Brief paper

High-degree cubature Kalman filter✩

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A R T I C L E   I N F O
Article history:
Received 30 January 2012
Received in revised form
27 August 2012
Accepted 7 September 2012
Available online 14 December 2012
Keywords:
Cubature Kalman filter
Nonlinear estimation
Kalman filter
Gaussian quadrature
Numerical integration

A B S T R A C T

The cubature Kalman filter (CKF), which is based on the third degree spherical–radial cubature rule, is numerically more stable than the unscented Kalman filter (UKF) but less accurate than the Gauss–Hermite quadrature filter (GHQF). To improve the performance of the CKF, a new class of CKFs with arbitrary degrees of accuracy in computing the spherical and radial integrals is proposed. The third-degree CKF is a special case of the class. The high-degree CKFs of the class can achieve the accuracy and stability performances close to those of the GHQF but at lower computational cost. A numerical integration problem and a target tracking problem are utilized to demonstrate the necessity of using the high-degree cubature rules to improve the performance. The target tracking simulation shows that the fifth-degree CKF can achieve higher accuracy than the extended Kalman filter, the UKF, the third-degree CKF, and the particle filter, and is computationally much more efficient than the GHQF.

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1. Introduction

Nonlinear filtering and estimation based on the Bayesian framework have been intensively studied and many filters have been proposed in recent decades. They can be roughly classified into two categories. The first category contains filters that are capable of handling non-Gaussian systems such as particle filters (Gordon & Salmond, 1993) and filters based on the numerical solutions to the Fokker–Planck equation (Challa & Bar-Shalom, 2000; Kumar & Chakravorty, 2010). The second category contains a variety of Gaussian approximation filters based on the Gaussian assumption of the probability density functions. Typical filters in the second category are the extended Kalman filter (EKF) (Gelb, 1974), the unscented Kalman filter (UKF) (Julier, Uhlmann, & Durrant-Whyte, 2000), and the Gauss–Hermite quadrature filter (GHQF) (Ito & Xiong, 2000). Although the filters in the first category are able to solve any non-Gaussian estimation problems, the computational load is usually prohibitive (Daum, 2005). In contrast, the filters in the second category are computationally more efficient. In addition, the Gaussian approximation filters can be used to assist particle filters to achieve higher sampling efficiency (Van der Merwe, de Freitas, Doucet, & Wan, 2000) and be combined with a Gaussian sum model to solve non-Gaussian estimation problems based on the Gaussian sum approximation (Alspach & Sorenson, 1972).

Among the Gaussian approximation filters, the GHQF based on Gauss–Hermite quadrature (GHQ) is more accurate and stable and has been shown to achieve higher accuracy than the UKF (Ito & Xiong, 2000; Jia, Xin, & Cheng, 2012). It is, however, difficult to use the GHQF to solve high-dimensional filtering problems since the number of quadrature points and computational complexity of the GHQF increase exponentially with the dimension. Nevertheless, the GHQF can be used as the benchmark filter for comparing different Gaussian approximation filters.

Recently, a cubature Kalman filter (CKF) (Arasaratnam & Haykin, 2009) based on the third-degree spherical–radial cubature rule (Genz & Monahan, 1999; Lu & Darmofal, 2005) has been proposed and used in many applications, such as positioning (Pesonen & Piché, 2010), sensor data fusion (Fernandez-Prades & Vila-Valls, 2010), and attitude estimation (Li & Ge, 2011). From computational perspective, the third-degree cubature rule is a special form of the unscented transformation (UT) with better numerical stability (Arasaratnam & Haykin, 2009). However, like the UT, the third-degree cubature rule has limited accuracy. For example, it cannot compute exactly the Gaussian weighted integrals of such simple polynomial functions as $x_1^2x_2$, where $x_1$ and $x_2$ are two components of a Gaussian random vector. For nonlinear dynamical systems with large uncertainties, for example, long-term orbit uncertainty propagation (Horwood & Poore, 2011;
Nevels, Jia, Turnowicz, Xin, & Cheng, 2011) and magnetometer-based spacecraft attitude estimation (Jia, Xin, & Cheng, 2011), higher-degree CKFs are necessary to obtain more accurate results.

The contribution of this paper is to propose a new generalized class of CKFs with arbitrary degree of accuracy to improve the performance of the third-degree CKF and the UKF, and achieve performance close to that of the GHQF with less computational complexity.

The rest of this paper is organized as follows: the point-based Gaussian approximation filtering is briefly reviewed in Section 2; Section 3 presents the arbitrary high-degree cubature rules. In Section 4, the high-degree cubature rules are applied to a numerical integration problem and the fifth-degree CKF is applied to a target tracking problem and compared with the EKF, the UKF, the third-degree CKF, as well as the GHQF. Conclusion remarks are given in Section 5.

2. Point-based Gaussian approximation filters

In this section, point-based Gaussian approximation filters are briefly reviewed. Consider a class of nonlinear discrete-time dynamical systems described by:

\[ \dot{x}_k = f(x_{k-1}) + v_{k-1} \]
\[ y_k = h(x_k) + n_k \]

where \( x_k \in \mathbb{R}^n; y_k \in \mathbb{R}^m; v_{k-1} \) and \( n_k \) are independent Gaussian white process noise and measurement noise with covariance \( Q_{k-1} \) and \( R_k \), respectively.

The optimal Bayesian filtering for discrete-time dynamic systems includes two steps: prediction and update, governed by the Chapman–Kolmogorov equation and Bayes rule, respectively. Both steps involve using multivariate integrals to obtain the probability density functions (pdfs) of the state. However, these integrals are generally intractable (Gelf, 1974). Under the assumption of the Gaussian approximate filters that the pdfs of the state are Gaussian, only the means and covariances of functions of the Gaussian state need to be computed. The computation of the mean and covariance still requires multivariate integrals, but they are of the tractable Gaussian weighted form \( \int_{\mathbb{R}^n} g(x)N(\mathbf{x}; \tilde{\mathbf{x}}, \mathbf{P}) \, dx \), where \( g(x) \) is a general nonlinear function that has different forms in different filtering steps and \( N(\mathbf{x}; \tilde{\mathbf{x}}, \mathbf{P}) \) denotes the Gaussian distribution with mean \( \tilde{\mathbf{x}} \) and covariance \( \mathbf{P} \) (Arasaratnam & Haykin, 2009; Ito & Xiong, 2000; Jia et al., 2012). This type of Gaussian weighted integral can be approximated efficiently by different quadrature rules that eventually lead to different Gaussian approximation filters, such as, the UKF based on UT and the CKF based on the cubature rule. The integral with respect to \( N(\mathbf{x}; \mathbf{0}, I) \) can be approximated by the quadrature rule

\[ \int_{\mathbb{R}^n} g(x)N(\mathbf{x}; \mathbf{0}, I) \, dx \approx \sum_{i=1}^{N_p} W_i g(y_i) \]

where \( N_p \) is the total number of points; \( y_i \) and \( W_i \) are the integral points and weights, respectively, corresponding to \( N(\mathbf{x}; \mathbf{0}, I) \).

The integral with respect to the more general Gaussian distribution \( N(\mathbf{x}; \tilde{\mathbf{x}}, \mathbf{P}) \) can be approximated by

\[ \int_{\mathbb{R}^n} g(x)N(\mathbf{x}; \tilde{\mathbf{x}}, \mathbf{P}) \, dx = \int_{\mathbb{R}^n} g(Sx + \tilde{x}) N(x; \mathbf{0}, I) \, dx \approx \sum_{i=1}^{N_p} W_i g(Sy_i + \tilde{x}) \]

where \( \mathbf{P} = SS^T; S \) can be obtained by the Cholesky decomposition or the singular value decomposition.

The Gaussian approximation filters based on the numerical integration method in Eq. (4) can be summarized as follows (Ito & Xiong, 2000).

**Prediction:**

\[ \hat{x}_{k|k-1} = \sum_{i=1}^{N_p} W_i f(\xi_i) \]  (5)

\[ P_{k|k-1} = \sum_{i=1}^{N_p} W_i (f(\xi_i) - \hat{x}_{k|k-1})(f(\xi_i) - \hat{x}_{k|k-1})^T + Q_{k-1} \]  (6)

where \( \xi_i \) is the transformed point obtained from the covariance decomposition, i.e.

\[ P_{k-1|k-1} = SS^T; \quad \xi_i = SY_i + \hat{x}_{k-1|k-1} \]

and \( W_i \) is the weight for \( \xi_i \).

**Update:**

\[ \hat{x}_{k|k} = \hat{x}_{k|k-1} + L_k (y_k - z_k) \]
\[ P_{k|k} = P_{k|k-1} - L_k P_{xz} \]

where \( L_k = P_{xz}(R_k + P_{zz})^{-1} \)

\[ z_k = \sum_{i=1}^{N_p} W_i h(\tilde{\xi}_i) \]

\[ P_{xz} = \sum_{i=1}^{N_p} W_i (\tilde{\xi}_i - \hat{x}_{k|k-1})(h(\tilde{\xi}_i) - z_k)^T \]

\[ P_{zz} = \sum_{i=1}^{N_p} W_i (h(\tilde{\xi}_i) - z_k)(h(\tilde{\xi}_i) - z_k)^T \]

The quadrature points \( y_i \), and \( W_i \) can be obtained by many numerical rules, such as, the UT, the GHQ rule, and the cubature rule.

The UT with 2n+1 symmetric points \( y_i \) and weights \( W_i \) is given by

\[ y_i = [0, \ldots, 0]^T, \quad W_i = \kappa / (n + \kappa) \quad i = 1 \]
\[ y_i = \sqrt{n + \kappa} \epsilon_{i-1} \quad W_i = 1 / (2(n + \kappa)) \quad 2 \leq i \leq n + 1 \]
\[ y_i = -\sqrt{n + \kappa} \epsilon_{i-n-1} \quad W_i = 1 / (2(n + \kappa)) \]
\[ n + 2 \leq i \leq 2n + 1 \]

where \( \epsilon_{i-1} \) is the unit vector in \( \mathbb{R}^n \) with the \((i-1)\)th element being 1 and \( \kappa \) is a scaling parameter with the suggested optimal value of \( \kappa = 3 - n \) for Gaussian distributions (Julier et al., 2000). Note that the third-degree cubature rule has the identical form with the UT if \( \kappa = 0 \) (Arasaratnam & Haykin, 2009).

3. Arbitrary-degree cubature Kalman filter

The arbitrary-degree CKF has the same structure as the general Gaussian approximation filters, i.e., (5)–(14), but uses the arbitrary-degree spherical–radial cubature rule to compute the Gaussian type integrals in Eqs. (5)–(6) and (11)–(13). Before introducing the arbitrary-degree cubature rule, we first define the dth-degree rule.
**Definition 3.1.** For the integral
\[
\int_{\mathbb{R}^n} g(x)w_g(x)dx \approx \sum_{i} W_i g(y_i)
\]
(16)
where \( x = [x_1, x_2, \ldots, x_n]^T \in \mathbb{R}^n \) and \( w_g(x) \) is a given weighting function, Eq. (16) is a \( d \)-th-degree rule if it is exact for \( g(x) \) whose components are linear combinations of monomials \( x_1^{g_1}x_2^{g_2} \cdots x_n^{g_n} \) with the total degree up to \( d \) \((\alpha_1, \alpha_2, \ldots, \alpha_n) \) are nonnegative integers and \( 0 \leq \alpha_1 + \alpha_2 + \cdots + \alpha_n \leq d \) and there is at least one monomial of degree \( d + 1 \) for which Eq. (16) is not exact (Stroud, 1971).

In the cubature rule, the following integral is considered (Arasaratnam & Haykin, 2009):
\[
I(g) = \int_{\mathbb{R}^n} g(x) \exp \left(-x^T \mathbf{x}ight) dx
\]
(17)
where, the weighting function is \( w_g(x) = \exp \left(-x^T \mathbf{x}\right) \). Let \( x = rs \) with \( s \cdot s = 1 \) and \( r = \sqrt{x^T x} \). Eq. (17) can be transformed in the spherical–radial coordinate system
\[
I(g) = \int_{0}^{\infty} \int_{U_n} g(r)s^{|s|-1} \exp \left(-r^2 \right) d\sigma(s) dr
\]
(18)
where \( s = [s_1, s_2, \ldots, s_n]^T, U_n = \{ s \in \mathbb{R}^n : s_1^2 + s_2^2 + \cdots + s_n^2 = 1 \}, \) and \( \sigma(\cdot) \) is the spherical surface measure or the area element on \( U_n \).

Eq. (18) contains two types of integrals: the radial integral \( \int_{0}^{\infty} g_r(r) r^{|s|-1} \exp \left(-r^2 \right) dr \) with the weighting function \( w_g(r) = r^{|s|-1} \exp \left(-r^2 \right) \), and the spherical integral \( \int_{U_n} g_s(s) d\sigma(s) \) with the weighting function \( w_g(s) = 1 \).

The spherical–radial cubature rule of the CKF is based on the combination of these two types of integrals and the following fact (Arasaratnam & Haykin, 2009; Lu & Darmofal, 2005): if the radial integral can be approximated by an \( N_r \)-point radial rule
\[
\int_{0}^{\infty} g_r(r) r^{|s|-1} \exp \left(-r^2 \right) dr \approx \sum_{i=1}^{N_r} w_{r,i} g_r r_i
\]
(19)
and the spherical integral can be approximated by an \( N_s \)-point spherical rule,
\[
\int_{U_n} g_s(s) d\sigma(s) \approx \sum_{j=1}^{N_s} w_{s,j} g_s(s_j)
\]
(20)
then Eq. (17) can be approximated by,
\[
I(g) \approx \int_{0}^{\infty} \int_{U_n} g(r,s) d\sigma(s) dr
\]
\[
= \int_{0}^{\infty} r^{|s|-1} \exp \left(-r^2 \right) \sum_{j=1}^{N_s} w_{s,j} g(s_j) dr
\]
\[
= \sum_{j=1}^{N_s} \sum_{i=1}^{N_r} w_{r,i} w_{s,j} g_r r_i s_j
\]
(21)
where \( r_i \) and \( w_{r,i} \) are the points and weights for calculating the radial integral; \( s_j \) and \( w_{s,j} \) are the points and weights for calculating the spherical integral. The total number of points of \( I(g) \) is \( N_r N_s \) if \( r_i \neq 0 \); it is \( (N_r - 1) N_s + 1 \) if one of \( r_i \) is zero.

**Proposition 3.1.** The spherical–radial rule using the \( d \)-th-degree radial rule and the \( d \)-th-degree spherical rule has the \( d \)-th-degree accuracy.

**Proof.** Let a component of \( g(x) \) be \( g(x) = x_1^{g_1}x_2^{g_2} \cdots x_n^{g_n} \).
\[
I(g) = \int_{\mathbb{R}^n} x_1^{g_1}x_2^{g_2} \cdots x_n^{g_n} \exp \left(-x^T \mathbf{x}\right) dx
\]
\[
= \int_{0}^{\infty} \int_{U_n} (rs_1)^{g_1} (rs_2)^{g_2} \cdots (rs_n)^{g_n} r^{|s|-1} \times \exp \left(-r^2 \right) d\sigma(s) dr
\]
\[
= \left( \int_{0}^{\infty} r^{g_1+g_2+\cdots+g_n} r^{|s|-1} \exp \left(-r^2 \right) dr \right)
\]
\[
\times \left( \int_{U_n} s_1^{g_1} s_2^{g_2} \cdots s_n^{g_n} d\sigma(s) \right).
\]
(22)
If the radial rule and the spherical rule both have the \( d \)-th-degree accuracy, by definition they can compute the spherical integral \( \int_{U_n} s_1^{g_1} s_2^{g_2} \cdots s_n^{g_n} d\sigma(s) \) and the radial integral \( \int_{0}^{\infty} r^{g_1+g_2+\cdots+g_n} r^{|s|-1} \exp\left(-r^2 \right) dr \) exactly with \( 0 \leq \alpha_1 + \alpha_2 + \cdots + \alpha_n \leq d \). It can be easily seen that not all \( g(x) \) with \( \alpha_1 + \alpha_2 + \cdots + \alpha_n > d \) can be computed exactly. For example, \( g(x) = x_1^{g_1} \) cannot be exactly computed by any radial integral rule of accuracy degree \( d \). Hence, \( I(g) \) has the \( d \)-th-degree accuracy. \( \square \)

Note that for the spherical–radial cubature rule to have the \( d \)-th-degree accuracy, higher than \( d \)-th-degree spherical or radial rules may be used. In other words, the degrees of the spherical and radial rules do not need to be the same.

**Remark 3.1.** A cubature rule is said to be fully symmetric if the points generated from the rule are fully symmetric. If a spherical rule is fully symmetric, the corresponding spherical–radial cubature rule in Eq. (21) is fully symmetric as well. A fully symmetric cubature rule is exact for any odd function \( g(x) \) in Eq. (17) automatically. If a fully symmetric spherical rule is used in the spherical–radial cubature rule, which is the case in this paper, the radial rule only needs to be exact for even-degree polynomials in \( r \), that is, \( \alpha_1 + \alpha_2 + \cdots + \alpha_n \) is even, because the spherical rule vanishes when \( g_s(s) \) are polynomials of odd degrees.

### 3.1. Spherical rules

A variety of spherical rules can be found in (Cools, 1999, 2003; Cools & Rabinowitz, 1993; Genz, 2003; Mysovskikh, 1980, 1981; Stoyanova, 1997; Stroud, 1966, 1967, 1971; Stroud & Secrest, 1963; Xu, 1998). Despite the large number of spherical rules, efficient and arbitrary-degree spherical rules for arbitrary dimensions are few. In this section, Genz’s method for constructing the arbitrary-degree spherical rule is introduced and explicit formulae for the third- and fifth-degree spherical rules are given.

Genz proposed a framework based on Silvester’s integration rules (Silvester, 1970) to obtain spherical rules with arbitrary accuracies (Genz, 2003), which is given by the following theorem.

**Theorem 3.1** (Genz, 2003). For the spherical integral \( I_{U_n}(g_s) \triangleq \int_{U_n} g_s(s) d\sigma(s) \),
\[
I_{U_n,2m+1}(g_s) = \sum_{\left| p \right|=m} w_p G\{u_p\}
\]
(23)
is a \((2m+1)\)-th-degree rule. \( I_{U_n} \) denotes a spherical integral and \( I_{U_n,2m+1} \) denotes the \((2m+1)\)-th-degree spherical rule used to approximate the integral. The 1st-degree spherical rule is trivial. So \( m \geq 1 \) is assumed. Here \( w_p \) and \( G\{u_p\} \) are defined as
\[
w_p \triangleq I_{U_n}\left( \prod_{i=1}^{n} \prod_{j=1}^{p_i-1} \frac{s_j^2 - u_i^2}{p_i^2 - u_i^2} \right)
\]
(24)
\[
G\{u_p\} \triangleq 2^{-c(u_p)} \sum_{v} g_v\{u_1 u_{p_1}, u_2 u_{p_2}, \ldots, u_n u_{p_n}\}.
\]
(25)
The right-hand side of Eq. (24) is a spherical integral with the integral variables \( s_i \). The subscripts \( p_i \) in Eqs. (24) and (25) are nonnegative integers with \( p = [p_1, p_2, \ldots, p_n] \) and \( |p| = p_1 + p_2 + \cdots + p_n \). The superscript \( (\mathbf{p}) \) in Eq. (25) is the number of nonzero entries in \( u_\mathbf{p} = (u_{p_1}, u_{p_2}, \ldots, u_{p_n}) \). The points of the spherical rule \( I_{n_{2m+1}} \) are given by \( \left[ v_1 u_{p_1}, v_2 u_{p_2}, \ldots, v_n u_{p_n} \right]^T \), where \( v_i = \pm 1 \). In order to use the fewest points, \( u_\mathbf{p} \) are chosen to be \( u_{p_i} = \sqrt{p_i/m} \) \((p_i = 0, \ldots, m)\) (Genz, 2003). The weight on the point \( \left[ v_1 u_{p_1}, v_2 u_{p_2}, \ldots, v_n u_{p_n} \right]^T \) is \( 2^{-\mathcal{C}(\mathbf{p})} w_\mathbf{p} \).

**Remark 3.2.** The \((2m + 1)\)-th degree spherical rule constructed using the above procedure is fully symmetric. By Remark 3.1, it is exact for any \( g \), of odd degrees.

**Remark 3.3.** If \( \mathbf{q} \) is one of the permutations of \( \mathbf{p} \), then \( w_{\mathbf{q}} = w_\mathbf{p} \).

The following formula is helpful for computation of \( w_\mathbf{p} \) (Stroud, 1971).

\[
\int_{\mathcal{U}_n} s_1^{j_1} s_2^{j_2} \cdots s_n^{j_n} d\mathbf{s} = \begin{cases} \frac{2}{\Gamma((k_1 + 1)/2) \Gamma((k_2 + 1)/2)} & \text{if } \Gamma((k_1 + 1)/2) \text{ and } \Gamma((k_2 + 1)/2) \text{ are defined}, \\ 0 & \text{otherwise} \end{cases}
\]

where \( \Gamma(z) \) is the gamma function defined by the integral \( \Gamma(z) = \int_0^\infty \exp(-x) x^{z-1} \, dx \).

To construct a \((2m + 1)\)-th degree rule, feasible \( \mathbf{p} \) are determined from \( |\mathbf{p}| = m \). Then, \( u_\mathbf{p} \) and \( w_\mathbf{p} \) are calculated for each \( \mathbf{p} \) using \( u_{p_i} = \sqrt{p_i/m} \) and Eq. (24), respectively. The points \( \left[ v_1 u_{p_1}, v_2 u_{p_2}, \ldots, v_n u_{p_n} \right]^T \) and their associated weights \( 2^{-\mathcal{C}(\mathbf{p})} w_{\mathbf{p}} \) are then determined accordingly.

Now, the specific third- and fifth-degree rules are given. When \( m = 1, I_{n_{2m+1}}(\mathbf{g}_1) \) is a third-degree rule and \( p_1 + p_2 + \cdots + p_n = m = 1 \), which implies that \( p_1 = 0 \) or 1 and only one of them is 1. Hence, there are two different \( u_\mathbf{p} \), i.e. \( u_0 \) and \( u_1 \). In fact,

\[
u_0 = \sqrt{0/1} = 0 \quad \text{and} \quad u_1 = \sqrt{1/1} = 1.
\]

For a specific \( \mathbf{p} \), e.g. \( \mathbf{p} = [1, 0, \ldots, 0] \),

\[
w_\mathbf{p} = I_{u_{p_0}} \left( \prod_{i=1}^{n} \prod_{j=0}^{p_i-1} s_i^{2j} \prod_{i=1}^{p_i} u_i^{2j} - u_i^2 \right) = \int_{U_n} s_i^2 d\mathbf{s}.
\]

From Eqs. (24) and (26),

\[
w_\mathbf{p} = 2 \frac{\Gamma((3/2) \Gamma((n+1)/2) / \Gamma(n/2) = \frac{A_0}{n}}
\]

where \( A_0 = 2 \Gamma((3/2) / \Gamma(n/2) = 2 \sqrt{\pi}/ \Gamma(n/2) \) is the surface area of the unit sphere.

There are two points \( \left[ v_1 u_{p_1}, v_2 u_{p_2}, \ldots, v_n u_{p_n} \right]^T \) corresponding to \( \mathbf{p} = [1, 0, \ldots, 0] \), i.e. \([1, 0, \ldots, 0]^T \) and \([1, 0, \ldots, 0]^T \). Since there are \( n \) permutations of \( \mathbf{p} \), \( I_{u_{p_0}}(\mathbf{g}_1) \) has \( 2n \) points. By Remark 3.3, all weights are identical and they are \( 2^{-\mathcal{C}(\mathbf{p})} w_{\mathbf{p}} = 2^{-1} w_{\mathbf{p}} = \frac{A_0}{2n} \).

Hence, the third-degree spherical rule is given by

\[
I_{u_{p_0}}(\mathbf{g}_1) = \frac{A_0}{2n} \sum_{i=1}^{n} \left( \mathbf{g}(\mathbf{e}_i) + \mathbf{g}(\mathbf{-e}_i) \right).
\]

Following the similar procedure, the fifth-degree spherical rule when \( m = 2 \) can be obtained by Eq. (30):
where \( g_r(r) = r^l \) is a monomial in \( r \), with \( l \) an even integer. Note that for \( g_r(r) = r^l \), the right-hand side of Eq. (38) reduces to \( \frac{1}{2} \Gamma \left( \frac{n+1}{2} \right) \). Only even-degree monomials need to be matched because the spherical rule and the resultant spherical–radial cubature rule are fully symmetric. To obtain a \((2m+1)\)th-degree radial rule for the \((2m+1)\)th-degree spherical–radial cubature rule, Eq. (38) needs to be exact for \( l = 0, 2, \ldots , 2m \), which contains \((m+1)\) equations. The minimum number of points needed to satisfy the \((m+1)\) equations is \((m+1)/2\) (for \( m \) odd) or \((m/2+1)\) (for \( m \) even). In the derivation of the radial rule that follows, the minimum number of radial quadrature points is used so that the number of points of the spherical–radial cubature rule is minimized.

To construct the \((2m+1)\)th-degree radial rule, the GGLQ rule and the moment matching method lead to identical points and weights when \( m \) is odd and \((m+1)/2\) points are used. However, they may be different when \( m \) is even and \((m/2+1)\) points are used. In this case, the GGLQ rule satisfies Eq. (38) for \( l = 0, 2, \ldots , 2m \). Using the moment matching method, one has the freedom to choose whether Eq. (38) is exact for \( l = 0, 2, \ldots , 2m \) or \( l = 0, 2, \ldots , 2m, 2m+2 \). This allows for more flexible construction and reduced number of points of the spherical–radial cubature rule, which will be illustrated in the construction of the \((2m+1)\)th-degree radial rule.

Now, the third-degree and the fifth-degree radial rules are derived using the moment matching method with the minimum number of points. For the third-degree radial rule \((N_r = 1)\), the following equations need to be satisfied:

\[
\begin{align*}
    w_{r,1}^0 & = \frac{1}{2} \Gamma \left( \frac{1}{2} n \right) \\
    w_{r,2}^2 & = \frac{1}{2} \Gamma \left( \frac{1}{2} n + 1 \right) = \frac{n}{4} \Gamma \left( \frac{1}{2} n \right)
\end{align*}
\]

where the last equality follows from the identity \( \Gamma (z+1) = z \Gamma (z) \).

Solving Eq. (39) gives the point and weight for the third-degree radial rule,

\[
r_1 = \sqrt{\frac{n}{2}} \quad \text{(40)}
\]

\[
w_{r,1} = \frac{\Gamma (n/2)}{2} \quad \text{(41)}
\]

Note that the point and weight are identical with those obtained by the GGLQ.

For the fifth-degree radial rule \((N_r = 2)\), we require that the points and weights satisfy the following three equations:

\[
\begin{align*}
    w_{r,1}^0 + w_{r,2}^2 & = \frac{1}{2} \Gamma \left( \frac{1}{2} n \right) \\
    w_{r,1}^2 + w_{r,2}^4 & = \frac{1}{2} \Gamma \left( \frac{1}{2} n + 1 \right) = \frac{n}{4} \Gamma \left( \frac{1}{2} n \right) \\
    w_{r,1}^4 + w_{r,2}^6 & = \frac{1}{2} \Gamma \left( \frac{1}{2} n + 2 \right)
\end{align*}
\]

(42)

Since there are three equations and four variables in Eq. (42), there is one free variable. We can choose \( r_1 \) as the free variable and set it to 0. Solving these three equations gives the points and weights for the fifth-degree radial rule,

\[
\begin{align*}
    r_1 & = 0 \\
    r_2 & = \sqrt{\frac{n}{2} + 1}
\end{align*}
\]

(43)

Remark 3.4. The free variable \( r_1 \) can take on other values than 0. In fact, the fifth-degree radial rule obtained by the third-degree GGLQ rule uses two nonzero points. However, the number of points of the spherical–radial cubature rule is minimum when \( r_1 = 0 \). For example, the fifth-degree cubature rule based on the two-point GGLQ rule has \( 4n^2 \) points while the cubature rule based on the two-point moment matching with \( r_1 = 0 \) uses only \( 2n^2 + 1 \) points. Of course, the two-point GGLQ rule has higher “radial” accuracy than the two-point moment matching method, but that alone is insufficient to raise the accuracy of the spherical–radial cubature rule to the seventh-degree.

3.3. The cubature rules

The arbitrary-degree cubature rule is obtained by combining the spherical and radial rules in the previous sections using Eq. (21).

3.3.1. The third-degree cubature rule

Combining Eqs. (21), (29), (40) and (41), the third-degree cubature rule \((N_r = 1, N_i = 2n)\) is given by

\[
\int_{\mathbb{R}^d} g(x) N(x; 0, I) dx \\
= \frac{1}{\pi^{n/2}} \int_{\mathbb{R}^d} g(\sqrt{2}x) \exp(-x^t x) dx \\
= \frac{1}{\pi^{n/2}} \int_0^\infty \int_{\mathbb{R}^d} g(\sqrt{2}rs) r^{n-1} \exp(-r^2) dr ds \\
\approx \frac{1}{\pi^{n/2}} \sum_{i=1}^{N_r} \sum_{j=1}^{N_i} w_{r,i} w_{s,j} g(\sqrt{2}r_is_j) \\
= \frac{1}{\pi^{n/2}} \sum_{i=1}^{N_r} \Gamma(n/2) A_n \\
\times \left[ g(\sqrt{2} \cdot \sqrt{\frac{n}{2}} \cdot e_j) + g(-\sqrt{2} \cdot \sqrt{\frac{n}{2}} \cdot e_j) \right] \\
= \frac{1}{2n} \sum_{j=1}^{2n} [g(\sqrt{n} \cdot e_j) + g(-\sqrt{n} \cdot e_j)]. \quad (45)
\]

3.3.2. The fifth-degree cubature rule

Combining Eqs. (21), (30), (43) and (44), the fifth-degree cubature rule \((N_r = 2, N_i = 2n^2)\) is given by

\[
\int_{\mathbb{R}^d} g(x) N(x; 0, I) dx \\
\approx \frac{2}{n+2} g(0) + \frac{1}{(n+2)^2} \sum_{j=1}^{n(n+1)/2} \left( g(\sqrt{n+2} \cdot s_j^*) + g(-\sqrt{n+2} \cdot s_j^*) \right) \\
+ \frac{1}{(n+2)^2} \sum_{j=1}^{n(n+1)/2} \left( g(\sqrt{n+2} \cdot s_j^*) \right) \\
= \frac{2}{n+2} g(0) + \frac{1}{(n+2)^2} \sum_{j=1}^{n(n+1)/2} \left( g(\sqrt{n+2} \cdot s_j^*) + g(-\sqrt{n+2} \cdot s_j^*) \right) \\
+ \frac{1}{(n+2)^2} \sum_{j=1}^{n(n+1)/2} \left( g(\sqrt{n+2} \cdot s_j^*) \right). \quad (46)
\]
\[ + g(-\sqrt{n+2 \cdot \mathbf{e}}) + \frac{4-n}{2(n+2)^2} \sum_{j=1}^{n} (g(\sqrt{n+2 \cdot \mathbf{e}})) \]

Note that \( w_{ij} = \bar{w}_{ij} \) for \( j = 1, \ldots, 2n(n-1) \) and \( w_{ij} = \bar{w}_{ij} \) for \( j = 2n(n-1) + 1, \ldots, 2n^2 \). The number of points of the fifth-degree cubature rule is \( 2n^2 + 1 \).

**Remark 3.5.** The third-degree cubature rule used in the CKF of (Arasaratnam & Haykin, 2009) has the same form as Eq. (45) and can be viewed as a special case of the proposed cubature rules.

Compared to the Gauss–Hermite quadrature (GHQ) rule, the cubature rules can be viewed as a special case of the proposed cubature rules.

**Proposition 3.2.** For given \((2m + 1)\)th degree accuracy, the number of points of the cubature rule increases polynomially with the increase of the dimension \(n\) and the highest degree of the polynomial is \(m\).

**Proof.** As described after Eq. (21), the number of points of the cubature rule is \( N = N_i \), if \( r_i \neq 0 \) or \((N_i - 1) = N_i + 1\) if one of \( r_i \) is zero. \( N_i \) is independent from the dimension \(n\) for given \( m.\) \( N_i \) depends on the number of nonzero elements in \( \{ v_1, v_2, \ldots, v_n \} \) in the chiral spherical rule Eq. (25). If there are \( m \) nonzero elements in \( p \) satisfying \( |p| = m \), the number of possible \( [v_1, v_2, \ldots, v_n] \) is \( 2^m \), which is a polynomial in \( n \) with degree \( m \). Note that the factor \( 2^m \) comes from the different \( \pm \) combinations of \( v_i \). For a given \( m \), there are at most \( m \) nonzero elements in \( p \). In this case, \( N_i = 2^m \) \( \leq \) \( N_i \) (\( m = 1 \)) where \( N_i \) is a polynomial with highest degree less than \( m \). It is clear that \( N_i \) is a polynomial in the dimension \(n\) and the highest degree of this polynomial is \(m\). Therefore, the number of points of the cubature rule increases polynomially with the dimension \(n\) and the highest degree of the polynomial is \(m\).

**Remark 3.6.** The use of high-degree cubature rules is motivated by the fact that the attainable accuracy of the third-degree cubature rule and the UT are markedly lower than that of the Gauss–Hermite quadrature (GHQ) rule in problems involving high nonlinearities and large uncertainties (Jia et al., 2011, 2012). One possible concern about higher-degree cubature rules is that because some of their weights are negative, they are less stable than the third-degree cubature, just as the UT with \( \kappa = 3 \) is less stable than the third-degree cubature rule, especially for large \( n \). However, the high-degree cubature rules behave very differently from the UT with \( \kappa = 3 \) as \( n \to \infty \), the negative weight in the UT (see Eq. (15)) goes to \( -\infty \), but the negative weight in the fifth-degree cubature rule goes to 0 (see Eq. (46)). The weights of the cubature rules are therefore much more balanced. The examples in the next sections further show that the high-degree cubature rules do not suffer numerical instability problems.

### 4. Numerical results and analysis

Two numerical examples are given to show the performance of high-degree cubature rules and filters.

#### 4.1. Numerical computation of integral

In this section, the cubature rules, the UT, the GHQ rules, and the Monte-Carlo method are compared by calculating the integral of a highly nonlinear function with respect to the standard Gaussian distribution. Note that all of these numerical rules compute the integral with the same form of quadrature approximation shown in Eq. (3). The only difference is the strategies of selecting the quadrature points \( \gamma_i \) and the weights \( W_i \).

The integral considered is given by

\[ \int_\mathbb{R}^n \cos\left(\|x\|_2\right) N(x; 0, 1) \, dx \]  

where \( n = 6 \) is used in this example. The exact solution is \(-3Dawson(\sqrt{2}/2)\) where \( Dawson(x) = e^{-x^2} \int_0^x e^{t^2} \, dt \). The integral (47) is approximately equal to \( I_1 = -0.543583844 \), which is considered the true value.

The relative error is defined by

\[ R_{err} = \left| I_1 - I_{\text{appr}} \right| / |I_1| \]  

where \( I_{\text{appr}} \) is the integral results computed by different numerical integral rules.

Table 1 shows the integral results using different rules. Four different parameters are used for the UT: UT1–UT4 with \( \kappa = -3, 1, 2, 3 \), respectively. Up to 11th-degree cubature rules (CRs) are used. The \( m \)-point univariate GHQ is used to construct the multi-dimensional GHQ by the tensor product with three GHQ rules being used: GHQ1–GHQ3 with \( m = 3, 4, 5 \), respectively.

In this example, the computer code to obtain the spherical rules with different degrees of accuracy is adapted from Genz’s code (Genz, a). The radial rules with different degrees of accuracy are obtained by solving the moment-matching equations that has been discussed in Section 3.2.

It can be seen from Table 1 that the UT with different parameters generate quite different results and all of them are not accurate (more than 10% error). The cubature rules perform much better than the UT and the errors converge very fast with the increase of accuracy degrees. The high-degree (greater than 3) cubature rules are more accurate than the third-degree cubature rules. Generally, if \( d_1 > d_2 \), the \( d_1 \)-th degree cubature rule is more accurate than the \( d_2 \)-th-degree cubature rule. This example substantiates the necessity of using higher degree cubature rules. The MC method provides satisfying results only when a large number of points are used. Furthermore, the convergence rate of the MC method is much slower than the cubature rule. The standard deviation of the MC estimate is \( \sigma/\sqrt{N} \) (Robert & Casella, 2004), where \( \sigma \) is the standard deviation of \( \cos(\|x\|_2) \) and \( N \) is the number of MC points. For this problem, \( \sigma \) is approximately equal to 0.417736.
As indicated by Eq. (3), the computation complexity of the different rules can be evaluated by the number of points they demand. It can be seen from Table 1 that the higher-degree cubature rules need much fewer points than the GHQ rules and the MC method to achieve the same degrees of accuracy.

4.2. Target tracking

In this section, the fifth-degree cubature Kalman filter (CKF) obtained by Eq. (46) is compared with the EKF, the UKF, the third-degree CKF, the GHQF, and the particle filter (PF) in a target tracking application, which has been used as a benchmark problem to validate the performance of the third-degree CKF (Arasaratnam & Haykin, 2009). The PF used in this paper is the sampling importance resampling filter (SIR) (Arulampalam, Maskell, Gordon, & Clapp, 2002).

The dynamical equation is given by

\[
x_k = \begin{bmatrix}
1 & \frac{\sin(\omega_{k-1} \Delta t)}{\omega_{k-1}} & 0 & \frac{\cos(\omega_{k-1} \Delta t) - 1}{\omega_{k-1}} & 0 \\
0 & \cos(\omega_{k-1} \Delta t) & 0 & -\sin(\omega_{k-1} \Delta t) & 0 \\
0 & 1 - \cos(\omega_{k-1} \Delta t) & 1 & \sin(\omega_{k-1} \Delta t) & 0 \\
0 & \frac{\sin(\omega_{k-1} \Delta t)}{\omega_{k-1}} & 0 & \cos(\omega_{k-1} \Delta t) & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix} \times x_{k-1} + v_{k-1}
\]  

(49)

where \( x_k = [x_k, \dot{x}_k, y_k, \dot{y}_k, \omega_k]^T \); \([x_k, y_k]\) and \([\dot{x}_k, \dot{y}_k]\) are the position and velocity at time \( k \), respectively; \( \omega_k \) is the unknown turn rate at time \( k \); \( v_{k-1} \) is the white Gaussian noise with mean zero and covariance \( Q_{k-1} \).

The measurement equation is given by

\[
y_k = \begin{bmatrix}
\sqrt{x_k^2 + y_k^2} \\
\text{atan2}(y_k, x_k)
\end{bmatrix} + n_k
\]  

(51)

where \( \text{atan2} \) is the four-quadrant inverse tangent function; \( n_k \) is the white Gaussian measurement noise with zero mean and covariance \( R_k = \text{diag}(100 \text{ m}^2, 100 \text{ m rad}^2) \). The measurement sampling interval is \( \Delta t = 1 \text{ s} \).

The simulation results are based on 100 Monte Carlo runs. The initial estimate \( \hat{x}_0 \) is generated randomly from the normal distribution \( N(\hat{x}_0; x_0, P_0) \) with \( x_0 \) being the true initial value \( x_0 = [1000 \text{ m}, 300 \text{ m/s}, 1000 \text{ m}, 0, -3^\circ/\text{s}]^T \) and \( P_0 \) being the initial covariance \( P_0 = \text{diag}(100 \text{ m}^2, 10 \text{ m}^2/\text{s}^2, 100 \text{ m}^2, 10 \text{ m}^2/\text{s}^2, 100 \text{ m rad}^2/\text{s}^2) \).

The metrics used to compare the performance of various filters is the root mean square error (RMSE). The RMSEs of the position, velocity, and turn rate using different filters are shown in Figs. 1–3, respectively. The results of the EKF are not shown because it fails to converge in many runs. There is not much difference among the filters in the interval of \( 0–40 \text{ s} \). Hence, simulation results in an interval of \( 40–100 \text{ s} \) are shown in Figs. 1–3.
Table 2
RMSEs of different filters.

<table>
<thead>
<tr>
<th>Filters</th>
<th>Position</th>
<th>Velocity</th>
<th>$\omega$</th>
</tr>
</thead>
<tbody>
<tr>
<td>UKF ($\kappa = -2$)</td>
<td>115.4779</td>
<td>67.7711</td>
<td>5.5909</td>
</tr>
<tr>
<td>UKF ($\kappa = 1$)</td>
<td>63.2685</td>
<td>38.5835</td>
<td>4.2351</td>
</tr>
<tr>
<td>UKF ($\kappa = 2$)</td>
<td>60.8488</td>
<td>37.8964</td>
<td>4.2294</td>
</tr>
<tr>
<td>3rd-degree CKF</td>
<td>69.1524</td>
<td>40.8675</td>
<td>4.2712</td>
</tr>
<tr>
<td>5th-degree CKF</td>
<td>58.9539</td>
<td>36.9493</td>
<td>4.2130</td>
</tr>
<tr>
<td>GHQF</td>
<td>58.9646</td>
<td>36.9513</td>
<td>4.2128</td>
</tr>
<tr>
<td>PF</td>
<td>469.1847</td>
<td>62.2377</td>
<td>4.6485</td>
</tr>
</tbody>
</table>

Table 3
Number of points and computation time of different filters.

<table>
<thead>
<tr>
<th>Methods</th>
<th>Number of points</th>
<th>Time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>EKF</td>
<td>–</td>
<td>0.0268</td>
</tr>
<tr>
<td>UKF</td>
<td>11</td>
<td>0.2952</td>
</tr>
<tr>
<td>3rd-degree CKF</td>
<td>10</td>
<td>0.2855</td>
</tr>
<tr>
<td>5th-degree CKF</td>
<td>51</td>
<td>0.9226</td>
</tr>
<tr>
<td>GHQF</td>
<td>243</td>
<td>3.9057</td>
</tr>
<tr>
<td>PF</td>
<td>1500</td>
<td>49.0733</td>
</tr>
</tbody>
</table>

From Figs. 1(a), 2(a), and 3, it can be seen that all CKFs outperform the PF. In addition, the 5th-degree CKF outperforms the 3rd-degree CKF considerably and shows indistinguishable difference with the GHQF. From Figs. 1(b), 2(b), and 3, it can be seen that no UKFs are as accurate as the 5th-degree CKF. The UKF using the suggested parameter ($\kappa = 3 - n = -2$) is worse than using $\kappa = 1$ or $2$. One explanation is that the positive $\kappa$ can avoid possible loss of positive definiteness of the covariance matrices. It is also shown that the UKF after tuning the parameter $\kappa$ can outperform the 3rd-degree CKF. The PF with 1500 particles does not give a good performance for this application. The results contain large errors compared with GHQF or the 5th-degree CKF. Although the PF can be improved by increasing the number of particles or using some advanced techniques, the computational complexity will increase. PF is not further studied since it is out of the scope of this paper.

The RMSEs averaged over the time interval in Figs. 1–3 are given in Table 2 for comparison. The same conclusion can be obtained that the 5th-degree CKF is very close to the GHQF and better than the 3rd-degree CKF and UKFs.

The numbers of points and average computation time of 100 runs for these filters are shown in Table 3.

The filtering algorithms were coded with MATLAB and the simulations were run on a computer with Intel Core Duo CPU at 2.40 GHz. The computation time of the filters is approximately proportional to the number of points. For this specific problem, we conclude that the 5th-degree CKF performs the best in terms of the balance between estimation accuracy and computational complexity.

To test the robustness of the 5th-degree CKF, we deliberately make the measurement noise to be non-Gaussian

$$ n_k \sim 0.5N(0, R_1) + 0.5N(0, R_2) $$

with

$$ R_1 = \begin{bmatrix} 1000 m^2 & 150 m \text{ m rad} \\ 150 m \text{ m rad} & 100 m \text{ rad}^2 \end{bmatrix}, $$

$$ R_2 = \begin{bmatrix} 50 m^2 & 100 m \text{ m rad} \\ 100 m \text{ m rad} & 1000 m \text{ rad}^2 \end{bmatrix}. $$

A similar setup has been used to test the robustness of the 3rd-degree CKF in Arasaratnam and Haykin (2009).

In this scenario, the RMSEs of the position, velocity, and turn rate using different filters are shown in Figs. 4–6, respectively. The PF, the 3rd-degree CKF, and UKFs with $\kappa = -2$, 1 exhibit much larger error than the 5th-degree CKF or GHQF. The UKF with $\kappa = 2$ maintains a good performance but is less accurate than the 5th-degree CKF, which shows the best performance and maintains an indiscernibly close performance to the GHQF.

5. Conclusion

In this paper, a new class of high-degree cubature Kalman filters is proposed to enhance the estimation accuracy of the CKF. Two numerical examples are utilized to demonstrate that the proposed high-degree cubature rules and filters can indeed improve the accuracy of the numerical integration and estimation, and they exhibited more accurate results than the EKF, the
third-degree CKF, the UKF, and the PF. In addition, the proposed high-degree filters can achieve the close performance to the GHQF but is computationally much more efficient since the computational complexity of the proposed filters scales polynomially with the dimension.

References


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