td>1.2777</td>
</tr>
<tr>
<td>1.2</td>
<td>1.12778</td>
<td>1.27917</td>
<td>1.1278</td>
<td>1.27918</td>
</tr>
</tbody>
</table>

Comparison of the critical value for various values of the power law index.\[44\]:

\[
\text{Table 4} 
\]

Comparison of the critical value for various values of the power law index.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0.8</td>
<td>1.67</td>
<td>1.35</td>
<td>4/3</td>
<td>1.3333</td>
<td></td>
</tr>
<tr>
<td>1.0</td>
<td>2.50</td>
<td>3.03</td>
<td>3</td>
<td>2.0000</td>
<td></td>
</tr>
<tr>
<td>1.2</td>
<td>2.50</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Comparison of the critical value for various values of the power law index.

\[
\text{Figure 2.} 
\]

Variations of film thickness with unsteadiness parameter at selected $n$. In terms of standard definition of stream function such that $u = \phi \partial y/\partial y$ and $v = -\partial \phi/\partial x$, the following transformation variables are introduced:

\[
\psi(x, y, t) = b^{(2-n)/(n+1)} y_f^{-1/2} \left(1 - \frac{x}{a}\right)^{-2} y_f^{-1}, 
\]

\[
\eta = b^{(2-n)/(n+1)} y_f^{-1/2} \left(1 - \frac{x}{a}\right)^{-2} \psi(x, y), 
\]

\[
T = T_0 - T_{ref} b^{-\gamma_f} y_f^{-2/3} \left(1 - \frac{x}{a}\right)^{-1} \gamma_f^{-1}, 
\]

\[
\beta = b^{2-n} y_f^{-1/2} \left(1 - \frac{x}{a}\right)^{-2} \beta_f(x, t), 
\]

\[
\text{Re}_x = \frac{u}{\nu} \gamma_f^{-1/2} \left(1 - \frac{x}{a}\right)^{n-2}, 
\]

\[
\text{Pr}_{nf} = \frac{\mu_{nf} / \rho_{nf} [\partial u / \partial y]^n}{\alpha_{nf} [\partial u / \partial y]^{n-1} / \alpha_{nf} (\mu C_p)_{nf}}, 
\]

\[
\text{Pr} = \frac{\mu_f / \rho_f (\mu C_p)_f}{k_f} = \frac{\gamma_f (\mu C_p)_f}{k_f}, 
\]

\[
S = \frac{a}{b} E_c = \frac{\mu_f}{(\mu C_p)_f} T_{ref}. 
\]

\[
\text{Table 5} 
\]

Comparison of the critical value of unsteadiness for different CMC-water base fluids, between Wang and Pop [8] and ours. There is total agreement.

<table>
<thead>
<tr>
<th>CMC-water base fluids</th>
<th>0.4%, $n = 0.76$</th>
<th>0.3%, $n = 0.81$</th>
<th>0.2%, $n = 0.85$</th>
<th>0.1%, $n = 0.91$</th>
<th>0.0%, $n = 1.00$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Analytical results [8]</td>
<td>$S$</td>
<td>$S$</td>
<td>$S$</td>
<td>$S$</td>
<td>$S$</td>
</tr>
<tr>
<td>Our results</td>
<td>1.2258</td>
<td>1.3613</td>
<td>1.4783</td>
<td>1.6697</td>
<td>2.0000</td>
</tr>
</tbody>
</table>
Using Eq. (15) and Eqs. (28)–(29), we obtain
\[ \text{Nu}_k = -\frac{q_w(x)}{k(T(x,0)-T(x,m))} = \frac{q_w(x)}{k(T_w-T_0)} \]
where \( q_w(x) \) is heat flux of nanoliquid film

Using Eq. (15) and Eqs. (28)–(29), we obtain
\[ \text{Nu}_k = -\frac{q_w(x)}{k(T(x,0)-T(x,m))} = \frac{q_w(x)}{k(T_w-T_0)} \]

Here, \( \theta'(0) \) represents the local Nusselt number.

### 3. Numerical techniques

In order to obtain numerical solutions for Eqs. (21)–(24), we transform the problems to a system of first-order differential equations, denoting by \( f, f', f'', \theta \) and \( \theta' \) variables as \( f, p, q, \theta \) and \( g \), resulting in
\[ f' = p. \]
\[ \rho' = q, \]  
\[ q' = f = \frac{A}{n}(q)^{1-n} \left[ S(\beta + \frac{2-n}{n+1}q) + p^2 - \frac{2n}{n+1}f^q \right], \]  
\[ \theta' = g, \]  
\[ g' = \theta' = (-q)^{1-n}Pr\left[ S(\beta + \frac{2-n}{n+1}q) + (p^2 - \frac{2n}{n+1}f^q) \right] + Ec(1-\phi)^{2.5}q^{n-1} \left[ S(\beta + \frac{2-n}{n+1}q) + (p^2 - \frac{2n}{n+1}f^q) \right], \]  
\[ f(0) = 0, p(0) = 1, f(\beta) = \frac{2-n}{2n}jS, q(\beta) = 0, \]  
\[ \theta(0) = 0, g(\beta) = 0, \]  
\[ A = (1-\phi)^{2.5} \left[ (1-\phi) + \phi \frac{\rho_i}{\rho_f} \right]. \]

Fig. 7. Variations of \(-\theta(0)\) with the solid volume fraction at selected values of \(n\).

Fig. 8. Effects of different nanoparticles on velocity profiles.

Fig. 9. Effects of different nanoparticles on temperature profiles.

Fig. 10. Variations of \(-\theta(0)\) with the solid volume fraction for different nanoparticles.
The dependence relationship of $\beta$ and $S$ is now solved by computer program BVP4C in MATLAB. For example, when $n = 0.76$, Eqs. (31)-(33) and Eq. (36) reduce to

$$
\begin{align*}
  f(0) & = 0, \\
  p(0) & = 1, \\
  q(0) & = 0.81579 S, \\
  q(\beta) & = 0.
\end{align*}
$$

(Eqs. (41)-(42))

Eqs. (41)-(42) are two-point boundary value problems with an unknown parameter $S$. When $\beta = 1$ (CuCMC; $\phi = 5\%$, moreover, $A = 1.22970$, $B = 0.86420$, $C = 0.99127$), using the program BVP4C in MATLAB [45], we obtain $S = 0.8130$.

Similarly, let $\beta$ ranges from 0.01 to 1.00, $N = 100$. We can obtain the dependence relationship of $\beta$ and $S$. It should be noted that the initial value of $S$ is $S = 1.0$ and the initial solutions of Eqs. (41)-(42) (when $0.01 \leq \beta \leq 1.00$) are

$$
\begin{align*}
  f(\eta) & = \eta + \frac{1}{2} \eta^2 + \frac{1}{6} \eta^3, \\
  p(\eta) & = 1 + \eta + \frac{1}{2} \eta^2, \\
  q(\eta) & = i + j \eta.
\end{align*}
$$

(Fig. 11)

4. Results and discussion

In this study, the flow and heat transfer of pseudo-plastic nanoliquid (CMC-water base fluids) in a finite thin film over an unsteady stretching surface with variable thermal conductivity and viscous dissipation effects are investigated. Effects of the solid volume fraction, power law index, unsteadiness parameter, the modified Prandtl number and the Eckert number on flow and heat transfer of the thin nanoliquid film are considered.

For hydrodynamic problems, there exists a critical value for unsteadiness parameter $S_0$ above which no solution could be obtained [5-8]. For positive values of $S$, $S \to 0$ corresponds to the case of $\beta \to +\infty$, while $S \to S_0$ corresponds to that of $\beta \to 0$. To verify the accuracy and effectiveness of the present method, the results for degradation ($n = 1.0$, $\phi = 0.0\%$) are presented with those in References [3,5-8] in Table 3. For different values of the power law

$$(43)$$

(Fig. 13, Fig. 14)
index, the values of the critical value obtained by the present method are compared with previous results [1,3,5–8] in Table 4. In addition, for different CMC–water base fluids, a comparison is also presented for the critical value of unsteadiness parameter obtained in this paper with the analytical results by Wang and Pop [8] in Table 5. It is seen that the present results agree totally with [8]. Figs. 2–4 illustrate, in sequential order, the nanoliquid thickness $\beta$ varying with the unsteadiness parameter $S$ at selected values of the power law index, the solid volume fraction and different nanoparticles. As the unsteadiness parameter increases from 0 to $S_0$, the thickness of nanoliquid film decreases from $+\infty$ to 0. The results imply that the nanoliquid film thickness increases as the power law index increases, while decreases as the solid volume fraction increases. It should be noted that the critical value is the same for different values of the solid volume fraction or different nanoparticles.

Figs. 5–7 show the effects of different CMC–water base fluids (the power law index) on the flow and heat transfer of the Cu/CMC nanoliquid thin film, in sequential order, the dimensionless velocity $f'(\eta)$, the dimensionless temperature $\theta(\eta)$, and the dimensionless local Nusselt number $-\theta'(0)$, while other physical parameters are fixed. For particular values of unsteadiness parameter and parameters, the pseudo-plastic nanoliquid film has a smaller thickness, associated with a larger free-surface velocity, than that of a Newtonian nanoliquid. It can be further obtained that the velocity at $S = 1.0, \phi = 3.0\%$ for Cu/CMC nanoliquid varies by 25.91% across the nanoliquid film for $n = 0.76$ and by as much as 69.89% for $n = 1.00$ in Fig. 5. Also, the temperature at $S = 1.0, \phi = 3.0\%, Ec = 0$, and $Pr = 0.78$ for the film varies by 24.73% across the film for $n = 0.76$ and by as much as 75.06% for $n = 1.00$ in Fig. 6. The results show that the velocity and the temperature increase, while the local Nusselt number decreases as the base fluids shearing thinner, i.e., as the power law index decreases. In addition, Fig. 7 shows that the CMC–water base fluids with high power law index have a better enhancement effect for in heat transfer.

Figs. 8 and 9 show the influences of different nanoparticles on the dimensionless velocity and the dimensionless temperature for the nanoliquid thin film with 0.2% CMC base fluids at selected of other parameters such as $\phi = 4.5\%, S = 1.0, Ec = 0$ and $Pr = 2.0$. The results show that the velocity and temperature profiles are well behaved and very little change occurs in the shapes of profiles with different types of nanoparticles. Furthermore, both the velocity and the temperature profiles decrease as the location similarity variable increases for all four nanoparticles. For different nanoparticle films with the same base fluids, the rate of decline of the velocity is: $Al_2O_3 < TiO_2 < CuO < Cu$, while the rate of decline of the temperature is: $Cu < CuO < Al_2O_3 < TiO_2$. Also, typical variations of the dimensionless local Nusselt number, in terms of the wall temperature gradient $-\theta(0)$, is presented for different nanoparticles in Fig. 10. The results show that for all four nanoparticle values of $-\theta(0)$ are always all positive, i.e., the heat is transferred from a hot surface to a cold surface. Thus, nanoparticles with low thermal conductivity, TiO$_2$ has better enhancement on heat transfer than Cu, CuO and Al$_2$O$_3$ in this problem.

Figs. 11 and 12 show the influence of the solid volume fraction on the dimensionless velocity and the dimensionless temperature for the 0.4% Cu/CMC thin film, while other physical parameters are fixed as $n = 0.76, S = 1.0, Ec = 0$ and $Pr = 5$. The results show that as the solid volume fraction of the film increases, both the thickness and the velocity decrease, while the temperature increases. Furthermore, the free-surface velocity varies by about 24.91% across the Cu/CMC nanoliquid thin film for all different values of the solid volume fraction (0.0%, 1.5%, 3.0%, 4.5%), i.e., the effect of the solid volume fraction on the velocity difference between the wall and free surfaces of the film is not obvious. However, the free-surface temperature at $n = 0.76, S = 1.0, Pr = 5, Ec = 0$ for Cu/CMC nanoliquid varies by 71.21% across the nanoliquid film for $\phi = 4.5\%$ and by 79.64% for $\phi = 0\%$, i.e., the temperature difference between the wall and free surfaces decreases as the solid volume fraction increases.

Figs. 13–15 show the influences of the Eckert number and the modified Prandtl number on heat transfer characteristic of the film, i.e., the dimensionless temperature $\theta(\eta)$. The results show that heat transfer behaviors are strongly dependent on the values of the Eckert number and the modified Prandtl number. Figs. 13–14 show that the temperature profiles rapidly decreases from the wall surface to the free surface of the film for all values of the Eckert number. It also can be seen that the temperature decreases, while the temperature difference between the wall and free surfaces increases as the Eckert number increases for both the CuO/CMC and Al$_2$O$_3$/CMC thin films with $n = 0.76, \phi = 5.0\%, S = 1.0, and Pr = 0.78$. Fig. 15 shows that the temperature decreases as the modified Prandtl number increases for TiO$_2$/CMC film and particular values of other parameters $n = 0.76, \phi = 5.0\%, S = 0.8, and Ec = 0$. The temperature profiles are reduced more rapidly by increasing the modified Prandtl number. It is also interesting to find that a nearly uniform distribution, $\theta(\eta) = 1$ or $T = T_w$, is observed in the liquid film at a very low generalized Prandtl number $Pr \to 0$ (e.g., $Pr = 0.01$).

5. Conclusions

Viscous dissipation effects on the flow and heat transfer of finite pseudo–plastic nanoliquid film over an unsteady stretching surface with variable thermal conductivity are studied in this paper. The governing PDEs are reduced into a system of coupled non-linear ODEs by proper similarity transformation. Numerical solutions are obtained using the MATLAB BVP4C and the shooting method coupled with the Runge–Kutta scheme and Newton’s method. The following results are established:

(a) The critical values are the same for different values of solid volume fraction or different types of nanoparticles for fixed other parameters.
(b) Power law index and solid volume fraction have significant effects on the film thickness and velocity profiles.
(c) Power law index, solid volume fraction, modified Prandtl number and Eckert number have extraordinary effects on temperature field heat transfer.
(d) The temperature decreases as the Eckert number increases. Base fluids with lower power law index have higher thermal conductivities than those of the corresponding base fluids.
Acknowledgments
The research was supported by the Scientific Research Funds of Huaqiao University (No. 14B5310), the National Natural Science Foundation of China (No. 51076012, 51276014), and Qatar National Research Fund (QNRF) grant National Priority Research Project (NPRP) #5-674-1-114.

Appendix A. Supplementary data
Supplementary data to this article can be found online at http://dx.doi.org/10.1016/j.powtec.2015.01.039.