Existence of solutions for some discrete boundary value problems with a parameter

Yang Yang\textsuperscript{a,b}, Jihui Zhang\textsuperscript{b,}\textsuperscript{*}

\textsuperscript{a}School of Science, Jiangnan University, Wuxi, People’s Republic of China
\textsuperscript{b}Institute of Mathematics, School of Mathematics and Computer Sciences, Nanjing Normal University, Nanjing, People’s Republic of China

\textbf{Abstract}

In this paper, the existence and multiplicity results of solutions are obtained for the discrete nonlinear two point boundary value problem (BVP) \[ -\Delta^2 u(k - 1) = \lambda f(k, u(k)) \quad k \in \mathbb{Z}(1, T); \]
\[ u(0) = 0 = \Delta u(T), \]
where $T$ is a positive integer, $\mathbb{Z}(1, T) = \{1, 2, \ldots, T\}$, $\Delta$ is the forward difference operator defined by $\Delta u(k) = u(k + 1) - u(k)$ and $f : \mathbb{Z}(1, T) \times \mathbb{R} \to \mathbb{R}$ is continuous, $\lambda \in \mathbb{R}^+$ is a parameter. By using the critical point theory and Morse theory, we obtain that the above (BVP) has solutions for $\lambda$ being in some different intervals.

\textbf{1. Introduction}

In this paper, we consider the existence and multiplicity of the solutions for the following discrete nonlinear two point boundary value problem:

\[ -\Delta^2 u(k - 1) = \lambda f(k, u(k)), \quad k \in \mathbb{Z}(1, T); \]
\[ u(0) = 0 = \Delta u(T), \]
where $T$ is a positive integer, $\mathbb{Z}(1, T) = \{1, 2, \ldots, T\}$, $\Delta$ is the forward difference operator defined by $\Delta u(k) = u(k + 1) - u(k)$ and $f : \mathbb{Z}(1, T) \times \mathbb{R} \to \mathbb{R}$ is continuous, $\lambda \in \mathbb{R}^+$ is a parameter.

Recent years, there have been many papers to study the existence of solutions on the nonlinear boundary value problems. By using various methods and techniques, such as Schauder fixed point theorem, the cone theoretic fixed point theorem, the method of upper and lower solutions and coincidence degree theory, a series of existence results of nontrivial solutions for difference equations have been obtained, we refer to [1–6] and references therein. There are also many authors who studied the discrete equations by using the critical point theory (see [7–9]). In particular, in paper [10], using the mountain pass lemma, Jiang considered the existence of nontrivial solutions of the (BVP) (1.1) and (1.2) with $\lambda = 1$. Motivated by his idea, in this paper, we shall study the (BVP) (1.1) and (1.2). By means of the Morse theory and local linking, when $f$ under other conditions, we obtain the existence of one solution or two nontrivial solutions for the discrete nonlinear two point boundary value problem.

\textbf{2. Preliminaries}

In this section, we give some notations and lemmas. Let $V = \{ u : \mathbb{Z}(1, T + 1) \to \mathbb{R}; u(0) = \Delta u(T) = 0 \}$. Then $V$ is a $T$-dimensional Hilbert space with the inner product.
\[ (u, v) = \sum_{k=1}^{T} u(k) v(k) \]
and we denote the induced norm by
\[ \|u\| = \left( \sum_{k=1}^{T} |u(k)|^2 \right)^{\frac{1}{2}}. \]

Let \( G(k, t) \) be Green's function for the following boundary value problem:
\[
\begin{aligned}
-\lambda^2 u(k-1) &= 0, \quad k \in Z(1, T); \\
u(0) = 0 &= \Delta u(T).
\end{aligned}
\]
Thus
\[ G(k, t) = \min\{k, t\} = \begin{cases} 
  k, & k \leq t; \\
  t, & t \leq k.
\end{cases} \]
It is easy to see that \( G(k, t) = G(t, k) \) for all \( t, k \in Z(1, T) \) and \( \max_{(k, t) \in Z(1, T)} G(k, t) = T. \)
Then the (BVP) (1.1) and (1.2) has a solution in \( V \) if and only if the following equation:
\[ u(k) = \lambda \sum_{t=1}^{T} G(k, t)f(t, u(t)), \quad k \in Z(1, T) \]
has a solution in \( V \).

Define the operators \( K, A_f : V \rightarrow V \) by
\[ Ku(k) = \sum_{t=1}^{T} G(k, t)u(t) \]
and
\[ A_f u(k) = f(k, u(k)). \]
Thus the (BVP) (1.1) and (1.2) has a solution in \( V \) if and only if the following equation:
\[ u = \lambda K A_f u \]
has a solution in \( V \).

It is well known that all eigenvalues of \( K \) are
\[ \{\lambda_i\}_{i \in Z(1, T)} = \frac{1}{4 \sin^2 \left( \frac{\pi}{4T} \right)} \]
and the corresponding orthonormal eigenfunctions are \( \{e_i\}_{i \in Z(1, T)} \), where
\[ e_i(k) = \sin \left( \frac{2(k-1)\pi}{2T+1} \right) \left( \sum_{n=1}^{T} \sin^2 \left( \frac{2n-1)\pi}{2T+1} \right) \right)^{\frac{1}{2}} > 0 \]
and \( \lambda_1 > \lambda_2 > \cdots > \lambda_T > 0. \)

We still have the following lemmas about the operator \( K \).

Lemma 2.1 [10]. \( K : V \rightarrow V \) is a linear completely continuous operator. Furthermore, \( K \) is symmetric definite, and any eigenvalue of \( K \) is larger than \( \frac{1}{4} \).

Lemma 2.2. \( K^2 : V \rightarrow V \) is a linear bounded symmetric finite positive operator.

Proof. The proof of the lemma can be found in [10]. As a complement, we give the proof.
Since \( K \) is a linear continuous and symmetric finite positive operator, the following formulas with \( K \) hold:
\[
\begin{align*}
u &= \sum_{t=1}^{T} (u, e_t) e_t, \quad u \in V; \\
\|u\|^2 &= \sum_{t=1}^{T} |(u, e_t)|^2, \quad u \in V; \\
K u &= \sum_{t=1}^{T} \frac{1}{4 \sin^2 \left( \frac{\pi}{4T} \right)} (u, e_t) e_t, \quad u \in V.
\end{align*}
\]
From (2.1), \( K^2 : V \rightarrow V \), the positive square root of \( K \) exists, and is unique, also bounded linear and symmetric. Then

\[
K^2 u = \sum_{i=1}^{r} \frac{1}{2\sin^{2}\left(\frac{2\pi - 1}{4r}\right)} (u, e_i)e_i, \quad u \in V, \tag{2.2}
\]

\[
(K^2 u, u) = \sum_{i=1}^{r} \frac{1}{2\sin^{2}\left(\frac{2\pi - 1}{4r}\right)} |(u, e_i)|^2, \quad u \in V. \tag{2.3}
\]

So \((K^2 u, u) > 0\) if \( u \neq 0, u \in V \), and the proof is completed. 

Remark 2.1. Equalities (2.2) and (2.3) imply that \( K^2 u \neq 0 \) for all \( u \in V \) with \( u \neq 0 \). Therefore, \( K^2 u_1 \neq K^2 u_2 \) for all \( u_1, u_2 \in V \) with \( u_1 \neq u_2 \).

In the next section, we will use the critical point theory and Morse theory to discuss the (BVP) (1.1) and (1.2). Here we state some necessary definitions and lemmas.

Definition 2.1 [11,12]. Let \( E \) be a real Banach space, and \( D \) an open subset of \( E \). Suppose that a functional \( J : D \rightarrow R \) is Fréchet differentiable on \( D \). If \( x_0 \in D \) and the Fréchet derivative \( J' (x_0) = 0 \), then we call that \( x_0 \) is a critical point of the functional \( J \) and \( c = J(x_0) \) is a critical value of \( J \).

Lemma 2.2 [11,12]. Let \( E \) be a real reflexive Banach space. If the functional \( J : E \rightarrow R \) is weakly lower semicontinuous and coercive, i.e. \( \lim_{||x|| \rightarrow \infty} J(x) = +\infty \). Then there exists an \( x_0 \) such that \( J(x_0) = \inf_{x \in E} J(x) \). Moreover, if \( J \) has bounded linear Gâteaux derivative on \( E \), the \( x_0 \) is also a critical point of \( J \), i.e. \( J'(x_0) = 0 \).

Let \( E \) be a real Banach space, let \( J \in C^1(E,R) \). For any \( c \in R \), now we recall some notions and results in Morse theory.

Definition 2.3 [13]. Let \( f = \{ u \in E : J(u) \leq c \} \), let \( u \) be an isolated critical point of \( J \) with \( J(u) = c \) and let \( U \) be a neighborhood of \( u \), containing the unique critical point. We call

\[
C_q(J, u) = H_q(f \cap U, f \cap U \setminus \{ u \}),
\]

the \( q \)th critical group of \( J \) at \( u \), \( q = 0, 1, 2, \ldots \) where \( H_q(\cdot, \cdot) \) stands for the \( q \)th singular relative homology group with integer coefficients. We say that \( u \) is a homological nontrivial critical point of \( J \), if at least one of its critical groups is non trivial.

Lemma 2.4 [14,15]. Let \( 0 \) be a critical point of \( J \) with \( J(0) = 0 \). Assume that \( J \) has a local linking at \( 0 \) with respect to \( E = V_1 \oplus V_2 \), \( k = \dim V_1 < \infty \). That is, there exists \( \rho > 0 \) small, such that

\[
\begin{align*}
J(u) &\leq 0, \quad u \in V_1, \quad ||u|| \leq \rho, \\
J(u) &> 0, \quad u \in V_2, \quad 0 < ||u|| \leq \rho.
\end{align*}
\]

Then \( C_1(J, 0) \neq 0 \). That is, \( 0 \) is a homological nontrivial critical point of \( J \).

Lemma 2.5 [15,16]. Assume that \( J \) satisfies the (PS) condition and is bounded from below. If \( J \) has a critical point that is homological nontrivial and is not the minimizer of \( J \), then \( J \) has at least three critical points.

Lemma 2.6 [10]. For a fixed \( \lambda \in R^+ \),

(i) the operator equation

\[
u = \lambda K^2 \lambda u \]

has a solution in \( V \) if and only if the operator equation

\[
u = \lambda K^2 \lambda K^2 \lambda \]

has a solution in \( V \).

(ii) If (1.5) has three solutions in \( V \), then (1.4) has at least three solutions in \( V \).
Lemma 2.7. Suppose that the functional
\[ J(u) = \frac{1}{2} (u, u) - \lambda \sum_{k=1}^{T} \int_{0}^{\beta(k)} f(k, s) ds, \quad u \in V \]
has a critical point \( u \in V \), then the (BVP) (1.1) and (1.2) has a solution in \( V \).

Proof. The proof is similar to that in [10], and we omit it here. \( \square \)

3. Main results and proofs

In this section, we give the existence results for the (BVP) (1.1) and (1.2), and make the following assumptions:

\( (f_0) \) There exists \( a > 0 \) such that \( \limsup_{|u| \to \infty} \frac{F(k, u)}{|u|^2} < a \), uniformly for \( k \in Z(1, T) \), where \( F(k, u) = \int_{0}^{1} f(k, s) ds \);

\( (f_1) \) There exists \( a > 0 \) such that \( \lim_{|u| \to \infty} \frac{F(k, u)}{|u|^2} = a \), uniformly for \( k \in Z(1, T) \);

\( (f_2) \) There exists \( a > 0 \) such that \( \limsup_{|u| \to \infty} \frac{F(k, u)}{|u|^2} \leq a \), uniformly for \( k \in Z(1, T) \);

\( (f_3) \) \( \limsup_{|u| \to \infty} (F(k, u) - au^2) = -\infty \), uniformly for \( k \in Z(1, T) \);

\( (f_4) \) \( \limsup_{|u| \to \infty} (f(k, u)u - 2F(k, u)) = +\infty \), uniformly for \( k \in Z(1, T) \);

\( (f_5) \) There exist \( \delta, A, B \in (0, +\infty) \) and an integer \( 1 \leq i < T \), which satisfy
\[ A > B > A \sin^2 \left( \frac{(2i+1)}{2} \right) > 0 \]
such that
\[ Bu^2 \leq F(k, u) \leq Au^2 \]
for all \( |u| \leq \delta \) and \( k \in Z(1, T) \);

\( (f_6) \) Let \( f(k, u) = 2au + g(k, u) \) for \( u_n = v_n + w_n \), \( v_n \in E \left( \frac{1}{4 \sin^2 \frac{\pi}{2}} \right) \) and \( w_n \in E \left( \frac{1}{4 \sin^2 \frac{\pi}{2}} \right) \). If \( \|u_n\| \to \infty \) and \( \frac{|u_n|}{|u_n|} \to 1 \), then there exist \( \delta > 0 \) and \( N > 0 \), such that
\[ \sum_{k=1}^{T} g(k, K^2 u_n(k)) k^2 v_n(k) \leq -\delta \]
for \( n \geq N \).

We have the following results:

**Theorem 3.1.** \( f \) satisfy \( (f_0) \), then for any \( \lambda \in (0, \frac{\pi}{2} \sin^2 \frac{\pi}{4T+2}) \), the (BVP) (1.1) and (1.2) has at least one solution.

**Theorem 3.2.** Let \( f \) satisfy \( (f_1) \) and \( (f_2) \), then for any \( \lambda \in (0, \frac{\pi}{2} \sin^2 \frac{\pi}{4T+2}) \), the (BVP) (1.1) and (1.2) has at least one solution.

**Theorem 3.3.** Let \( f \) satisfy \( (f_1) \) and \( (f_4) \), then for any \( \lambda \in (0, \frac{\pi}{2} \sin^2 \frac{\pi}{4T+2}) \), the (BVP) (1.1) and (1.2) has at least one solution.

**Theorem 3.4.** Let \( f \) satisfy \( (f_2) \) and \( (f_6) \), then for any \( \lambda \in (0, \frac{\pi}{2} \sin^2 \frac{\pi}{4T+2}) \), the (BVP) (1.1) and (1.2) has at least one solution.

**Theorem 3.5.** Let \( f \) satisfy \( (f_0) \) and \( (f_5) \), then for any
\[ \lambda \in \left[ 2 \sin^2 \left( \frac{(2i+1)}{2} \right), 2 \sin^2 \left( \frac{(2i+1)}{2} \right) \right] \subset \left( 0, \frac{2 a \sin^2 \frac{\pi}{4T+2}}{A} \right), \]
the (BVP) (1.1) and (1.2) has at least two nontrivial solutions.

**Theorem 3.6.** Let \( f \) satisfy \( (f_1) \), \( (f_3) \) and \( (f_5) \), then for any
\[ \lambda \in \left[ 2 \sin^2 \left( \frac{(2i+1)}{2} \right), 2 \sin^2 \left( \frac{(2i+1)}{2} \right) \right] \subset \left( 0, \frac{2 a \sin^2 \frac{\pi}{4T+2}}{A} \right), \]
the (BVP) (1.1) and (1.2) has at least two nontrivial solutions.

**Theorem 3.7.** Let \( f \) satisfy \( (f_1) \), \( (f_2) \) and \( (f_5) \), then for any
\[ \lambda \in \left[ \frac{2}{B} \sin^2 \left( \frac{(2i-1)\pi}{4T+2} \right), \frac{2}{A} \sin^2 \left( \frac{(2i+1)\pi}{4T+2} \right) \right] \subset \left( \frac{2}{A} \sin^2 \frac{\pi}{4T+2} \right). \]

The (BVP) (1.1) and (1.2) has at least two nontrivial solutions.

**Theorem 3.8.** Let \( f \) satisfy (f_2), (f_3) and (f_6), then for any

\[ \lambda \in \left[ \frac{2}{B} \sin^2 \left( \frac{(2i-1)\pi}{4T+2} \right), \frac{2}{A} \sin^2 \left( \frac{(2i+1)\pi}{4T+2} \right) \right] \subset \left( \frac{2}{A} \sin^2 \frac{\pi}{4T+2} \right), \]

the (BVP) (1.1) and (1.2) has at least two nontrivial solutions.

**Remark 3.1.** If \( a = 4 \sin^2 \left( \frac{(2i-1)\pi}{4T+2} \right) \), \( i \in \mathbb{Z} \cap (1, T) \), \( f_0 \) is a nonresonance condition, and \( f_1 \) means the problem is resonant at infinity, and we impose the condition \( f_2 \) or \( f_3 \) which are frequently used in resonance to get the compact condition, while \( f_2 \) can unify the conditions \( f_0 \) and \( f_1 \), but \( f_3 \) condition cannot be obtained directly. So we impose a more interesting condition \( f_0 \) to settle the problem.

Now we give the proofs of the theorems. Let the functional \( J : V \rightarrow \mathbb{R} \) be given by

\[ J(u) = \frac{1}{2} (u, u) - \lambda \sum_{k=1}^{T} \int_{0}^{\xi k u(k)} f(k, s)ds, \quad u \in V. \]

**Lemma 3.1.** Let \( f \) satisfy (f_3), when \( \lambda \in \left[ \frac{2}{B} \sin^2 \left( \frac{(2i-1)\pi}{4T+2} \right), \frac{2}{A} \sin^2 \left( \frac{(2i+1)\pi}{4T+2} \right) \right] \), the functional \( J \) has a local linking with respect to \( V = V_1 \oplus V_2 \), where \( V_1 = \text{span}\{e_1, e_2, \ldots, e_i\}, V_2 = V_1^\perp \).

**Proof.** Let \( V_1 = \text{span}\{e_1, e_2, \ldots, e_i\}, V_2 = V_1^\perp \). If \( u \in V_1 \), then \( (Ku, u) \geq \frac{1}{4 \sin^2 \frac{\pi}{4T+2}} \|u\|^2 \); and if \( u \in V_2 \), then \( (Ku, u) \leq \frac{1}{4 \sin^2 \frac{\pi}{4T+2}} \|u\|^2 \).

We can assume that for the given \( \delta > 0 \), there is a \( \rho = 2\delta \sin \frac{\pi}{4T+2} > 0 \), such that

\[ u \in V_1, \quad \|u\| \leq \rho \Rightarrow \|K^2 u(k)\| \leq \|K^2 u\| \leq \frac{1}{2 \sin \frac{\pi}{4T+2}} \|u\| = \delta \quad \text{for} \ k \in \mathbb{Z} \cap (1, T), \]

thus by (f_3), we have

\[ J(u) = \frac{1}{2} (u, u) - \lambda \sum_{k=1}^{T} \int_{0}^{\xi k u(k)} f(k, s)ds = \frac{1}{2} \|u\|^2 - \lambda \sum_{k=1}^{T} F(k, K^2 u(k)) \]

\[ \leq \frac{1}{2} \|u\|^2 - \lambda \sum_{k=1}^{T} B(K^2 u(k))^2 = \frac{1}{2} \|u\|^2 - B\lambda (Ku, u) \]

\[ \leq \frac{1}{2} \|u\|^2 - \frac{B\lambda}{4 \sin^2 \frac{(2i-1)\pi}{4T+2}} \|u\|^2 \leq 0, \quad \text{for} \ u \in V_1, \ |u| \leq \rho. \]

For \( u \in V_2 \), consider the above \( \rho \), we still have

\[ \|u\| \leq \rho \Rightarrow |K^2 u(k)| \leq \|K^2 u\| \leq \frac{1}{2 \sin \frac{\pi}{4T+2}} \|u\| = \delta \quad \text{for} \ k \in \mathbb{Z} \cap (1, T), \]

thus by (f_3), we have

\[ J(u) = \frac{1}{2} (u, u) - \lambda \sum_{k=1}^{T} \int_{0}^{\xi k u(k)} f(k, s)ds = \frac{1}{2} \|u\|^2 - \lambda \sum_{k=1}^{T} F(k, K^2 u(k)) \]

\[ \geq \frac{1}{2} \|u\|^2 - \lambda \sum_{k=1}^{T} A(K^2 u(k))^2 = \frac{1}{2} \|u\|^2 - A\lambda (Ku, u) \]

\[ \geq \frac{1}{2} \|u\|^2 - \frac{A\lambda}{4 \sin^2 \frac{(2i-1)\pi}{4T+2}} \|u\|^2 \geq 0 \quad \text{for} \ u \in V_2, \ |u| \leq \rho. \]

This implies that \( J \) has a local linking at 0 with respect to \( V = V_1 \oplus V_2 \). By Lemma 2.4, one has \( C_i(0) \neq 0 \), that is 0 is homological nontrivial. \( \Box \)
Lemma 3.2. If \( f \) satisfies one of the following conditions:

(a) \((f_0)\) and \( \lambda \in (0, \frac{a}{2} \sin^2 \frac{\pi}{4T+2}) \);
(b) \((f_1), (f_2)\) and \( \lambda \in (0, \frac{a}{2} \sin^2 \frac{\pi}{4T+2}) \);
(c) \((f_1), (f_2)\) and \( \lambda \in (0, \frac{\pi}{2} \sin^2 \frac{\pi}{4T+2}) \).

Then, we have

(i) \( J \) is coercive on \( V \), That is \( J(u) \to +\infty \), as \( \|u\| \to \infty \);
(ii) \( J \) satisfies the (PS) condition.

Proof. (i) Let (a) hold. It follows from \((f_0)\) that there is a constant \( c > 0 \), such that \( F(k, u) < au^2 + c, \forall u \in R, k \in Z(1, T) \). Therefore

\[
J(u) = \frac{1}{2} (u, u) - \lambda \sum_{k=1}^{R} F(k, K^2 u(k)) = \frac{1}{2} \|u\|^2 - \lambda \sum_{k=1}^{R} (a(K^2 u(k))^2 + c)
\]

\[
= \frac{1}{2} \|u\|^2 - \lambda a (Ku, u) - \lambda Tc \\
\geq \frac{1}{2} \|u\|^2 - \frac{\lambda a}{4 \sin^2 \frac{\pi}{4T+2}} \|u\|^2 - \lambda Tc
\]

\[
= \left( 1 - \frac{\lambda a}{4 \sin^2 \frac{\pi}{4T+2}} \right) \|u\|^2 - \lambda Tc \to +\infty \quad \text{as} \quad \|u\| \to \infty.
\]

Let (b) or (c) hold. Assume that \((f_1)\) is satisfied. Write \( f(k, u) = 2au + g(k, u) \), and \( F(k, u) = au^2 + G(k, u) \). If \((f_2)\) holds, then \( \lim_{\|u\| \to \infty} G(k, u) = -\infty \); if \((f_1)\) holds, then

\[
\lim_{\|u\| \to \infty} \frac{G(k, u)}{u^2} = 0. \quad \text{and} \quad \lim_{\|u\| \to \infty} \frac{G(k, u)u - 2G(k, u)}{u^2} = +\infty.
\]

It follows that for every \( M > 0 \), there is \( R_M > 0 \), such that \( \forall u \in R, |u| \geq R_M, k \in Z(1, T) \).

Integrating the equality

\[
\frac{d}{du} \frac{G(k, u)}{u^2} = \frac{g(k, u)u - 2G(k, u)}{u^3}
\]

over the interval \([u, U] \subset [R_M, +\infty)\), we have

\[
\frac{G(k, U)}{U^2} - \frac{G(k, u)}{u^2} \geq -M \left( \frac{1}{U^2} - \frac{1}{u^2} \right).
\]

Let \( U \to +\infty \), we see that \( G(k, u) \leq -\frac{M}{u^2} \) for \( k \in Z(1, T), u \geq R_M \). In a similar way, we have \( G(k, u) \leq -\frac{M}{u^2} \) for \( k \in Z(1, T), u \leq -R_M \). Hence \( \lim_{\|u\| \to \infty} G(k, u) = -\infty, k \in Z(1, T) \).

Let \( \{u_n\} \subset V \) be such that \( \|u_n\| \to \infty \) as \( n \to \infty \), and \( J(u_n) \leq C \), for some constant \( C \in R \). Taking \( v_n = \frac{u_n}{\|u_n\|} \), then passing to a subsequence, we may assume that there is some \( v_0 \in V \), such that \( v_n \to v_0 \) in \( V \) with \( \|v_0\| = 1 \), then \( K^2 v_n \to K^2 v_0 \) in \( V \).

Now

\[
\frac{C}{\|u_n\|^2} \geq \frac{J(u_n)}{\|u_n\|^2} = \frac{1}{2} \frac{\lambda a}{\|u_n\|^2} \sum_{k=1}^{R} (K^2 u_n(k))^2 - \frac{\lambda a}{\|u_n\|^2} \sum_{k=1}^{R} G(k, K^2 u_n(k)) = \frac{1}{2} \frac{\lambda a}{\|u_n\|^2} \sum_{k=1}^{R} \left( a(K^2 u_n(k))^2 + G(k, K^2 u_n(k)) \right)
\]

\[
\geq \frac{1}{2} - \lambda a \sum_{k=1}^{R} (K^2 u_n(k))^2 + \frac{\lambda a}{\|u_n\|^2} \sum_{k=1}^{R} G(k, K^2 u_n(k)) \geq \frac{1}{2} - \lambda a \sum_{k=1}^{R} (K^2 u_n(k))^2 + \frac{\lambda a}{\|u_n\|^2} \sum_{k=1}^{R} G(k, K^2 u_n(k))
\]

\[
\geq \frac{1}{2} - \lambda a \sum_{k=1}^{R} (K^2 u_n(k))^2 - \frac{C}{\|u_n\|^2} \geq \frac{1}{2} - 2 \sin^2 \frac{\pi}{4T+2} + \frac{\pi}{4T+2} \sum_{k=1}^{R} (K^2 v_n(k))^2 - \frac{C}{\|u_n\|^2}.
\]

Then

\[
(K v_0, v_0) = \lim_{n \to \infty} (K u_n, v_n) \geq \frac{1}{4 \sin^2 \frac{\pi}{4T+2}} = \frac{1}{4 \sin^2 \frac{\pi}{4T+2}} \|v_0\|^2.
\]
while for any \( v \in V \), we have

\[
(Kv, v) \leq \frac{1}{4} \sin^2 \frac{\pi}{4T} \|v\|^2.
\]

So

\[
(Kv_0, v_0) = \frac{1}{4} \sin^2 \frac{\pi}{4T} \|v_0\|^2.
\]

Hence \( v_0 = \pm \varepsilon_1 \), then \( K^2v_0(k) \neq 0 \) for all \( k \in Z(1, T) \), and

\[
|K^2u_n(k)| = |K^2(\|u_n\|v_n(k))| = \|u_n\| |K^2v_n(k)| \to \infty.
\]

So \( G(k, K^2u_n(k)) \to -\infty \), as \( n \to \infty \), for \( k \in Z(1, T) \).

Therefore

\[
C \geq J(u_n) = \frac{1}{2} \sum_{k=1}^{T} (K^2u(k))^2 - \lambda \sum_{k=1}^{T} G(k, K^2u(k)) \geq \frac{1}{2} \left( \|u_n\|^2 - 4 \sin^2 \frac{\pi}{4T} (Ku, u_n) \right) - \lambda \sum_{k=1}^{T} G(k, K^2u(k)) \geq -\lambda \sum_{k=1}^{T} G(k, K^2u_n(k)) \to +\infty, \quad \text{as} \quad n \to \infty.
\]

This is impossible, so \( J \) is coercive on \( V \).

(ii) It follows from (i) that if \( J(u_n) \) is bounded, then \( \{u_n\} \) is bounded in \( V \), which implies that \( \{u_n\} \) has a convergent subsequence, and the (PS) condition is satisfied. \( \square \)

**Proof of Theorem 3.1.** We shall verify that the functional \( J(u) \) defined in Lemma 2.7 has a critical point \( u \in V \). Since \( J(u) = \lambda K^2A_1Ku = (I - \lambda K^2A_1K)^{-1}u \), and \( K^2A_1K \) is completely continuous, then \( J : V \to R \) is \( C^1 \) functional and \( J \) is weakly semicontinuous on \( V \). By Lemma 3.2, \( J \) is coercive. In view of Lemma 2.3, \( J \) has a critical point \( u \in V \). The proof is completed. \( \square \)

The proofs of Theorems 3.2 and 3.3 are similar to the proof of Theorem 3.1.

**Proof of Theorem 3.5.** By Lemma 3.2, \( J \) is coercive and satisfies the (PS) condition. Hence \( J \) is bounded from below. By Lemmas 2.4 and 3.1, the trivial solution \( u = 0 \) is homological nontrivial. If \( \inf_{u \in V} J(u) \geq 0 \), then \( J(u) = \inf_{u \in V} J(u) = 0 \) for all \( u \in V \), with \( \|u\| \leq \rho \), which implies that all \( u \in V \), with \( \|u\| \leq \rho \) are solutions of the (BVP) (1.1) and (1.2). If \( \inf_{u \in V} J(u) < 0 \), that is, 0 is not a minimizer of \( J \). By Lemma 2.5, one has that \( J \) has at least three critical points. Thus the problem (1.1) and (1.2) has at least two nontrivial solutions. This completes the proof of Theorem 3.5. \( \square \)

The proofs of Theorems 3.6 and 3.7 are similar to the proof of Theorem 3.5.

**Proof of Theorem 3.8.** Firstly, we show that \( J \) satisfies the (PS) condition and is bounded from below. Define \( J \) by

\[
J(u) = \frac{1}{2} \sum_{k=1}^{T} (K^2u(k))^2 - \lambda \sum_{k=1}^{T} G(k, K^2u(k)),
\]

where \( \lambda \in (0, \frac{3}{4} \sin^2 \frac{\pi}{4T} \pi) \) and \( G(k, u) = F(k, u) - au^2 \).

Put \( u = v + w \), with \( v \in E(\frac{\sin^2 \frac{\pi}{4T} \pi}{\sin^2 \frac{\pi}{4T} \pi}), w \in E(\frac{1}{\sin^2 \frac{\pi}{4T} \pi}) \). By (f2), we note that for any given \( \varepsilon > 0 \), there is \( \gamma > 0 \) such that \( G(k, u) \equiv \varepsilon u^2 \) for \( |u| > \gamma \). So there exists a constant \( C > 0 \), such that \( G(k, u) \leq \varepsilon u^2 + C \) for \( u \in R \). Hence

\[
J(u) = \frac{1}{2} \sum_{k=1}^{T} (K^2u(k))^2 - \lambda \sum_{k=1}^{T} G(k, K^2u(k)) \geq \frac{1}{2} \left[ \|w\|^2 - 2 \sin^2 \frac{\pi}{4T} \|Ku\|^2 \right] - \lambda \sum_{k=1}^{T} \varepsilon (K^2u(k))^2 - \lambda CT \]

\[
= \frac{1}{2} \left[ \|w\|^2 - 4 \sin^2 \frac{\pi}{4T} \|Ku\|^2 \right] - \lambda \varepsilon (Ku, u) - \lambda CT \]

\[
\geq \frac{1}{2} \left( \|w\|^2 - \frac{\sin^2 \frac{\pi}{4T} \pi}{\sin^2 \frac{\pi}{4T} \pi} \|w\|^2 \right) - \frac{\lambda \varepsilon}{4 \sin^2 \frac{\pi}{4T} \pi} \|w\|^2 - \frac{\lambda \varepsilon}{4 \sin^2 \frac{\pi}{4T} \pi} \|w\|^2 - \lambda CT \]

\[
\geq \frac{1}{2} \left( 1 - \frac{\sin^2 \frac{\pi}{4T} \pi}{\sin^2 \frac{\pi}{4T} \pi} \frac{\varepsilon}{\lambda} \right) \|w\|^2 - \frac{\varepsilon}{2a} \|w\|^2 - \lambda CT.
\]
Taking $0 < \epsilon < a(1 - \sin^2 \frac{\sin \phi}{\sin^2 \sin \phi})$, we see that for a fixed $R > 0$, 
$$J(u) \to +\infty \quad \text{as} \quad \|u\| \to \infty \quad \text{with} \quad \|v\| = R.$$ 
Since $J$ is weakly lower semicontinuous on $V$, this shows that $J$ attains its minimum on the set 
$$C_R = \{ u \in V, u = v + w, \|v\| = R \}.$$ 
That is for any given $R > 0$, there is $u_R \in V$, such that 
$$J(u_R) \leq J(u) \quad \text{for} \quad u \in C_R.$$ 
Now take $\{ u_n \} \subset V$, such that $\|u_n\| \to \infty$, and 
$$J(u_n) \leq J(u) \quad \text{for} \quad u \in C_{R_n}.$$ 
Where we put $u_n = w_n + \tau_n e_1$ and $e_1 > 0$ is the first eigenfunction such that $\|e_1\| = 1$, $\tau_n \in R$ and $w_n \in E \left( \frac{1}{2 \sin \frac{\sin \phi}{\sin^2 \sin \phi}} \right)^{-1}$. If there is some $\theta \in [0, 1)$, such that $|\tau_n| \leq \theta |u_n|$ for large $n$, then $\|w_n\| \geq \sqrt{1 - \theta^2} |\tau_n|$, and we obtain 
$$J(u_n) \leq \frac{1}{2} \left( 1 - \frac{\sin^2 \frac{\sin \phi}{\sin^2 \sin \phi}}{2\sin \frac{\sin \phi}{\sin^2 \sin \phi}} - \frac{\epsilon}{\theta} \right) \|w_n\|^2 - \frac{\epsilon}{2 \theta \sqrt{1 - \theta^2}} \|w_n\|^2 - \lambda CT,$$
we see that $J(u_n) \to +\infty$ as $n \to \infty$, if $\epsilon > 0$ is small enough. Therefore, we consider the case where $\frac{\sin \phi}{\sin \sin \phi} \to 1$ as $n \to \infty$. Passing to a subsequence, we can choose $|\tau_n| < |\tau_{n+1}|$. Then 
$$J(u_{n+1}) - J(u_n) = J(w_{n+1} + \tau_{n+1} e_1) - J(w_n + \tau_n e_1) \geq J(w_{n+1} + \tau_{n+1} e_1) - J(w_{n+1} + |\tau_n|/|\tau_{n+1}|) \tau_{n+1} e_1$$ 
$$\geq \lambda \sum_{k=1}^{\tau_{n+1}} g(k, K^2(\tau_{n+1} k) + |\tau_n|/|\tau_{n+1}|) \tau_{n+1} e_1(k)) - \lambda \sum_{k=1}^{\tau_{n+1}} g(k, K^2(\tau_{n+1} k) + \tau_{n+1} e_1(k))$$ 
$$= \lambda \sum_{k=1}^{\tau_{n+1}} \int_{0}^{1} g(k, K^2(\tau_{n+1} k + \tau_n s \tau_{n+1} e_1(k))) \tau_{n+1} e_1(k) \tau_n(s) \tau_{n+1} K^2 e_1(k) \tau_n(s) ds,$$
where 
$$\tau_n(s) = |\tau_n| + (1 - s)|\tau_{n+1}|, \quad s \in (0, 1), \quad \tau_n(s) \leq 1.$$ 
Since 
$$\frac{|\tau_n|}{\|u_n\|} \to 1, \quad \text{then} \quad \frac{\|w_n\|}{|\tau_n|} \to 0$$
and 
$$\frac{\|w_{n+1} + \tau_n(s) \tau_{n+1} e_1\|^2}{\|\tau_n(s) \tau_{n+1} e_1\|^2} = \frac{\|w_{n+1}\|^2 + \|\tau_n(s) \tau_{n+1} e_1\|^2}{\|\tau_n(s) \tau_{n+1} e_1\|^2} = 1 + \frac{\|w_{n+1}\|^2}{\|\tau_{n+1}\|^2} \cdot \frac{1}{\|\tau_n(s)\|^2} \to 1 \quad \text{as} \quad n \to \infty,$$
so we get 
$$\frac{\|\tau_n(s) \tau_{n+1} e_1\|}{\|w_{n+1} + \tau_n(s) \tau_{n+1} e_1\|} \to 1 \quad \text{as} \quad n \to \infty$$
by $(f_0)$, there exist $\delta > 0$ and $N > 0$, such that 
$$\sum_{k=1}^{\tau_{n+1}} g(t, K^2(\tau_{n+1} k + \tau_n s \tau_{n+1} e_1(k))) K^2(\tau_n(s) \tau_{n+1} e_1(k)) \leq -\delta \quad \text{for} \quad n \geq N.$$ 
Thus 
$$J(u_{n+1}) - J(u_n) \geq -\delta \int_{0}^{1} \tau_n(s) \tau_n(s) ds = -\delta \int_{0}^{1} \frac{|\tau_n| - |\tau_{n+1}|}{s|\tau_n| + (1 - s)|\tau_{n+1}|} ds = \delta |\ln|\tau_{n+1}| - \ln|\tau_n||).$$
Hence 
$$J(u_{n+1}) = J(u_n) + \sum_{k=1}^{n} (J(u_{n+1}) - J(u_k)) \geq J(u_n) + \delta \sum_{k=1}^{n} (|\ln|\tau_{n+1}| - |\ln|\tau_n||$$
$$= J(u_n) + \delta \ln|\tau_{n+1}| - \ln|\tau_n|) \to +\infty \quad \text{as} \quad n \to \infty.$$ 
Therefore $J$ is coercive on $V$, and bounded from below. By Lemma 3.1, the trivial solution $u = 0$ is homological nontrivial. If $\inf_{w \in V} J(u) = 0$, then $J(u) = \inf_{w \in V} J(u) = 0$ for all $u \in V_1$ with $\|u\| \leq \rho$, which implies that all $u \in V_1$ with $\|u\| \leq \rho$ are solutions
of the (BVP) (1.1) and (1.2). If \( \inf_{x \in [a, b]} f(u) < 0 \), that is 0 is not a minimizer of \( f \). By Lemma 2.5, one has that \( f \) has at least three critical points. Thus the problem (1.1) and (1.2) has at least two nontrivial solutions. This completes the proof of Theorem 3.8. \( \square \)

The proof of Theorem 3.4 follows from the proof of Theorem 3.8 and Lemma 2.3.

Example 1. Consider the (BVP)
\[
- \Delta^2 u(k - 1) = \lambda \frac{u^3(k) - 1}{1 + u^2(k)}, \quad k \in Z(1, T);
\]
\[
u(0) = 0 = \Delta u(T).
\]
Set \( f(k, u) = \frac{u^2}{1 + u^2} \), and \( F(k, u) = \frac{1}{2} u^2 - \frac{1}{2} \ln(1 + u^2) - \arctan u \). By simple calculation, we have
\[
\lim_{|u| \to \infty} \frac{F(k, u)}{u^2} = 1 > 0, \quad \lim_{|u| \to \infty} (f(k, u) u - 2F(k, u)) = +\infty.
\]
Thus the condition \((f_1)\) and \((f_4)\) hold. Hence by Theorem 3.3, when \( \lambda \in (0, 4 \sin^2 \frac{\pi}{24}] \), the (BVP) has at least one nontrivial solution.

Example 2. Consider the (BVP)
\[
- \Delta^2 u(k - 1) = \lambda f(k, u(k)), \quad k \in Z(1, T);
\]
\[
u(0) = 0 = \Delta u(10),
\]
where
\[
f(k, u) = \begin{cases} 
12u, & -1 < u < 1, \quad k \in Z(1, 10); 
(u - 1)/(k + 1) + 12, & u \geq 1, \quad k \in Z(1, 10); 
(u + 1)/(k + 1) - 12, & u \leq -1, \quad k \in Z(1, 10).
\end{cases}
\]
and
\[
F(k, u) = \begin{cases} 
6u^2, & -1 < u < 1, \quad k \in Z(1, 10); 
(u - 1)^2/(2k + 2) + 12u - 6, & u \geq 1, \quad k \in Z(1, 10); 
(u + 1)^2/(2k + 2) - 12u - 6, & u \leq -1, \quad k \in Z(1, 10).
\end{cases}
\]
By simple calculation, we have
\[
\lim_{|u| \to \infty} \sup \frac{F(k, u)}{|u|^2} = 1/(2k + 2) < 101/400 \quad \text{for } k \in Z(1, 10)
\]
and there exist an integer \( i = 2 \) and \( \delta = 1 > 0 \), such that
\[
2.31u^2 \leq F(k, u) \leq 6u^2 \leq 6.4u^2 \quad \text{for } |u| \leq 1,
\]
set \( A = 6.4, B = 2.31 \), then \( A, B \) satisfy
\[
A > B > A \frac{\sin^2 \frac{\pi}{12}}{\sin^2 \frac{\pi}{24}}.
\]
Thus the condition \((f_5)\) and \((f_5)\) hold. Hence by Theorem 3.5, when \( \lambda \in [1.275 \times 10^{-5}, 1.33 \times 10^{-5}] \subset (0, 1.35 \times 10^{-5}) \), the (BVP) has at least two nontrivial solutions.

Acknowledgements

The project is supported by NSFC (No. 10871096), Foundation of Major Project of Science and Technology of Chinese Education Ministry, Project of Graduate Education Innovation of Jiangsu Province and Foundation for young teachers of Jiangnan University (No. 2008LQN008).

The authors thank the referee for his (her) valuable comments and suggestions.

References